# Short-sale Constraints, Bid-Ask Spreads, and Information Acquisition

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November 15, 2013

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## Abstract

Short-sale constraints are prevalent in many financial markets and have been actively adjusted by regulators to tackle various problems in markets. However, theories of the impact of short-sale constraints on market liquidity and information quality are largely absent. In this paper, we extend Liu and Wang (2013) to study this impact in the presence of information asymmetry, inventory risk, and imperfect competition among market makers. In contrast to Diamond and Verrecchia (1987), we show that shortsale ban decreases bid price, increases ask price, implying an increase in bid-ask spread. In addition, short-sale constraints increase the volatility of bid-ask spreads. The presence of asymmetry information can further magnify the adverse impact of short-sale constraints on market liquidity. On the other hand, the presence of short-sale constraints may *increase* investors' incentive to produce more precise information and thus improve aggregate information quality.



*JEL* Classification Codes: D42, D53, D82, G12, G18.

Keywords: Short-sale Constraints, Bid-Ask Spread, Information Acquisition, Imperfect Competition, Welfare.

# 1. Introduction

Short-sale constraints are prevalent in many financial markets and have been actively adjusted by regulators to alter market conditions. During the financial crisis of 2008, many countries impose bans or constraints on short-selling. There is a vast literature on how short-sale constraints affect portfolio choice, market prices, information efficiency, and social welfare.[1] However, theoretical analysis on how short-sale constraints affect market liquidity, especially bid-ask spreads and depths, and information acquisition is very limited. For example, Wang (2013) examines how short-sale constraints affect asset prices, price volatility and the price impact of a small shock to the aggregate demand. She finds that shortsale constraints increase price and can decrease price volatility. To our knowledge, Diamond and Verrecchia (1987) is the only theoretical paper in the existing literature that considers the effect of short-selling constraints on bid-ask spreads. They show that when both the informed and the uninformed are prohibited from short selling, neither the bid nor the ask changes, and thus the bid-ask spread stays the same. In contrast, one of the most robust results of empirical studies is that shortsale bans tend to significantly increase bid-ask spreads (e.g., Beber and Pagano (2011), Boehmer, Jones and Zhang (2012)).

Like most of the rational expectations models in market microstructure literature (e.g., Glosten and Milgrom (1985), Admati and Pfleiderer (1988)), Diamond and Verrecchia (1987) consider a market where market makers are risk neutral and where there is perfect competition among them, and thus impose a zero expected profit condition for *each* trade of a market maker. However, there are many other markets where market makers are risk-averse to carrying inventory (e.g., Garman (1976), Lyons (1995)), competition among market makers are imperfect (e.g., Christie and Schultz (1994), Chen and Ritter (2000), and Biais, Bisière and Spatt (2003)), and market makers can make offsetting trades to avoid any significant inventory position (e.g., Sofianos (1993), Shachar (2012)). In these markets, market makers may lose money in expectation on a particular trade, as long as they can make profit on

[1] See, for example, Miller (1977), Diamond and Verrecchia (1987), Harris and Kreps (1978), Bai, Chang, and Wang (2006), Asquith, Pathak and Ritter (2005), Boehme, Danielsen and Sorescu (2005), Chen, Hong and Stein (2002), Jones and Lamont (2002), Nagel (2005), and Wang (2013).

the offsetting trades. With the improvement in market transparency, liquidity, and trading technology, time between orders of opposite directions becomes shorter and thus the capability of making offsetting trades has become even more important in affecting market makers' trading and pricing decision.

In this paper, we extend the equilibrium model of Liu and Wang (2013) to study how short-sale constraints affect market prices, market liquidity and information acquisition in markets where market makers are risk averse, can frequently make offsetting trades, and the competition among market makers are imperfect (possibly due to heterogeneous speed of trading or limited access to potential counterparty). Although this model incorporates many important features in these markets, such as short-sale constraints, asymmetric information, inventory risk, imperfect competition, and risk aversion, and allows both bid/ask prices and depths as well as all demand schedules to be endogenous, the model is still tractable. Indeed, we solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form. These explicit solutions make it possible to conduct reliable comparative statics and to generate empirically testable implications. In contrast to Diamond and Verrecchia (1987), we find that shortsale constraints decrease bid, increase ask (thus also increase spread), and decrease both the bid and ask depths. Consistent with empirical findings (e.g., Ho (1996)), shortsale constraints also increase volatility of the bid-ask spread. Conditional on the informed shortselling, both the current bid and ask prices as well as the conditional expectation of future price become lower. In addition, the presence of shortsale constraints may motivate the informed to get more precise information and thus help improve aggregate information quality.

