Solving Life Cycle Problems with Biometric Risk by Artificial Insurance Markets

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Abstract: We study canonical consumption-savings problems of an individual involving uninsurable biometric risk. These problems are important in many applications from insurance economics and actuarial science. Since biometric risk is uninsurable, closed-form solutions do not exist and thus the problems must be approached by numerical methods. We propose a powerful approach where the solution is obtained by optimizing over a parametrized family of consumption strategies. In settings with mortality risk, critical illness risk, and habit formation, our solution method outperforms the well-established finite-difference approach both in run time and in precision. Our method also delivers a precision measure and closed-form representations of the optimal controls.

Keywords: dynamic programming, life-cycle models, biometric risk, insurance, habit formation

JEL subject codes: G10, D14, D91, E21, R21

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1 Introduction

Human wealth given by the present value of future income constitutes the most important asset for typical households. Biometric risk such as mortality risk and the risk of critical illness or disability is a severe threat since it potentially jeopardizes the ability to generate income. It is thus crucial to include biometric risk in life-cycle problems of households and a large strand of literature has developed in recent years (see, e.g., Gomes (2020) for a recent survey).

In practice, biometric risk cannot be perfectly insured (e.g., due to transaction costs) or insurance contracts do not exist at all (e.g., critical illness insurance in some countries). Therefore, the corresponding consumption-savings problem of a household involves unhedgeable shocks to labor income and cannot be solved in closed form. In turn, it is typically approached by grid-based numerical methods, which are computationally intesnive and cannot easily deal with a high number of state variables.

Our paper introduces a powerful method to solve consumption-savings problems of individuals who have no or limited access to insurance contracts for biometric risk. In settings with mortality risk, critical illness risk, and habit formation, our solution method outperforms the well-established finite difference approach both in run time and precision. In contrast to these approaches, our method also produces a closed-form consumption strategy and a precision measure. The latter can also be used to evaluate the performance of other numerical methods such as grid-based implementations.

To highlight our main ideas, we focus on a problem with no insurance contracts.[1] We refer to this situation as the true market. Our method builds upon the observation that we can explicitly solve the decision problem if we complete the market by adding continuously adjustable short-term insurance contracts. Since the premium attached to these contracts is not uniquely determined if they are not available, there are many complete markets that are consistent with the true market. We refer to all these markets as artificial markets, which is also motivated by the fact that the relevant contracts do not exist in practice.[2]

The explicit, optimal strategy in any of the artificial markets is generally infeasible in the true market as it involves some insurance demand, positive or negative, and may violate various constraints.[3] However, by minor modifications we can feasibilize the strategy, which leaves us with a

[1]Notice that our approach could also be applied to markets with partial access to insurance (e.g., term life insurance, but no critical illness insurance).

[2]In reality, short-selling of insurance contracts is not possible. Furthermore, only long-term contracts are available (if at all) that can only be revised at significant transaction costs. Therefore, real insurance markets are incomplete.

[3]From a formal point of view, infeasible means that a strategy is not admissible in the sense of Definition 2.1.

family of feasible consumption-savings strategies. The utility generated by each strategy is evaluated using Monte Carlo simulation. By embedding the simulation into a standard optimization routine, we determine the best of these strategies. We refer to our method as SAIMS, Simulation of Artificial Insurance Market Strategies.

First, we use the SAIMS method to solve a life-cycle consumption-savings problem with mortality risk if the individual has preferences involving habit formation in consumption. These preferences are well-established in the economic literature and have been applied in several papers, e.g., Ingersoll (1992), Gomes and Michaelides (2003), Browning and Collado (2007), Polkovnichenko (2007), and Munk (2008). Habit formation is numerically challenging since it leads to an additional state variable. For a wide range of calibrations (14 cases), we compare our approach with two grid-based implementations relying on a coarse and a fine grid. The coarse grid consists of 20 steps per year, 150 steps in the direction of wealth, and 75 steps in the direction of habit. The fine grid has 20 steps per year, 600 steps in the direction of wealth, and 300 steps in the direction of habit.[4] We find that our approach outperforms both implementations in all cases.

Next, we challenge our method by adding another layer of complexity: We add a third biometric state in which the individual is critically ill implying lower income and higher mortality risk. In this richer setup, our approach still outperforms the coarser implementation in all cases. The implementation with the fine grid is beat by our method in 11 out of 14 cases. In the remaining cases, our precision measure shows that this happens by an extremely small margin.

Conceptually, the SAIMS method is a generalization of an approach suggested and tested by Bick, Kraft, and Munk (2013) for life-cycle consumption-investment problems without biometric risk,[5] which is inspired by theoretical work of Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas (1992). Our paper makes a methodological contribution by showing how their idea can be extended to consumption-savings problems involving biometric risk.

In insurance economics and actuarial science, Markov chains are widely used to model all kinds of biometric risk (see, e.g., Bruhn and Steffensen (2011) and the references therein). For example, with mortality risk as the sole biometric risk the corresponding Markov chain has two states (alive and dead). In a continuous-time setting, the optimal consumption-savings strategy can be found by solving a system of Hamilton-Jacobi-Bellman (HJB) equations, one equation for each possible biometric state. A closed-form solution is only available if there is a complete insurance market. Extending the work by Merton (1969), Richard (1975) was the first to add mortality risk

[4]Notice that to what we refer as a coarse grid is actually pretty fine as it involves 11,250 nodes per time step. It is just coarser than fine grid that has 180,000 nodes per time step.

[5]Kraft and Steffensen (2013) consider an alternative approach to deal with portfolio constraints.

to the optimal consumption-investment problem of a power-utility investor. He derives a closed-form solution under the assumption that the decision-maker has continuously unrestricted access to instantaneous life insurance contracts. Extending Richard's idea, Kraft and Steffensen (2008) show that with a general finite-state Markov chain representing the biometric state, the problem amounts to solving a system of ordinary differential equations, again assuming the existence of continuously adjustable short-term insurance contracts for all Markov chain transitions, i.e., for all types of biometric risk. As emphasized above, this crucial assumption lacks realism as in practice insurance products are not available for all contingencies or are long-term contracts that can only be adjusted at significant costs.

Several papers have numerically solved life-cycle consumption problems with mortality risk or other types of biometric risk in the absence of insurance contracts. For example, Cocco, Gomes, and Maenhout (2005) consider a discrete-time model with deterministic mortality risk and a labor income process that features non-hedgeable diffusion risk. Cocco and Gomes (2012) allow for stochastic mortality risk and De Nardi, French, and Jones (2010) introduce health shocks. Hambel, Kraft, Schendel, and Steffensen (2017) study a setting with three biometric states (healthy, unhealthy, dead) and allow the individual to purchase long-term life insurance from a short menu of contracts that are costly to adjust.

The paper proceeds as follows. Section 2 specifies the dynamic optimization problem to be solved. Section 3 explains the details of our solution approach. Section 4 presents the benchmark calibration. Section 5 discusses numerical results for selected examples. We compare the performance of our method to the performance of finite-difference approaches. Section 6 introduces our precision measure, which can be used to evaluate any numerical method. Finally, Section 7 concludes. An appendix provides additional material such as proofs and technical details.

2 Framework

This section presents the finite-state Markov model and the agent's utility maximization problem.

2.1 Markov Chain Model

We consider a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{\{t\geq 0\}}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}_{\{t\geq 0\}}$ is generated by a discrete state variable $I = (I_t)_{t\geq 0}$ taking values in a finite set $\mathcal{K} = \{0, \dots, K\}$ of possible states and starting in state $I_0 = 0$ at time $t = 0$. We define the $K+1$-dimensional counting process $N = (N_t^0, \dots, N_t^K)_{t \geq 0}$ by[6]

$$N_t^k = \big|\{s \in (0,t] \mid I_{s^-} \neq k, I_s = k\}\big|, \tag{1}$$

which counts the number of jumps into state k until time t. We assume the existence of sufficiently smooth, age-dependent, but deterministic functions $\lambda^{j,k} : \mathbb{R}^+ \to \mathbb{R}^+$, $j,k \in \mathcal{K}$, such that N^k admits the stochastic intensity process $(\lambda^{I_{t^-},k}(t))_{t \geq 0}$ for $k \in \mathcal{K}$. We interpret the states of the Markov chain I as the agent's states of life which will be specified below. The state $I_t = K$ is assumed to be absorbing and corresponds to the agent's death. We call the transition intensity from a given state j to the state of death K, $\lambda^{j,K}$, the *hazard rate of death*. The random variable τ models the time at which the agent dies and is thus defined as the first jump of N^K or T, whichever occurs first, i.e.,

$$\tau = \inf_{t \geq 0}\{N_t^K > 0\} \wedge T. \tag{2}$$

In other words, the agent dies the latest at time T. This framework is the canonical model used in actuarial science.[7] It enables us to consider an optimal control problem with unspanned biometric risk and a stochastic planning horizon determined by the agent's time of death.