Specifically, we consider a one-period setting with three types of risk averse investors: informed investors, uninformed investors, and market makers who are also uninformed. On date 0, all investors optimally choose how to trade a risk-free asset and a risky security to maximize their expected constant absolute risk averse (CARA) utility from the terminal wealth on date 1 and all are endowed with some shares of the risky security. On date 0 informed investors can observe a private signal about the date 1 payoff of the security before trading and thus they have trading demand motivated by private information. They are also subject to a liquidity shock that is realized on date 0 before trading. We model the liquidity shock as a random endowment of a nontradable asset whose payoff is correlated with that of the security. Accordingly, informed investors also have trading demand motivated by the liquidity needs for hedging.[2] There is a continuum of the informed and the uninformed and thus neither the informed nor the uninformed trade strategically. Due to high search costs or market makers' greater trading speed, informed and uninformed investors trade only through market makers. Both the uninformed and the informed are subject to shortsale constraints, with possibly different stringency.

Similar to Kyle (1989), Glosten (1994), and Biais, Martimort, and Rochet (2000), other investors submit demand schedules to market makers. Market makers determine how much to buy at the bid and how much to sell at the ask, taking into account the adverse price impact implied by the demand schedules: As market makers buy more, the bid price becomes higher and as they sell more, the ask price becomes lower. The equilibrium bid and ask prices are determined by the market clearing conditions at the bid and at the ask, i.e., the total amount market makers buy (sell) at the bid (ask) is equal to the total amount other investors sell (buy). In equilibrium, the risk-free asset market also clears. Although bid and ask prices and trading amounts are determined after the submissions of orders, because of rational expectations, the equilibrium is equivalent to a set up where market makers post bid and ask prices and depths *before* observing the order flow, as in Glosten and Milgrom (1986). The main difference from Glosten and Milgrom (1986) is that the posted bid and ask prices depend on realized signed order sizes of other investors.[3] This order size dependence is consistent with the bargaining feature in a less liquid market such as a bond market.[4]

[2] Alternatively, one can model the informed's trade as only motivated by private information, but there is liquidity traders who randomly submit buy or sell orders. The results are qualitatively the same, but the analysis would be more complicated.

[3] In Glosten and Milgrom (1986), the order size is restricted to one unit, so the quoted prices only depend on the trading directions (buy or sell). See Biais, Martimort, and Rochet (2000) for the equivalence of order size dependent quotes and a sequence of limit orders posted by a market maker as in an order-driven market (e.g., the Paris Bourse, or the Tokyo Stock Exchange).

[4] Indeed, in Liu and Wang (2013) we show that the equilibrium outcome is equivalent to the solution to a Nash bargaining game between investors and the market maker where the market maker has all the bargaining power. We also solve the case where both investors and the market maker have bargaining power. The qualitative results are the same. For example, the equilibrium bid-ask spread is still proportional to the absolute value of the reservation price difference between

Because short-sale constraints restrict selling at the bid, one may expect that the equilibrium bid price increases, which is exactly the opposite to what we find. The intuition for our results is as follows. Because of market makers' market power, the equilibrium bid and ask prices are set to non market makers' reservation prices at the equilibrium trading amount, i.e., conditional expectation of the security payoff minus the risk premium for carrying inventory to date 1. If the inventory level is negative (i.e., a net short position) then the bid price is above the conditional expected payoff to compensate the investor for bearing this inventory risk. Consider the case where market makers are extremely risk averse and thus do not carry any inventory. When shortsale constraints bind, the shortsellers carry smaller amount of short inventory, their required risk premium is lower, and thus the equilibrium bid price is lower. On the other hand, market makers buy less from the shortsellers because of the binding shortsale constraints. In order to avoid any inventory, they must sell less at the ask, thus the ask price becomes higher, and so does the spread. In addition, we show that short-sale constraints increase liquidity risk measured by the volatility of bid-ask spreads, because short-sale constraints increase ask price and decrease bid price and thus always make the bid-ask spread more volatile.

In addition to the above predictions on the impact of shortsale constraints on bid, ask, spread, depths, and spread volatility, our model also implies that the expected increase in the spread and expected decrease in the trading volume due to shortsale constraints are greater when the informed's private signal is more volatile, or the volatility of liquidity shock is greater, or the correlation between the hedging demand and the security payoff is greater, because all of these changes make the trading demand more volatile and thus increase the probability of shortsale constraints binding. In addition, the presence of asymmetric information can magnify the adverse impact of short-sale constraints on market liquidity and liquidity risk, because information asymmetry can increase the reservation price difference between the informed and the uninformed.

We further study how short-sale constraints affect information acquisition and thus the aggregate quality of information in the market. To this end, we assume that the

the informed and the uninformed.

Table 1: Comparison of Predictions on Average Bid, Ask and Spread

| | This Paper | | | Diamond and Verrecchia (1987) | | |
|---|---|---|---|---|---|---|
| Cases | Bid | Ask | Spread | Bid | Ask | Spread |
| Base case to Case 1 | $\downarrow$ | $\uparrow$ | $\uparrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ |
| Base case to Case 2 | $\downarrow$ | $\uparrow$ | $\uparrow$ | $\downarrow$ | $\rightarrow$ | $\uparrow$ |
| Case 2 to Case 1 | $\downarrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\rightarrow$ | $\downarrow$ |
| Base case to Case 3 | $\downarrow$ | $\downarrow$ | $\uparrow$ or $\downarrow$ | $\downarrow$ | $\rightarrow$ | $\uparrow$ |

Base case: Unconstrained; Case 1: Shortsale prohibition for both the informed and the uninformed; Case 2: Only the informed can short; Case 3: Only the informed can short, conditional on shorting.

cost for the private signal about the risky security payoff is an increasing and convex function of the signal precision. Informed investors choose an optimal level of precision to trade off the benefit from trading on the more precise private information and the higher cost for the more precise private information. With asymmetric information, the welfare loss of the informed from binding short-sale constraints tends to decrease with aggregate information quality, because of the lower risk premium required by the market makers and the uninformed. Therefore, in the presence of short-sale constraints, informed investors may choose to acquire more precise information to reduce their welfare loss from the constraints and thus short-sale constraints may increase investors' incentive to produce more precise information and thus may improve the aggregate information quality in the market.

Our model generates several empirically testable implications that differ from those of Diamond and Verrecchia (1987). Table 1. summarizes the difference that may help empirical tests differentiate these two theories. For example, Table 1 implies that if before a shortsale ban it is mostly the informed who can short, then after imposing a shortsale ban Diamond and Verrecchia (1987) implies that bid price goes up (because of less bad information revealed), the ask stays the same, and thus the bid-ask spread decreases, while our model predicts that bid price goes down, ask price goes up and so does spread.

The remainder of the paper proceeds as follows. In Section 2 we present the model. In Section 3 we derive the equilibrium with and without short-sale constraints. In

Section 4 we examine the effect of short-sale constraints on bid-ask spread, liquidity risk, welfare, and investors' incentive to acquire information ex-ante. We conclude in Section 5. All proofs are in the Appendix.

## 2. The Model

We consider a one period setting with trading dates 0 and 1. There are a continuum of identical informed investors with mass $N_I$, a continuum of identical uninformed investors with mass $N_U$, and $N_M$ identical designated market makers ($M$) who are also uninformed. They can trade one risk-free asset and one risky security on date 0 and date 1 to maximize their expected constant absolute risk aversion (CARA) utility from the terminal wealth on date 1. There is a zero net supply of the risk-free asset, which also serves as the numeraire and thus the risk-free interest rate is normalized to 0. The total supply of the security is $N \times \bar{\theta} > 0$ shares where $N = N_I + N_U + N_M$ and the date 1 payoff of each share is $\tilde{V} \sim \mathbf{N}(\bar{V}, \sigma_V^2)$, where $\bar{V}$ is a constant, $\sigma_V > 0$, and $\mathbf{N}(\cdot)$ denotes the normal distribution. The aggregate risky asset endowment is $N_i\bar{\theta}$ shares for type $i \in \{I, U, M\}$ investors, but no investor is endowed with any risk-free asset.