We assume that the agent receives an age- and state-dependent income stream $Y = (Y_t)_{t \geq 0}$ with $Y_t \geq 0$ for all $t \in [0,T]$. The agent retires at a known time $\widehat{T}$. In her active phase, income is assumed to follow the dynamics

$$\mathrm{d}Y_t = Y_{t^-}\mu(t, I_{t^-})\mathrm{d}t + \sum_{k: k \neq I_{t^-}} Y_{t^-}\big[p(t, I_{t^-}, k) - 1\big]\mathrm{d}N_t^k, \qquad t < \widehat{T}, \tag{3}$$

where μ and p are deterministic functions. Here, $\mu(t,j)$ is the growth rate of income given that the agent is in state j at time t and $p(t,j,k) > 0$ is the fraction of income that remains after a transition at time t from state j to state k.

In retirement, income is assumed to be risk-free and given by a state-dependent fraction $\Gamma(j) \in (0,1]$ (the replacement ratio) of income just prior to retirement, i.e.,

$$Y_t = \Gamma(I_{\widehat{T}})Y_{\widehat{T}}, \qquad t > \widehat{T}. \tag{4}$$

[6]Notice that $|A|$ denotes the cardinality of the set A.

[7]See, e.g., Kraft and Steffensen (2008) and the references therein.

2.2 Canonical Examples

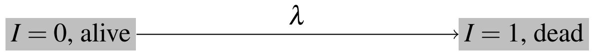

Figure 1: Survival Model.

The standard model involving biometric risk is the so-called *survival model* with only two states, called *alive* and *dead*. In this setup, the Markov chain jumps from the state 0 to state 1 upon death of the agent. The mortality intensity λ is age-dependent. Figure 1 illustrates the structure of the model.

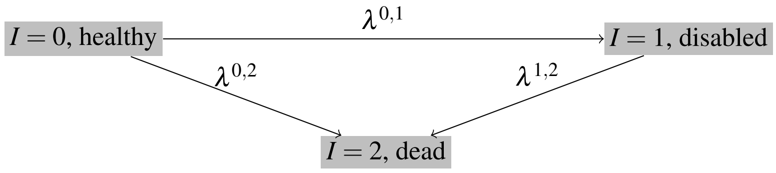

Figure 2: Critical Illness Model.

As a generalization, Figure 2 depicts the three-state Markov model studied by Hambel et al. (2017) and Hambel (2020). In this setup there are several potential transitions: In state 0, the agent can get critically ill. Then the chain jumps into state 1 with a certain age-dependent jump intensity $\lambda^{0,1}$. Second, the agent can die in each of these states. This happens with an age- and state-dependent intensity $\lambda^{j,2}$, $j \in \{0, 1\}$. We refer to this model as *critical illness model.*

2.3 Optimization Problem

Since Bick et al. (2013) have already analyzed models with diffusive risk, we focus on a framework with a Markov chain and abstract from risky financial assets. Therefore, the control variable of the agent is the consumption rate $c = (c_t)_{t \geq 0}$. The agent's financial wealth is kept on a bank account earning a constant interest rate of r. Her financial wealth evolves according to

$$\mathrm{d}X_t = (rX_t + Y_t - c_t)\mathrm{d}t, \tag{5}$$

where the initial financial wealth is positive, $X_0 > 0$. To demonstrate the richness of our approach, we allow the agent's preferences to exhibit internal habit formation as, for instance, in

Constantinides (1990) and Gomes and Michaelides (2003). The agent's habit level satisfies the dynamics

$$\mathrm{d}H_t = (\alpha c_t - \beta H_t)\mathrm{d}t, \tag{6}$$

where $\beta \geq 0$ measures the habit's persistence and $\alpha \geq 0$ is a scaling parameter. As Kraft, Munk, Seifried, and Wagner (2017) point out, the difference $\beta - \alpha$ measures the strength of the habit. The smaller the difference, the stronger the habit. We focus on the case $\beta > \alpha$ since otherwise, the habit level would increase over the life-cycle irrespective of the chosen consumption strategy. The agent's optimization problem is thus characterized by four state variables: financial wealth X, income Y, habit level H, and state of life I. We now define the set of admissible strategies in the true market as follows.

Definition 2.1 (Admissible Strategy)**.** *A consumption strategy c is called admissible if*

(i) c is progressively measurable w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$,

(ii) the differential equation (5) has a unique solution for all $X_0 > 0$,

(iii) $c_t > H_t$ for all $t \geq 0$, and

(iv) $X_\tau > 0$ (a.s.) where τ is given by (2).

We denote the set of admissible consumption strategies attainable at time t by $\mathcal{A}_t$.

Remark. We will consider two types of setups: the true market where the agent cannot insure biometric risks and artificial markets where these risks are perfectly insurable. Since the agent can die with a positive probability at any point in time, the constraint $X_\tau > 0$ implies that $X_t > 0$ for all $t \in [0, \tau]$ if, first, the agent has a bequest motive satisfying the Inada condition as assumed in (7) below and, second, there are no life insurance contracts. In the articial markets, we can relax condition (iv) and financial wealth X_t is allowed to become negative as long as the agent buys the right amount of life insurance contracts such that the sum of financial wealth and insurance sum is positive. See, e.g., Richard (1975).

The agent's preferences are captured by time-additive expected utility of surplus consumption, $s_t = c_t - H_t$, and bequest. An admissible consumption strategy c generates the utility index

$$J^c(t, x, y, h, j) = \mathbb{E}_{t,x,y,h,j}\left[\int_t^\tau e^{-\delta(s-t)}\frac{1}{1-\gamma}(c_s - H_s)^{1-\gamma}\mathrm{d}s + \varepsilon e^{-\delta(\tau-t)}\frac{1}{1-\gamma}X_\tau^{1-\gamma}\right],$$

where $\gamma > 1$ controls the agent's risk aversion, $\delta > 0$ is the time preference rate, and

$$\varepsilon > 0 \tag{7}$$

is the weight of the bequest motive. The optimization problem is thus given by

$$\sup_{c \in \mathcal{A}_0} J^c(0, x, y, h, 0) \tag{8}$$

and the agent's value function (indirect utility function) is

$$J(t, x, y, h, j) = \sup_{c \in \mathcal{A}_t} J^c(t, x, y, h, j). \tag{9}$$

It is well-known that the above optimization problem cannot be solved in closed form unless the agent can fully hedge the transitions between life states as assumed, for instance, in Kraft and Steffensen (2008). Only if the agent has access to a frictionless insurance market that provides fully flexible insurance products to hedge the transitions between the states of life, the model is complete and the above optimization problem can be solved in closed form. In this case, the value function has the representation

$$J^{\mathrm{com}}(t, x, y, h, j) = \frac{1}{1-\gamma}(x + yF^{\mathrm{com}}(t, j) - hB^{\mathrm{com}}(t, j))^{1-\gamma} G^{\mathrm{com}}(t, j)^{\gamma}. \tag{10}$$

where B^{com}, F^{com}, and G^{com} satisfy systems of ordinary differential equations. For the more realistic case where such a perfect insurance market does not exist, a closed-form solution for optimal consumption and the value function is not available. In particular, a separation of the form (10) does not hold since, among other issues, labor income risk cannot be hedged by available contracts.

3 SAIMS Solution Approach

3.1 Artificial Insurance Markets

Karatzas et al. (1991) and Cvitanić and Karatzas (1992) show how to construct the relevant artificial markets for a number of different portfolio constraints including the constraint that the financial market is incomplete. Applying this construction, Bick et al. (2013) develop an efficient solution approach to a broad class of portfolio-consumption problems where incompleteness comes from portfolio constraints or unspanned Brownian motions. Since these papers disregard biometric risk, we now generalize their approach.

First, we construct an artificial insurance market in which the agent can fully hedge all transitions between the different states of life. As mentioned above, this requires a frictionless insurance market with fully flexible insurance products that can be bought and sold without frictions such as leverage constraints or transaction costs. In particular, an artificial insurance market gives access to short-term insurance contracts for each type of biometric risk. These contracts provide coverage over the infinitesimal time interval $[t, t + \mathrm{d}t]$ at any point in time t. If the agent suffers a life shock of type k at time τ^k, i.e., if $I_{\tau^k -} \neq k$ and $I_{\tau^k} = k$, while holding the corresponding insurance with notional ι^k, the insurance company pays ι^k such that

$$X_{\tau^k} = X_{\tau^k -} + \iota^k_{\tau^k -}. \tag{11}$$

Following Richard (1975) and Pliska and Ye (2007), among others, we refer to ι^K as a life insurance sum. For the special case $k = K$, equation (11) is the agent's total bequest in the event of a death at time τ. For critical illness model, we interpret ι^1 as a critical illness insurance sum.

These insurances are available continuously and the agent buys them by paying an insurance premium at a rate $\pi^{j,k}_t$ that depends on both current age and state of life. The premium $\pi^{j,k}_t$ is proportional to the chosen notional ι^k_t. We refer to the ratio $\widehat{\lambda}^{j,k}_t = \pi^{j,k}_t / \iota^k_t$ as the unit premium for insurance against a transition into state k while being in state j at age t. The artificial insurance market is thus characterized by the set of unit premiums $\widehat{\lambda}^{j,k}_t$ for all pairs (j, k) with $j \neq k$ and $\lambda^{j,k}_t > 0$, i.e., all possible transitions of the Markov chain representing the different biometric states. This leads to the following definition of an artificial insurance market.

Definition 3.1 (Artificial Insurance Market)**.** *An artificial insurance market is characterized by a set of insurance contracts where the agent can fully flexible choose the corresponding notionals* $\iota^0, \dots, \iota^K$. *The agent pays an insurance premium for holding the insurance at time t at a rate*

$$\pi^k_t = \iota^k_t \widehat{\lambda}^{j,k}_t, \qquad j \neq k.$$

Here, $\widehat{\lambda}^{j,k}_t$ *is a non-negative stochastic process satisfying suitable integrability conditions and*

$$\widehat{\lambda}^{j,k}_t > 0 \quad \textit{if and only if} \quad \lambda^{j,k}_t > 0. \tag{12}$$

This process is called the artificial unit premium for a transition from state j to k.

Remarks. (a) Note that the insurances are actuarially fair if $\widehat{\lambda}_t^{j,k} = \lambda^{j,k}(t)$. (b) In an artificial insurance market, an admissible strategy will in general not satisfy $X_t > 0$, but the weaker condition $X_t + \iota_t^K > 0$ for all $t \geq 0$, since the agent can buy life insurance contracts.

In an artificial insurance market characterized by a set of artificial transition intensities $\widehat{\lambda} = (\widehat{\lambda}^{j,k})_{j \neq k, j,k=0}^K$, the agent must choose an insurance strategy $\iota = (\iota_t^0, \ldots, \iota_t^K)_{t \geq 0}$ and a consumption rate c in order to maximize her utility index. Let us denote by $J(t, x, y, h, j \mid \widehat{\lambda})$ the value function in the artificial market characterized by $\widehat{\lambda}$. We emphasize that the value function in each artificial market is an upper bound for the true, unknown value function associated with problem (9), i.e.,

$$J(t, x, y, h, j \mid \widehat{\lambda}) \geq J(t, x, y, h, j),$$

since the agent can choose zero insurance.

3.2 Analytical Results

There is not a closed-form solution of the value function $J(t, x, y, h, j \mid \widehat{\lambda})$ for every choice of $\widehat{\lambda}$. However, for artificial insurance markets that are characterized by a deterministic $\widehat{\lambda}$, the value function $J(t, x, y, h, j \mid \widehat{\lambda})$ can be calculated analytically, as the following theorem shows.

Theorem 3.2 (Explicit Solution to Artificial Market Problem)**.** *The value function of the artificial market characterized by the deterministic unit premiums $\widehat{\lambda} = (\widehat{\lambda}^{j,k})_{j \neq k, j,k=0}^K$ is given by*

$$J(t, x, y, h, j \mid \widehat{\lambda}) = \frac{1}{1-\gamma} \left(x + yF(t,j) - hB(t,j)\right)^{1-\gamma} G(t,j)^{\gamma}, \tag{13}$$

where both B and G satisfy a system of linear ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} B(t,j) = \Big[r + \sum_{k \neq j} \widehat{\lambda}^{j,k}(t) - \alpha + \beta\Big] B(t,j) - 1 - \sum_{k \neq j} \widehat{\lambda}^{j,k}(t) B(t,k),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} G(t,j) = \Delta(t,j) G(t,j) - 1 - \sum_{k \neq j} \lambda^{j,k}(t) \Big(\frac{\widehat{\lambda}^{j,k}(t)}{\lambda^{j,k}(t)}\Big)^{1-1/\gamma} G(t,k)$$

with boundary condition $B(t,K) = B(T,j) = 0$ and $G(t,K) = G(T,j) = \varepsilon^{1/\gamma}$. Here

$$\Delta(t,j) = \frac{1}{\gamma}\Big[\delta + \sum_{k \neq j} \lambda^{j,k}(t)\Big] + \frac{\gamma - 1}{\gamma}\Big[r + \sum_{k \neq j} \widehat{\lambda}^{j,k}(t)\Big].$$

Moreover, we obtain $F(t,j) = F^a(t,j)\mathbb{1}_{\{t<\widehat{T}\}}(t) + F^r(t,j)\mathbb{1}_{\{t\geq\widehat{T}\}}(t)$ *where* F^r *and* F^a *satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t}F^r(t,j) = \Big[r + \sum_{k\neq j}\widehat{\lambda}^{j,k}(t)\Big]F^r(t,j) - 1 - \sum_{k\neq j}\widehat{\lambda}^{j,k}(t)F^r(t,k),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}F^a(t,j) = \Big[r + \sum_{k\neq j}\widehat{\lambda}^{j,k}(t) - \mu(t,j)\Big]F^a(t,j) - 1 - \sum_{k\neq j}\widehat{\lambda}^{j,k}(t)p(t,j,k)F^a(t,k)$$

with boundary condition $F^r(t,K) = F^r(T,j) = F^a(t,K) = 0$ *and* $F^a(\widehat{T},j) = \Gamma(j)F^r(\widehat{T},j)$*. The optimal consumption-insurance strategy is given by the following feedback functions*

$$c(t,x,y,h,j) = h + \frac{x + yF(t,j) - hB(t,j)}{G(t,j)}\left(1 + \alpha B(t,j)\right)^{-1/\gamma}, \tag{14}$$

$$\iota^k(t,x,y,h,j) = \frac{x + yF(t,j) - hB(t,j)}{G(t,j)}\left(\frac{\widehat{\lambda}^{j,k}}{\lambda^{j,k}}\right)^{-1/\gamma}G(t,k) - x + hB(t,k) - yp(t,j,k)F(t,k). \tag{15}$$

The proof is given in Appendix A.

The value function and the consumption strategy are of the same form as in other settings settings with habit formation, labor income, and complete market. The term $yF(t,j)$ represents the agent's *human wealth* defined as the present value of future income [8] *Total wealth* is thus given by $x + yF(t,j)$. Following Kraft et al. (2017), the term $hB(t,j)$ is referred to as *tied-up wealth*. It denotes the wealth that is necessary to finance minimum future consumption dictated by the agent's habit level. Therefore, the term $x + yF(t,j) - hB(t,j)$ can be interpreted as the agent's *free wealth*. The function G is determined by the agent's preferences and the investment opportunity set which includes the risk-free asset and the artificial insurance contracts.

In the special case where the underlying Markov chain is directed, i.e., if $\lambda^{j,k} = 0$ for $k \leq j$, the functions B, F, and G have explicit representations as shown in the following corollary.

Corollary 3.3 (Directed Markov Chain)**.** *Suppose* $\lambda^{j,k} = 0$ *for* $k \leq j$*, then* B*,* F*, and* G *in Theorem 3.2 are explicitly given by*

$$B(t,j) = \int_t^T \mathrm{e}^{-\int_t^s\left(r + \sum_{k>j}\widehat{\lambda}^{j,k}(u) + \beta - \alpha\right)\mathrm{d}u}\Big(1 + \sum_{k>j}\widehat{\lambda}^{j,k}(s)B(s,k)\Big)\mathrm{d}s,$$
$$G(t,j) = \int_t^T \mathrm{e}^{-\int_t^s \Delta(u,j)\mathrm{d}u}\Big(1 + \sum_{k>j}\lambda^{j,k}(s)\Big(\frac{\widehat{\lambda}^{j,k}(s)}{\lambda^{j,k}(s)}\Big)^{1-1/\gamma}G(s,k)\Big)\mathrm{d}s + \varepsilon^{1/\gamma}\mathrm{e}^{-\int_t^T \Delta(u,j)\mathrm{d}u},$$

[8] Formally, human wealth is $\mathcal{H}_t = \mathbb{E}_t^{\mathbb{Q}}\big[\int_t^{T\wedge\tau}\mathrm{e}^{-r(s-t)}Y_s\mathrm{d}s\big]$, where $\mathbb{Q}$ is the risk-neutral measure, which is unique due to market completeness. The risk-neutral transition intensities are equal to the artificial unit premiums. It is straightforward to check that $\mathcal{H}_t = Y_tF(t,j)$.