On date 0, informed investors observe a private signal

$$\hat{s} = \tilde{V} - \bar{V} + \tilde{\varepsilon} \tag{1}$$

about the payoff $\tilde{V}$, where $\tilde{\varepsilon}$ is independently normally distributed with mean zero and variance $\sigma_\varepsilon^2$.[5] In addition to the security, every informed investor is also subject to a liquidity shock that is modeled as a random endowment of $\hat{X}_I \sim \mathbf{N}(0, \sigma_X^2)$ units of a non-tradable risky asset on date 0, with $\hat{X}_I$ realized on date 0 and only known to informed investors.[6] The non-tradable asset has a per-unit payoff of $\tilde{N} \sim \mathbf{N}(0, \sigma_N^2)$

[5] Throughout this paper, "bar" variables are constants, "tilde" random variables are realized on date 1 and "hat" random variables are realized on date 0. Observing the private signal may also be reinterpreted as extracting more precise information from public news (e.g., Engelberg, Reed, and Ringgenberg (2012)).

[6] The random endowment can represent any shock in the demand for the security, such as a

that has a covariance of $\sigma_{VN}$ with the payoff of the risky asset $\tilde{V}$. The payoff of the nontradable asset is realized and becomes public on date 1. The correlation between the non-tradable asset and the security results in a liquidity demand for the risky asset to hedge the non-tradable asset payoff. In a model with private information sources such as a private signal $\hat{s}$ and a private liquidity shock (e.g., Grossman and Stiglitz (1980), Wang (1994), O'Hara (2003), and Vayanos and Wang (2012)), assuming that all investors who are subject to liquidity shock also observe $\hat{s}$ is only for simplicity: even if they do not observe $\hat{s}$, they can infer it perfectly from the equilibrium price, because the equilibrium price is an invertible function of the weighted sum of $\hat{s}$ and the private liquidity shock (and they observe the liquidity shock). In this type of models asymmetric information can therefore exist only if some investors who do not have any liquidity shock do not observe $\hat{s}$ either and are thus uninformed, as in Vayanos and Wang (2012) for example. We assume that these investors are all uninformed for simplicity. In contrast to most of the existing literature that assume uninformed investors cannot observe any signal about the informed's private signal, we assume that there is a public signal

$$\hat{S}_s = \hat{s} + \hat{\eta} \tag{2}$$

about the informed's private signal $\hat{s}$ that all investors (i.e., the uninformed, market makers, and the informed) can observe, where $\hat{\eta}$ is independently normally distributed with mean zero and variance $\sigma_\eta^2$.[7] This public signal represents public news about the asset payoff determinants, such as macroeconomic conditions, cash flow news and regulation shocks, which is correlated with but less precise than the informed's private signal.[8] While this additional signal $\hat{S}_s$ is not critical for our main results, it allows us to model different degrees of information asymmetry in one unified setting.

All trades go through the designated market makers (dealers) whose market making

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liquidity shock or a change in the needs for rebalancing an existing portfolio or a change in a highly illiquid asset.

[7]While this public signal can also be observed by the informed, it is useless to them because they can already perfectly observe it privately.

[8]For example, one can regress historical public news on the historical private signal to obtain the historical relationship between public news and the private signal and then the uninformed use this relationship and the observed public news to estimate the private signal.

cost is assumed to be 0, either because the counterparty search cost is high (e.g., in some OTC markets) or because market makers can trade faster than other investors (e.g., high frequency traders in some liquid markets). Specifically, $I$ and $U$ investors sell to market makers at the bid $B$ or buy from them at the ask $A$ or do not trade at all. Given that there is a continuum of informed and uninformed investors, we assume that they are price takers and there are no strategic interactions among them or with market makers. We assume that both $I$ and $U$ investors are subject to short-sale constraints, *i.e.*, $\theta_i \geq -\lambda_i \bar{\theta}, \quad i = I, U$, where $\lambda_i \geq 1$ can be different for the informed and the uninformed, a smaller $\lambda_i$ means a more stringent short-selling constraint. If $\lambda_i = 1$, then type $i$ investors are prohibited from shortselling. Heterogeneous shortsale constraint stringency for the informed and the uninformed captures the essence of possible heterogeneous shortsale costs across them (e.g., Kolasinki, Reed and Ringgenberg (2013)).