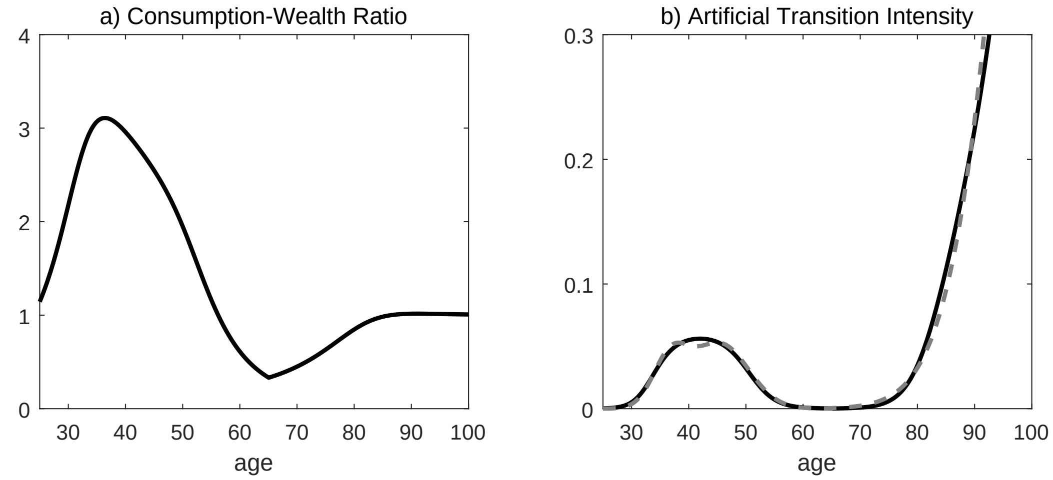

Figure 3: Survival Model without Habit Formation. Panel (a) depicts the optimal consumption-wealth ratio in a survival model without habit formation for the calibration in Table 1. The graph is produced by solving the optimization problem (8) with a grid-based approach. Panel (b) depicts the optimal artificial hazard rate of death $\widehat{\lambda}^{0,1}$ for which the optimal insurance demand is zero (black line), and the corresponding parametrization via (17) (gray dashed line). The black line is computed using a grid-based approach.

$$F^r(t,j) = \int_{\widehat{T}}^{T} \mathrm{e}^{-\int_t^s \left(r+\sum_{k>j}\widehat{\lambda}^{j,k}(u)\right)\mathrm{d}u}\Big(1+\sum_{k>j}\widehat{\lambda}^{j,k}(s)F^r(s,k)\Big)\mathrm{d}s$$

$$F^a(t,j) = \int_{t}^{\widehat{T}} \mathrm{e}^{-\int_t^s \left(r+\sum_{k>j}\widehat{\lambda}^{j,k}(u)-\mu(u,j)\right)\mathrm{d}u}\Big(1+\sum_{k>j}\widehat{\lambda}^{j,k}(s)p(s,j,k)F^a(s,k)\Big)\mathrm{d}s$$
$$+\Gamma(j)F^r(\widehat{T},j)\mathrm{e}^{-\int_t^{\widehat{T}} \left(r+\sum_{k>j}\widehat{\lambda}^{j,k}(u)-\mu(u,j)\right)\mathrm{d}u}$$

3.3 Parametrization of the Unit Premiums

In the previous section, we have solved the problem for any artificial market with *deterministic* specification of the unit premiums. To approximate the true solution well, we must find a suitable parametrization of these premiums that gets us close to the unknown solution of the true market. Bick et al. (2013) consider income diffusion risks and parametrize the artificial markets via affine functions of time. In our setting, however, such affine forms for $\widehat{\lambda}^{j,k}$ are too simplistic. Intuitively, the desire to insure biometric risks varies over life in a more complex way than the desire to fully hedge diffusive income risk.

The value function of every artificial insurance market constitutes an upper bound for the value function in the true market. To obtain a tight upper bound, we shall thus find a parametrization where the minimum over the corresponding artificial insurance markets is sufficiently low. To conjecture a suitable functional form of the artificial unit premiums, we first consider the special case without habit formation, $\alpha = \beta = H_0 = 0$, where the state variable H is redundant. Cvitanić and Karatzas (1992) show that in their setting the solution to the true market is obtained if the demand for the artificial contracts is zero. Applying these ideas to our framework, we thus conclude that a suitable parametrization is characterized by an optimal insurance demand equal or at least close to zero, i.e., $\iota^k \approx 0$ for all $k = 0, \ldots, K$. According to (15) this requires

$$\widehat{\lambda}^{j,K} \approx \varepsilon \lambda^{j,K} \Big(\frac{c(t,x,y,j)}{x}\Big)^{\gamma}, \qquad \widehat{\lambda}^{j,k} \approx \lambda^{j,k} \Big(\frac{c(t,x,y,j)}{c(t,x,y,k)}\Big)^{\gamma}. \tag{16}$$

The optimal artificial hazard rate of death thus involves the consumption-wealth ratio. For illustrative purposes, Panel (a) of Figure 3 depicts the optimal consumption-wealth ratio over the life cycle for an initial wealth-to-income ratio of $x/y = 1$ in a survival model without habit formation; this solution is calculated by applying a grid-based method to the utility maximization problem where the calibration is summarized in Table 1. We emphasize that the consumption-wealth ratio and thus the optimal artificial hazard rate of death depend on the initial wealth-to-income ratio. Figure 3 shows that the optimal consumption-wealth ratio c/x is not a "simple" function of time, which is very different from the results in Bick et al. (2013). Therefore, we use the following parametrization of the artificial insurance markets[9]

$$\widehat{\lambda}^{j,k}(t) = \mathbf{1}_{\{t<\widehat{T}\}} \sum_{i=1}^{2} \vartheta^{j,k}_{i,1} \exp\Big\{ - \Big[\frac{t-\vartheta^{j,k}_{i,2}}{\vartheta^{j,k}_{i,3}}\Big]\Big\} + \mathbf{1}_{\{t\geq\widehat{T}\}} \sum_{i=3}^{4} \vartheta^{j,k}_{i,1} \exp\Big\{ - \Big[\frac{t-\vartheta^{j,k}_{i,2}}{\vartheta^{j,k}_{i,3}}\Big]\Big\}, \tag{17}$$

which takes the pronounced non-linear shape of the consumption-wealth ratio into account. With this specification, the corresponding strategies and the value function are parametrized by a set of constants $\theta = (\vartheta^{j,k}_{i,n})_{i=1,\ldots,4, n=1,2,3, k\neq j,k,j\leq K}$. We denote the associated value function by $J^\theta(t,x,y,h,j)$. Similarly, we use the notation F^θ, G^θ, and B^θ to stress the dependence on the parameter set θ. As each artificial market defines an upper bound on the value function (9) in the true model, we apply a standard numerical optimization algorithm to determine a tight upper bound, $\overline{J}$, by a minimization over the parametrized artificial transition intensities, i.e.,

$$\overline{J}(t,x,y,h,j) = J^{\overline{\theta}}(t,x,y,h,j) = \min_{\theta} J^\theta(t,x,y,h,j). \tag{18}$$

For $x/y = 1$, the black line in Panel (b) of Figure 3 depicts the optimal artificial unit life insurance premium $\widehat{\lambda}^{0,1}$ for which optimal insurance demand is zero, whereas the gray dashed line shows the best corresponding parametrization (17) in the sense of (18). Apparently, parametrization (17) yields a close fit of the optimal artificial unit life premium.

[9] Of course, we have to calibrate (17) such that (12) is satisfied.

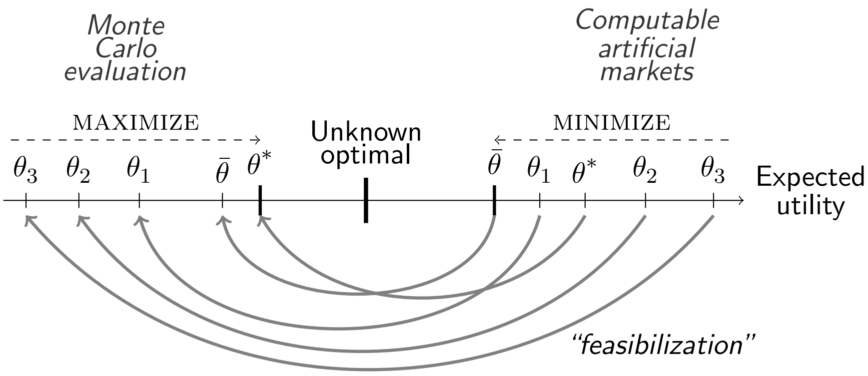

Figure 4: Solution Approach. The axis shows the agent's expected utility. *Unknown optimal* represents the indirect utility in the original problem, i.e., the expected utility generated by the unknown optimal consumption strategy. Each point to the right corresponds to the indirect utility in an artificial insurance market with deterministic insurance premiums characterized by some parameter set θ. The corresponding strategy is transformed into an admissible strategy in the true market which generates an expected utility on the left part of the axis. The best of these strategies, parametrized by θ^*, is derived from the optimal strategy in an artificial insurance market for some θ.

3.4 Best Strategy for the True Market

Recall that in an artificial insurance market, we do not require that financial wealth stays strictly positive because the agent can decide to buy insurance contracts that compensate for a potentially negative wealth. The optimal consumption strategy of an artificial market is thus in general not admissible in the true market, but it can be transformed into an admissible strategy following Bick et al. (2013).

Without habit formation we consider modified consumption strategies of the form

$$c^{\theta,\eta}(t,x,y,j) = \frac{x + yF^{\theta}(t,j)\big(1-\mathrm{e}^{-\eta_j x}\big)}{G^{\theta}(t,j)} \tag{19}$$

for a parameter set θ and a set of positive constants $\eta = (\eta_j)_{j=0,\ldots,K}$. Compared to (14), the human wealth $yF^{\theta}(t,j)$ is scaled down by the factor $1-\mathrm{e}^{-\eta_j x}$, which becomes negligible as financial wealth x approaches zero. In particular, this factor turns (14) into an admissible strategy in the true market as the following proposition shows. It implies that $X_t > 0$ for all $t \in [0,\tau]$.[10]

Proposition 3.4 (Feasibilization without Habit Formation)**.** *The strategy (19) is an admissible consumption strategy in the true market for any choice of θ and $\eta_j > 0$.*

The proof is given in Appendix A where we in particular show that the consumption rate becomes approximately linear in financial wealth if financial wealth becomes small.

[10] See also the remark following Definition 2.1.

With habit formation, the situation is more involved since the consumption rate does not show this behavior. This is because the habit level is in general not linear in financial wealth for small financial wealth levels. Consequently, we must ensure that the agent is able to finance her habit level. Define

$$c^{\theta,\eta}(t,x,y,h,j) = h + \frac{x + \left[yF^{\theta}(t,j) - hB^{\theta}(t,j)\right]\left(1 - \mathrm{e}^{-\eta_j x}\right)}{G^{\theta}(t,j)}\left(1 + \alpha B^{\theta}(t,j)\right)^{-1/\gamma},$$

which is a modified version of (14). We thus consider the following consumption strategy where the agent "plays safe" below a certain wealth level:

$$c_t = \begin{cases} c^{\theta,\eta}(t,X_t,Y_t,H_t,I_t) & \text{for } X_t > \underline{X}_t \text{ and } c^{\theta,\eta}(t,X_t,Y_t,H_t,I_t) > H_t, \\ kH_t & \text{else,} \end{cases} \tag{20}$$

where $k \in (1, \beta/\alpha)$ can be chosen arbitrarily. We will show below that (20) is admissible given that we choose $\underline{X}_t$ appropriately. For our calibrations discussed in Section 5, the suitable choice of $\underline{X}_t$ is sufficiently small such that in our numerical examples the financial wealth of the agent never reaches this level. Besides, the second condition $c^{\theta,\eta}(t,X_t,Y_t,H_t,I_t) > H_t$ is also never binding. Therefore, both conditions is practically less relevant albeit theoretically necessary.

Proposition 3.5 (Feasibilization with Habit Formation)**.** *Define the threshold $\underline{X}_t$ as in (23) and assume that $X_0 \geq \underline{X}_0$.*[11] *The strategy (20) is an admissible consumption strategy in the true market for any choice of θ and $\eta_j > 0$.*

We determine our best strategy by maximizing the indirect utility generated by $c^{\theta,\eta}$. For this purpose, we compute the corresponding utility index $J^{\theta,\eta}(t,x,y,h,j)$ via Monte Carlo simulation and maximize over the parameter set (θ,η). Figure 4 depicts the general structure of our solution approach that we refer to as SAIMS, short for Simulation of Artificial Insurance Market Strategies.

4 Calibration

The results presented in Section 5 are based on Monte Carlo simulations using 10 000 paths where the strategies are updated 20 times per year. The initial time is $t = 0$ and corresponds to an initial age of 25 years. Table 1 summarizes the model parameters.

[11] The condition $X_0 \geq \underline{X}_0$ ensures that the habit is initially financiable.

General Parameters		
α	habit scaling parameter	0.1
β	habit persistence parameter	0.174
γ	risk aversion parameter	4
δ	time preference rate	0.03
ε	bequest weight	1
r	risk-free rate	0.01
X_0	initial financial wealth	2
Y_0	initial income	2
H_0	initial habit	1
$\widehat{T}$	retirement date	40
Γ	replacement ratio	0.6
μ_y	expected income growth	0.01
a	age at $t = 0$	25
Survival Model		
m	x-axis displacement	84.56
b	steepness parameter	8.8
Critical Illness Model		
m	x-axis displacement	89.45
b	steepness parameter	6.5
k_1	constant impact of a health shock	0.048
k_2	age-dependent impact of a health shock	0.0008
$\widehat{a}$	scaling parameter	0.02489
$\widehat{b}$	x-axis displacement	66.96
$\widehat{c}$	steepness parameter	29.42
$p^{0,1}(t < \widehat{T})$	income level after a health shock while working	0.8
$p^{0,1}(t \geq \widehat{T})$	income level after a health shock during retirement	1
$p^{0,2}$	income level at death without previous health shock	0
$p^{1,2}$	income level at death with previous health shock	0

Table 1: Benchmark Calibration Parameters. The initial wealth $X_0 = 2$ and annual income $Y_0 = 2$ are interpreted as \$20 000.

4.1 General Parameters

We choose a risk-free rate of $r = 0.01$. The preference parameters are standard values in the life-cycle literature. The agent has a risk aversion parameter of $\gamma = 4$, a time preference rate of $\delta = 0.03$, and a bequest weight of $\varepsilon = 1$. We consider models with and without habit formation. For models with habit formation, we follow Kraft et al. (2017), and choose for the scaling parameter $\alpha = 0.1$ and for the persistence parameter $\beta = 0.174$. The agent retires at an age of 65 years corresponding to $\widehat{T} = 40$. The replacement ratio is set to $\Gamma = 0.6$. In the active phase, income grows at a constant rate of $\mu_y = 0.01$.

4.2 Biometric Risk

Survival Model We assume that the hazard rate of death $\lambda^{0,1}(t)$ follows a Gompertz mortality law of the form

$$\lambda^{0,1}(t) = \frac{1}{b}\exp\left\{\frac{a+t-m}{b}\right\}$$

where a denotes the agent's age at $t = 0$. Our calibration to the life tables from Germany as of 2010 yields $b = 8.8$, $m = 84.56$.

Critical Illness Model The calibration of the critical illness model is taken from Hambel et al. (2017).The hazard rate of death follows a generalized Gompertz mortality model of the form

$$\lambda^{j,2}(t) = \begin{cases} \frac{1}{b}\exp\left\{\frac{a+t-m}{b}\right\} & \text{for } j = 0, \\ \frac{1}{b}\exp\left\{\frac{a+t-m}{b}\right\} + k_1 + k_2(a+t) & \text{for } j = 1. \end{cases}$$

Notice that in case of a health shock the hazard rate of death increases by a constant term k_1 and an age-dependent term $k_2 t$. The model is calibrated to German mortality and cancer data, which yields $m = 89.45$, $b = 6.5$, $k_1 = 0.032$, $k_2 = 0.0008$. We assume that the health shock intensity has the following age-dependent functional form

$$\lambda^{0,1}(t) = \widehat{a}\exp\left\{-\left(\frac{\min(t,65)-\widehat{b}}{\widehat{c}}\right)^2\right\}.$$

The calibration to German cancer data yields $\widehat{a} = 0.02489$, $\widehat{b} = 66.95$, $\widehat{c} = 29.42$.

5 Numerical Results

This section discusses our numerical results. We firstly present our benchmark results with and without habit formation. Then, we compare the performance with a well-established finite-differences approach.