Following Kyle (1989), we assume that informed and uninformed investors submit their demand schedules before trading. Different from Kyle (1989), however, the demand schedules are submitted not to an auctioneer but to designated market makers who then determine how to trade, and the schedules depend on both the bid and ask prices rather than a single trading price as assumed in Kyle (1989). This assumption is consistent with a market microstructure where designated market makers observe order flows before determining bid and ask prices and bid and ask depths.

For each $i \in \{I, U, M\}$, investors of type $i$ are ex ante identical. Accordingly, we restrict our analysis to symmetric equilibria where all investors of the same type adopt the same trading strategy. Let $\mathcal{I}_i$ represent a type $i$ investor's information set on date 0 for $i \in \{I, U, M\}$.

For $i \in \{I, U\}$, a type $i$ investor's problem is to choose the (signed) demand schedule $\theta_i(A, B)$ to

$$\max E[-e^{-\delta \tilde{W}_i} | \mathcal{I}_i], \tag{3}$$

where

$$\tilde{W}_i = \theta_i^- B - \theta_i^+ A + (\bar{\theta} + \theta_i)\tilde{V} + \hat{X}_i \tilde{N}, \tag{4}$$

$\hat{X}_U = 0$, $\delta > 0$ is the absolute risk-aversion parameter, $x^+ := \max(0, x)$, and $x^- :=$

$\max(0, -x)$.[9] In addition, a type $i$ investor is subject to the short-sale constraint

$$\theta_i \geq -\lambda_i \bar{\theta}. \tag{5}$$

Since $I$ and $U$ investors buy from market makers at ask and sell to them at bid, we can view these trades occur in two separate markets: the "ask" market and the "bid" market. In the "ask" market, market makers are suppliers and other investors are demanders and the opposite is true in the "bid" market. As market makers supply (sell) more in the "ask" market, the ask price goes down and as market makers demand (buy) more in the "bid" market, the bid price goes up. Accordingly, in contrast to the standard microstructure literature where market makers directly choose market prices, we assume market makers directly choose how much to buy at bid given the inverse supply function (a function of the market makers' purchasing quantity) of all other participants and how much to sell at ask given the inverse demand function (a function of the market makers' selling quantity) of all other participants. The posted bid and ask prices are the required prices to achieve the optimal amount market makers choose to trade. Since all trades go through market makers, market makers can have market powers especially when the number of market makers is small. To model the oligopolistic competition among the market makers, we use the notion of the Cournot competition that is well studied and understood. Specifically, we assume that market makers simultaneously choose the optimal number of shares to sell at ask and to buy at bid, taking into account the price impact of their trades. *Note that for the monopoly case where $N_M = 1$, the alternative Bertrand competition formulation is the same as the Cournot competition formulation.* Because as we show later our results hold in the special case where $N_M = 1$, it is not the choice of Cournot competition that drives our main results. In addition, with Bertrand competition, it yields the same outcome as either Cournot Competition (if $N_M = 1$) or perfect competition (if $N_M \geq 2$).

[9]We have solved the more general case where all investors have different liquidity shocks and different risk aversions. Qualitative results on the impact of shortsale constraints are the same. We focus on the current case where all investors have the same risk aversion and only $I$ investors have liquidity shocks to make the main intuitions as clear as possible and to save space.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{N_M})^\top$ and $\beta = (\beta_1, \beta_2, ..., \beta_{N_M})^\top$ be the vector of the number of shares market makers sell at ask (i.e., ask depth) and buy at bid (i.e., bid depth) respectively.[10] Given the demand schedules of the informed and the uninformed ($\theta_I^*(A, B)$ and $\theta_U^*(A, B)$), the bid price $B(\beta)$ (i.e., the inverse supply function in the bid market) and the ask price $A(\alpha)$ (i.e., the inverse demand function in the ask market) can be determined by the following security market clearing conditions at the bid and ask prices.[11]

$$\sum_{j=1}^{N_M} \alpha_j = \sum_{i=I,\ U} N_i \theta_i^*(A, B)^+, \ \sum_{j=1}^{N_M} \beta_j = \sum_{i=I,\ U} N_i \theta_i^*(A, B)^-, \tag{6}$$

where the left-hand sides represent the total sales and purchases by market makers respectively and the right-hand sides represent the total purchases and sales by other investors respectively.

Then for $j = 1, 2, ..., N_M$, the designated market maker $M_j$'s problem is to choose ask depth $\alpha_j \geq 0$ and bid depth $\beta_j \geq 0$ to

$$\max E\left[-e^{-\delta \tilde{W}_{M_j}} | \mathcal{I}_M\right], \tag{7}$$

where

$$\tilde{W}_{M_j} \quad = \quad \alpha_j A(\alpha) - \beta_j B(\beta) + (\bar{\theta} + \beta_j - \alpha_j)\tilde{V}. \tag{8}$$

Note that different from other investors, a market maker takes into account the price impact of her own trades, i.e., recognizing that both $A$ and $B$ will be affected by her trades.

This leads to our definition of the Nash equilibrium of the Cournot competition.[12]

[10] To help remember, Alpha denotes Ask depth and Beta denotes Bid depth.

[11] The risk-free asset market will be automatically cleared by the Walras' law. A buyer's (seller's) trade only depends on ask $A$ (bid $B$). So $A$ only depends on $\alpha$ and $B$ only depends on $\beta$.

[12] Deviations by undercutting prices can be prevented by matching prices by other market makers in subsequent periods in a repeated-game setting. As in standard Cournot competition models, varying prices is not in the strategy space.

**Definition 1** *An equilibrium* $(\theta_I^*(A,B), \theta_U^*(A,B), A^*, B^*, \alpha^*, \beta^*)$ *is such that*

1. *given any* $A$ *and* $B$, $\theta_i^*(A,B)$ *solves a type* $i$ *investor's Problem* (3) *for* $i \in \{I, U\}$;

2. *given* $\theta_I^*(A,B)$ *and* $\theta_U^*(A,B)$, $\alpha_j^*$ *and* $\beta_j^*$ *solve potential market maker* $M_j$*'s Problem (7), for* $j = 1, 2, ..., N_M$*; and*

3. $A^* := A(\alpha^*)$ *and* $B^* := B(\beta^*)$ *clear both the security and the risk-free asset markets, where* $A(\alpha)$ *and* $B(\beta)$ *solve Equation (6).*

# 3. The equilibrium

In this section, we solve the equilibrium bid and ask prices, bid and ask depth and trading volume in close form with and without short-sale constraints. We first present the results without short-sale constraints.

## *A.* The equilibrium without short-sale constraints

In this subsection, we solve the equilibrium bid and ask prices, bid and ask depth and trading volume in closed form.

Given $A$ and $B$, the optimal demand schedule for a type $i$ investor for $i \in \{I, U\}$ is

$$\theta_i^*(A,B) = \begin{cases} \frac{P_i^R - A}{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_i]} & A < P_i^R, \\ 0 & B \leq P_i^R \leq A, \\ -\frac{B - P_i^R}{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_i]} & B > P_i^R, \end{cases} \tag{9}$$

where

$$P_i^R = E[\tilde{V}|\mathcal{I}_i] - \delta \mathrm{Cov}[\tilde{V}, \tilde{N}|\mathcal{I}_i]\hat{X}_i - \delta \mathrm{Var}[\tilde{V}|\mathcal{I}_i]\bar{\theta} \tag{10}$$

is the reservation price of a type $i$ investor (i.e., the critical price such that non-market-makers buy (sell, respectively) the security if and only if the ask price is lower (the bid price is higher, respectively) than this critical price).