5.1 Benchmark Results

Survival Model Figure 5 depicts the best consumption strategy derived by our method (black lines) and the corresponding habit level (gray lines). The light dashed lines show the best consumption strategies for models without habit formation. Panel (a) depicts the results for the

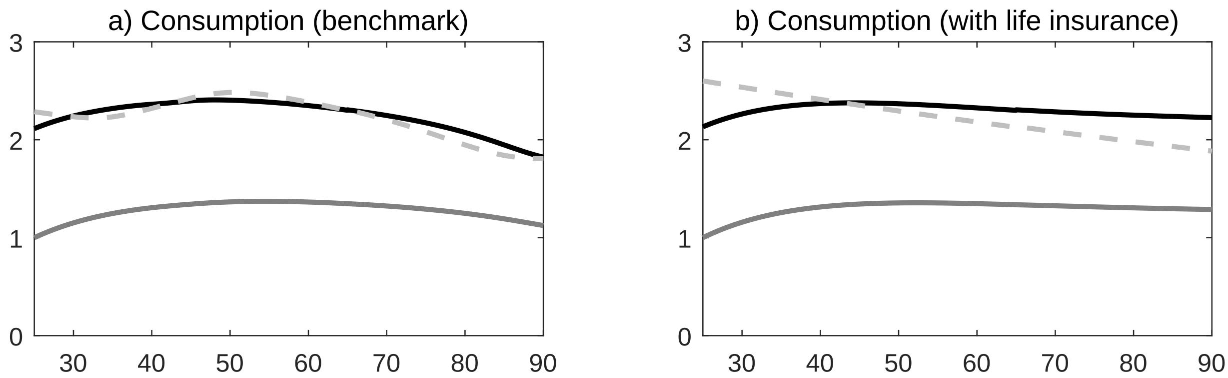

Figure 5: Optimal Consumption in the Survival Model. The figure shows the best consumption strategy derived by our method (black lines) and the corresponding habit level (gray lines). The light dashed lines show optimal consumption for models without habit formation. Panel (a) depicts the results for the benchmark model. Panel (b) depicts the results for a model where the agent has access to a perfect insurance market with actuarially fair priced life insurance as in Richard (1975).

benchmark model. Note that consumption is smoother if the agent's preferences exhibit habit formation. Interestingly, the consumption strategy involves a hump around the age of 45, which is in line with empirical evidence, see, e.g., Gourinchas and Parker (2002). Panel (b) depicts the results for a model where the agent has access to a perfect life insurance market as in Richard (1975), i.e., the agent can continuously purchase and sell fairly priced life insurance. This panel suggests that with access to a perfect life insurance market, the consumption pattern less in line with real-world consumption data.

Critical Illness Model Figure 6 depicts the best consumption strategy conditional on staying healthy. The shape of the curve is similar to the survival model with a hump at an age of about age 45. The risk of becoming critically ill and thus receiving lower income induces the agent to cut down on consumption while healthy in order to avoid a large reduction in the event of illness. This is because the agent wants to smooth consumption across states. On the other hand, when becoming critically ill, the mortality risk increases, which reduces the expected time span for which consumption has to be financed and thus reduces the necessary savings.

5.2 Comparison with finite-differences

This section reports the numerical results for several model specifications and compares the performance of our approach with a finite-difference implementation of the HJB equation, see, e.g., Brennan, Schwartz, and Lagnado (1997) and Munk and Sørensen (2010). It is well known that both the precision and the run time of this popular method strongly depend on the number of state variables and grid points. To challenge our approach, we speed up our implementation of the

Survival Model without Habit Formation						
	SAIMS		**Fine Grid**		**Coarse Grid**	
	J^{θ^*}	P^{θ^*}	J^{fi}	P^{fi}	J^{co}	P^{co}
Benchmark	-5.996**	1.0000	-6.003	0.9996	-6.008	0.9993
$\gamma = 2$	-23.164**	0.9995	-23.168	0.9994	-23.172	0.9992
$\gamma = 6$	-2.859**	0.9999	-2.865	0.9995	-2.871	0.9991
$\delta = 0.01$	-9.682**	0.9999	-9.670	0.9995	-9.703	0.9992
$\delta = 0.05$	-4.124**	1.0000	-4.129	0.9999	-4.133	0.9996
$\varepsilon = 0.2$	-5.885**	0.9999	-5.892	0.9995	-5.897	0.9992
$\varepsilon = 5$	-6.173**	1.0000	-6.181	0.9996	-6.185	0.9993
Survival Model with Habit Formation						
	SAIMS		**Fine Grid**		**Coarse Grid**	
	J^{θ^*}	P^{θ^*}	J^{fi}	P^{fi}	J^{co}	P^{co}
Benchmark	-66.660**	1.0000	-66.970	0.9984	-67.187	0.9974
$\gamma = 2$	-51.541**	0.9999	-51.614	0.9985	-51.650	0.9978
$\gamma = 6$	-159.916**	0.9999	-161.239	0.9983	-162.273	0.9970
$\delta = 0.01$	-112.816**	1.0000	-113.200	0.9989	-113.419	0.9982
$\delta = 0.05$	-43.051**	1.0000	-43.274	0.9983	-43.459	0.9969
$\varepsilon = 0.2$	-66.178**	1.0000	-66.498	0.9984	-66.724	0.9972
$\varepsilon = 5$	-67.437**	1.0000	-67.736	0.9985	-67.944	0.9975
Critical Illness Model without Habit Formation						
	SAIMS		**Fine Grid**		**Coarse Grid**	
	J^{θ^*}	P^{θ^*}	J^{fi}	P^{fi}	J^{co}	P^{co}
Benchmark	-6.161**	0.9990	-6.165	0.9987	-6.171	0.9984
$\gamma = 2$	-23.483**	0.9859	-23.484	0.9859	-23.490	0.9856
$\gamma = 6$	-2.987**	0.9981	-2.993	0.9977	-2.999	0.9973
$\delta = 0.01$	-10.114**	0.9992	-10.117	0.9991	-10.125	0.9989
$\delta = 0.05$	-4.181*	0.9983	-4.177	0.9986	-4.182	0.9982
$\varepsilon = 0.2$	-6.057**	0.9992	-6.064	0.9988	-6.071	0.9985
$\varepsilon = 5$	-6.323**	0.9985	-6.327	0.9983	-6.333	0.9979
Critical Illness Model with Habit Formation						
	SAIMS		**Fine Grid**		**Coarse Grid**	
	J^{θ^*}	P^{θ^*}	J^{fi}	P^{fi}	J^{co}	P^{co}
Benchmark	-70.234**	0.9984	-70.260	0.9983	-70.480	0.9972
$\gamma = 2$	-52.574*	0.9988	-52.548	0.9993	-52.585	0.9986
$\gamma = 6$	-175.346**	0.9989	-175.913	0.9982	-177.119	0.9969
$\delta = 0.01$	-118.747*	0.9988	-118.631	0.9991	-118.848	0.9985
$\delta = 0.05$	-44.952**	0.9982	-45.050	0.9975	-45.251	0.9960
$\varepsilon = 0.2$	-69.719**	0.9985	-69.778	0.9982	-70.005	0.9971
$\varepsilon = 5$	-71.037**	0.9984	-71.039	0.9984	-71.251	0.9974

Table 2: Numerical Results. The table reports the simulation results for several model specifications and for when one key input variable is varied and the other parameter values are the same as in the benchmark case. One star (*) indicates that the SAIMS method has a higher precision than the finite-difference approach on the coarse grid. Two stars (**) indicate that the method even beats the finite-difference approach on the fine grid. It also reports the values of the precision measure P discussed in Section 6. Here P^{θ^*} is the precision of SAIMS, whereas P^{fi} and P^{co} are the precisions of the grid-based methods (fine, coarse).

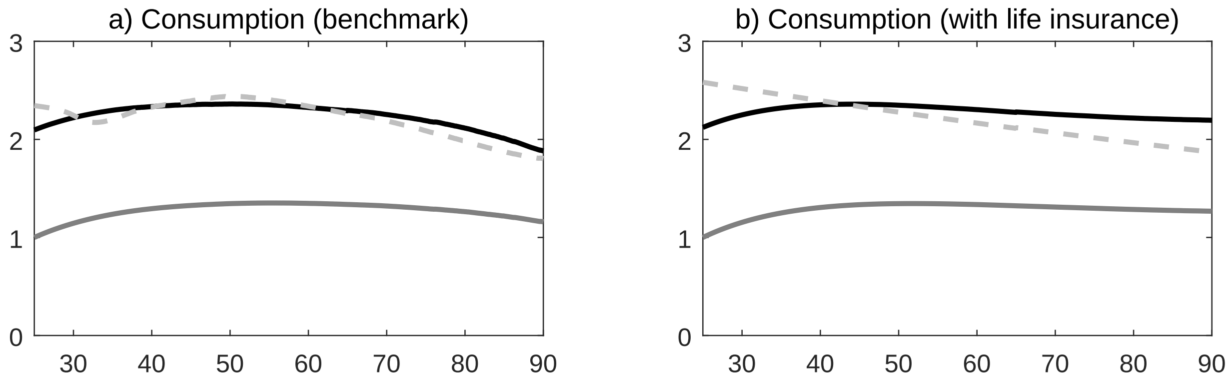

Figure 6: Optimal Consumption in the Critical Illness Model. The figure shows the best consumption strategy derived by our method (black lines) and the corresponding habit level (gray lines). The light dashed lines show consumption for models without habit formation. Panel (a) depicts the results for the benchmark model. Panel (b) depicts the results for a model where the agent has access to a perfect insurance market with actuarially fair priced life and critical illness insurance products.

grid-based method and exploit the homogeneity property of the value function, $J(t, x, y, h, j) = y^{1-\gamma} V(t, x/y, h/y, j)$, which reduces the number of state variables by one, see Lemma B.1. Solving for V requires a discretization of the simplified HJB equation (26), which leads to K different grids each of dimension three with variables t, $\widehat{x} = x/y$, and $\widehat{h} = h/y$. With only one state variable, the finite-difference approach is quite fast. The run time is below two minutes in our examples without habit formation. To compare our SAIMS method with the grid-based approach, we use two different grid sizes with 20 time steps per year. This is similar to Munk and Sørensen (2010), but on the higher side compared to the existing literature. For the state variables, the coarse grid involves $N_{\widehat{x}} = 150$ steps for normalized wealth and $N_{\widehat{h}} = 75$ steps for normalized habit. The fine grid uses $N_{\widehat{x}} = 600$, $N_{\widehat{h}} = 300$. Notice that the fine grid involves almost the double amount of grid points in the state-variable dimensions compared to Munk and Sørensen (2010).

Table 2 shows that the SAIMS method outperforms the finite-difference approaches in almost all cases. Although grid-based methods are supposed to deliver the optimal solution, the indirect utility achieved by the fine grid is worse in 25 out of 28 cases. Our numerical analyses also show that the run time of the finite-difference approach on the fine grid with two state variable is on average eight times higher than the the run time of our SAIMS method. For instance, the benchmark calibration of the critical illness model is solved in 39 minutes by the SAIMS approach and in 310 minutes on the fine grid. This indicates that the SAIMS method also outperforms the well-established finite-difference approach in run time.[12] Finally, by contrast to

[12]The run time of the SAIMS method can be sped up even further if one parallelizes the Monte Carlo simulations.

the finite-difference approach, the SAIMS method delivers closed-form solutions for the controls. This simplifies the economic analysis, which typically relies on comparative statics, heavily.

6 Precision Measure

In contrast to a grid-based method, our approach also delivers a precision measure. This is because we derive an upper bound $\overline{J}$ on the value function in the true market. We emphasize that this upper bound can also be used to evaluate the performance of any other numerical method such as grid-based implementations. Formally, our precision measure P is given as the solution to

$$J^{num}(t, x, y, h, j) = \overline{J}(t, xP, yP, hP, j),$$

where J^{num} is the estimate of the indirect utility derived form some numerical method such as SAIMS or a grid-based implementation. Intuitively, the definition of P is based on an indifference argument. The value $1 - P$ can be interpreted as the maximal percentage of initial wealth that the agent is willing sacrifice if she gets to know the theoretically optimal solution and not only the one derived via the numerical method under consideration (e.g., grid based or SAIMS). Therefore, a high value of P stands for a high precision.

For our paper, relative comparisons of the implementations are even more important, since we want to show that our method outperforms grid-based implementations. In this case, one can just compare the corresponding values of P. More precisely, the method that delivers the higher value of P outperforms the other method.

As can be seen from Table 2, SAIMS generates higher values of P in 25 out 28 cases compared to the implementation using a fine grid, which mirrors our earlier findings that our method is superior to the grid-based implementations. In the three cases where SAIMS is worse, one can however see that the relative margin, i.e., the difference of the values of P, is very small. In these few cases, the increase of precision is a dearly bought advantage since the run time of the fine grid is significantly longer than the run time of SAIMS.

A convenient feature of the precision measure P is its closed-form representation, which is given in the following proposition:

Proposition 6.1 (Closed-form Representation of P)**.** *The precision measure P for some numerical implementation delivering an indirect utility of J^{num} is given by*

$$P(t,x,y,h,j) = \Big(\frac{J^{num}(t,x,y,h,j)}{\overline{J}(t,x,y,h,j)}\Big)^{\frac{1}{1-\gamma}}.$$

The proof is given in Appendix A.

Notice that for a relative comparison of two approaches, $\overline{J}$ is actually irrelevant as the following argument shows: Let P^{SAIMS} be the precision of a SAIMS implementation and P^{grid} be the precision of a grid-based implementation and consider the ratio $RP = P^{SAIMS}/P^{grid}$. If RP is greater than one, then SAIMS outperforms the grid-based implementation. If it is smaller, than it is outperformed. Therefore, RP is a relative performance measure that can be applied to compare two implementations. Formally, we get

$$RP = \frac{P^{SAIMS}}{P^{grid}} = \Big(\frac{J^{SAIMS}}{J^{grid}}\Big)^{\frac{1}{1-\gamma}},$$

i.e., $\overline{J}$ drops out and is thus irrelevant for a comparison of two methods.

7 Conclusion

This paper proposes an efficient method to solve consumption-savings problems involving unhedgeable biometric risk. We analyze two canonical settings that are highly relevant in insurance economics and actuarial science. In total, we solve the settings for 28 calibrations. It turns out that our method outperforms standard grid-based solution techniques both in precision and run time. More precisely, our method delivers a superior value of the goal function in all cases for the coarse grid and in 25 out of 28 cases for the fine grid. In the remaining three cases, it is only beat by a very small margin. Another advantage of our approach is that, in contrast to grid-based methods, it delivers closed-form solutions of the best policies. This is very useful in applications and simplifies comparative statics. Furthermore, our approach provides a precision measure that can be used to evaluate any numerical method including grid-based implementations. Our approach can potentially be applied to many other consumption-savings problems that involve biometric risk beyond mortality or critical-illness risk.

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A Proofs

Proof of Theorem 3.2 The dynamics of financial wealth for a given artificial insurance market are

$$\mathrm{d}X_t = (X_t r + Y_t - c_t)\mathrm{d}t - \sum_{k \neq A_t} \iota_t^k \widehat{\lambda}_t^{I_t,k} \mathrm{d}t, \qquad X_{\tau^k} = X_{\tau^k -} + \iota_{\tau^k -}^k.$$

The Hamilton-Jacobi-Bellman (HJB) partial differential equation for the indirect utility function $J = J(t, x, y, h, j \mid \overline{\Theta})$, $j \neq K$, is given by

$$\begin{aligned}
\delta J = \sup_{c,(\iota_t^k)_{k=0}^K} \Big\{ & \tfrac{1}{1-\gamma}(c-h)^{1-\gamma} + J_t + J_x \Big[xr + y - c - \sum_{k \neq j} \iota_t^k \widehat{\lambda}_t^{j,k} \Big] + J_h(\alpha c - \beta h) \\
& + J_y y \mu(t, j) + \sum_{k \neq j, K} \lambda_t^{j,k} J(t, x + \iota^k, yp(t, j, k), h, k) + \lambda_t^{j,K} \tfrac{\varepsilon}{1-\gamma}(x + \iota^K)^{1-\gamma} - \sum_{k \neq j} \lambda_t^{j,k} J \Big\}
\end{aligned}$$

where subscripts denote partial derivatives and the terminal condition is $J(T, x, y, h, j) = \frac{\varepsilon}{1-\gamma} x^{1-\gamma}$. The first-order conditions for optimal consumption and insurance decisions imply

$$c = h + (J_x + \alpha J_h)^{-1/\gamma}, \qquad J_x(t, x + \iota^k, yp(t, j, k), h, k) = J_x(t, x, y, h, j) \frac{\widehat{\lambda}_t^{j,k}}{\lambda_t^{j,k}}. \tag{21}$$

We conjecture an indirect utility function of the form

$$J(t, x, y, h, j) = \frac{1}{1-\gamma}(x + yF(t, j) - hB(t, j))^{1-\gamma} G(t, j)^{\gamma}$$

where $G(T, j) = \varepsilon^{1/\gamma}$, $B(T, j) = 0$, $F(T, j) = 0$. After substituting this conjecture into the HJB equation, we collect terms involving $(x + yF(t, j) - hB(t, j))^{1-\gamma}$ and terms involving $(x + yF(t, j) - hB(t, j))^{-\gamma}$. This leads to the system of ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} F(t, j) &= \Big[r + \sum_{k \neq a} \widehat{\lambda}^{j,k}(t) - \mu(t, j) \Big] F(t, j) - 1 - \sum_{k \neq a} \widehat{\lambda}^{j,k}(t) p(t, j, k) F(t, k) \\
\frac{\mathrm{d}}{\mathrm{d}t} G(t, j) &= \Delta(t, j) G(t, j) - 1 - \sum_{k \neq j} \lambda^{j,k}(t) \Big(\frac{\widehat{\lambda}^{j,k}(t)}{\lambda^{j,k}(t)} \Big)^{1-1/\gamma} G(t, k) \\
\frac{\mathrm{d}}{\mathrm{d}t} B(t, j) &= \Big[r + \sum_{k \neq j} \widehat{\lambda}^{j,k}(t) - \alpha + \beta \Big] B(t, j) - 1 - \sum_{k \neq j} \widehat{\lambda}^{j,k}(t) B(t, k)
\end{aligned}$$

with $\Delta(t,j) = \frac{1}{\gamma}\left[\delta + \sum_{k\neq j}\lambda^{j,k}(t)\right] + \frac{\gamma-1}{\gamma}\left[r + \sum_{k\neq j}\widehat{\lambda}^{j,k}(t)\right]$. Inserting the conjecture into (21), we obtain the optimal consumption-insurance strategy

$$\begin{aligned} c(t,x,y,h,j) &= h + \frac{x + yF(t,j) - hB(t,j)}{G(t,j)}\left(1+\alpha B(t,j)\right)^{-1/\gamma}, \\ \iota^k(t,x,y,h,j) &= \frac{x + yF(t,j) - hB(t,j)}{G(t,j)}\left(\frac{\widehat{\lambda}^{j,k}}{\lambda^{j,k}}\right)^{-1/\gamma} G(t,k) \\ &\quad - x + hB(t,k) - yp(t,j,k)F(t,k). \end{aligned}$$

This finishes the proof. □

Proof of Proposition 3.4 Fix an arbitrary parameter set θ and a set of positive constants $\eta = (\eta_j)_{j=0,\ldots,K}$. Then, the consumption-wealth ratio is given by

$$\frac{c^{\theta,\eta}}{x} = \frac{1 + yF^\theta(t,j)\frac{1-\mathrm{e}^{-\eta_j x}}{x}}{G^\theta(t,j)} \tag{22}$$

Applying l'Hospital's rule yields

$$\lim_{x\to 0^+}\frac{c}{x} = \frac{1 + yF^\theta(t,j)\eta_j}{G^\theta(t,j)} > 0.$$

Therefore, the consumption rate (22) is approximately linear in x for small financial wealth x. This implies that financial wealth stays strictly positive implying that $c^{\theta,\eta}$ is admissible in the true market. □

Proof of Proposition 3.5 Fix an arbitrary parameter set θ and a set of positive constants $\eta = (\eta_j)_{j=0,\ldots,K}$ and let $k \in (1, \beta/\alpha)$ be a constant. Define the threshold

$$\underline{X}_t = \max\left\{\epsilon; \frac{1}{r}(H_t k - \underline{Y})\right\} \tag{23}$$

where $\epsilon > 0$ can be arbitrarily small and $\underline{Y}$ is given by

$$\underline{Y} = \inf_{t\leq\tau(\omega),\,\omega\in\Omega} Y_t,$$

i.e., $\underline{Y}$ is the minimum income across all paths as long as the agent is alive.[13] It is obvious that implementing the consumption strategy stated in the proposition implies $c_t > H_t$. By

[13] Typically, this is strictly positive since most developed countries have a welfare system.

construction, we also have $X_t > 0$ for all $t \in [0, \tau]$ since $X_t \geq \underline{X}_t > 0$. It remains to show that the strategy is financiable if $X_t = \underline{X}_t$ for some $t \in (0, \tau)$, i.e., that the financial wealth also stays positive going forward. Therefore, we distinguish two cases: (i) $X_t = \underline{X}_t > \epsilon$ and (ii) $X_t = \underline{X}_t = \epsilon$.

For (i) we have $rX_t + \underline{Y} - H_t k > 0$ and thus the consumption strategy satisfies

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = rX_t + Y_t - c_t = rX_t + Y_t - H_t k \geq rX_t + \underline{Y} - H_t k > 0$$

which implies that financial wealth stays strictly positive.

For (ii) we have $r\epsilon + \underline{Y} - H_t k \geq 0$ and thus

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = rX_t + Y_t - c_t = r\epsilon + Y_t - H_t k \geq r\epsilon + \underline{Y} - H_t k \geq 0$$

which also implies that financial wealth stays strictly positive.

Finally, notice that choosing $c = kH$ at the threshold (23) implies

$$\frac{\mathrm{d}H_t}{\mathrm{d}t} = (\alpha k - \beta)H_t < 0,$$

so that the habit level declines. This shows that the consumption strategy is financiable even if the agent's income jumps to its lower bound $\underline{Y}$. This finishes the proof. □

Proof of Theorem 6.1 The precision measure $P = P(t, x, y, h, j)$ is defined as

$$J^c(t, x, y, h, j) = \overline{J}(t, xP, yP, hP, j).$$

It follows from Theorem 3.2 that

$$\overline{J}(t, xP, yP, hP, j) = P^{1-\gamma}\overline{J}(t, x, y, h, j),$$

so the precision measure becomes

$$P = \Big(\frac{J^c(t, x, y, h, j)}{\overline{J}(t, x, y, h, j)}\Big)^{1/(1-\gamma)}.$$

This finishes the proof. □

B Finite Difference Approach

In Section 5.2 we compare the SAIMS method with a grid-based finite difference approach. This appendix briefly outlines the grid-based method. The HJB-equation corresponding to the optimization problem (13) reads

$$\delta J = \sup_c \Big\{ \frac{1}{1-\gamma}(c-h)^{1-\gamma} + J_t + J_x\big[xr + y - c\big] + J_y y\mu(t,j) + J_h(\alpha c - \beta h) \\ + \sum_{k\neq j} \lambda_t^{j,k}\big[J(t,x,yp(t,j,k),h,k) - J\big]\Big\}, \tag{24}$$

where subscripts of J denote partial derivatives. The following lemma establishes a standard separation result that reduces the number of state variables.

Lemma B.1. *The number of state variables in the optimization problem (13) can be reduced by one. The value function satisfies the following homogeneity property:*

$$J(t,x,y,h,j) = y^{1-\gamma} V(t,\widehat{x},\widehat{h},j) \tag{25}$$

with $\widehat{x} = \frac{x}{y}$ *and* $\widehat{h} = \frac{h}{y}$. V *is a* $C^{1,2}$*-function that satisfies the following HJB equation*

$$0 = \sup_{\widehat{c}} \Big\{ \frac{1}{1-\gamma}(\widehat{c}-\widehat{h})^{1-\gamma} + V_t + V_{\widehat{x}}\big[\widehat{x}\big(r-\mu(t,j)\big) + 1 - \widehat{c}\big] + V_{\widehat{h}}\big[\alpha\widehat{c} - \beta\widehat{h} - \mu(t,j)\widehat{h}\big] \\ + \sum_{k\neq j}\lambda_t^{j,k} V\Big(t, \frac{\widehat{x}}{p(t,j,k)}, \frac{\widehat{h}}{p(t,j,k)}, k\Big) - V\Big[\delta + \sum_{k\neq j}\lambda_t^{j,k} - (1-\gamma)\mu(t,j)\Big]\Big\} \tag{26}$$

subject to the terminal condition $V(T,\widehat{x},\widehat{h},j) = \frac{\varepsilon}{1-\gamma}\widehat{x}^{1-\gamma}$. *The optimal consumption strategy is given by* $c^* = \widehat{c}^* y$.

Proof. This can be proven by substituting the conjecture (25) into the HJB equation (24). □

We solve the simplified HJB equation (26) by an implicit finite difference method on equally spaced grids in the $(t,\widehat{x},\widehat{h},j)$-space. To improve stability, we use an "up-wind" scheme. Technical details on this method can be found in Munk and Sørensen (2010). For normalized wealth $\widehat{x}$ we span the grid in the range (0, 6] and for normalized habit $\widehat{h}$ in the range (0, 3]. We choose 20 steps per year and compare two different grid sizes. First, a coarse grid with $N_{\widehat{x}} = 300$ grid points for $\widehat{x}$ and $N_{\widehat{h}} = 150$ for $\widehat{h}$. Second, a fine grid with $N_{\widehat{x}} = 600$ grid points for $\widehat{x}$ and $N_{\widehat{h}} = 300$ for $\widehat{h}$. Starting at the terminal condition we work backwards through the time grid. We first calculate the solution for the state of critical illness $j = 1$ followed by the state of healthiness $j = 0$.