Because the informed know exactly $\{\hat{s}, \hat{X}_I\}$ while equilibrium prices $A^*$ and $B^*$ and the public signal $\hat{S}_s$ are only noisy signals about $\{\hat{s}, \hat{X}_I\}$, the information set of the informed in equilibrium is

$$\mathcal{I}_I = \{\hat{s}, \hat{X}_I\}, \tag{11}$$

which implies that

$$\mathrm{E}[\tilde{V}|\mathcal{I}_I] = \bar{V} + \rho_I \hat{s}, \quad \mathrm{Var}[\tilde{V}|\mathcal{I}_I] = \rho_I \sigma_\varepsilon^2, \quad \mathrm{Cov}[\tilde{V}, \tilde{N}|\mathcal{I}_I] = (1-\rho_I)\sigma_{VN}, \tag{12}$$

where

$$\rho_I := \frac{\sigma_V^2}{\sigma_V^2 + \sigma_\varepsilon^2}. \tag{13}$$

Equation (10) then implies that

$$P_I^R = \bar{V} + \hat{S} - \delta \rho_I \sigma_\varepsilon^2 \bar{\theta}, \tag{14}$$

where $\hat{S} := \rho_I \hat{s} + h\hat{X}_I$ and $h = -\delta(1-\rho_I)\sigma_{VN}$ represents the hedging premium per unit of liquidity shock.

While $\hat{s}$ and $\hat{X}_I$ both affect the informed investor's demand and thus the equilibrium prices, other investors can only infer the value of $\hat{S}$ from market prices because the joint impact of $\hat{s}$ and $\hat{X}_I$ on market prices is only in the form of $\hat{S}$. In addition to $\hat{S}$, other investors can also observe the public signal $\hat{S}_s$ about the private signal $\hat{s}$. Thus we conjecture that the equilibrium prices $A^*$ and $B^*$ depend on both $\hat{S}$ and $\hat{S}_s$. Accordingly, the information sets for the uninformed investors and the market maker are[13]

$$\mathcal{I}_U = \mathcal{I}_M = \{\hat{S}, \hat{S}_s\}. \tag{15}$$

Then the conditional expectation and conditional variance of $\tilde{V}$ for the uninformed

[13]Note that uninformed only need to observe their own trading price, i.e., $A^*$ or $B^*$, not both $A^*$ and $B^*$. For OTC markets, investors may not be able to observe trading prices by others, although with improving transparency, this has also become possible in some markets (e.g., TRACE system in the bond market).

and the market maker are respectively

$$E[\tilde{V}|\mathcal{I}_U] = \bar{V} + \rho_U(1-\rho_X)\hat{S} + \rho_U\rho_X\rho_I\hat{S}_s, \tag{16}$$

$$\mathrm{Var}[\tilde{V}|\mathcal{I}_U] = \rho_U\rho_I(\sigma_\varepsilon^2 + \rho_X\sigma_\eta^2), \tag{17}$$

where

$$\rho_X := \frac{h^2\sigma_X^2}{h^2\sigma_X^2 + \rho_I^2\sigma_\eta^2}, \quad \rho_U := \frac{\sigma_V^2}{\sigma_V^2 + \rho_X\rho_I\sigma_\eta^2} \leq 1. \tag{18}$$

It follows that the reservation price for a $U$ investor and the market maker is

$$P_U^R = P_M^R = \bar{V} + \rho_U(1-\rho_X)\hat{S} + \rho_U\rho_X\rho_I\hat{S}_s - \delta\rho_U\rho_I\left(\sigma_\varepsilon^2 + \rho_X\sigma_\eta^2\right)\bar{\theta}. \tag{19}$$

Let $\Delta$ denote the difference in the reservation prices of $I$ and $U$ investors. We then have

$$\Delta := P_I^R - P_U^R = (1-\rho_U)\left(\left(1 + \frac{\sigma_V^2}{\rho_I\sigma_\eta^2}\right)\hat{S} - \frac{\sigma_V^2}{\sigma_\eta^2}\hat{S}_s + \delta\rho_I\sigma_V^2\bar{\theta}\right). \tag{20}$$

Define

$$C_I := \frac{N_M(N_U + N_M + 1)}{(N_M+1)\left(\overline{N}+1\right)}, \quad C_U := \frac{\nu N_M N_I}{(N_M+1)\left(\overline{N}+1\right)}.$$

Let

$$\nu := \frac{\mathrm{Var}[\tilde{V}|\mathcal{I}_U]}{\mathrm{Var}[\tilde{V}|\mathcal{I}_I]} = \rho_U + \frac{\rho_U\rho_X\sigma_\eta^2}{\sigma_\varepsilon^2} \geq 1$$

be the ratio of the security payoff conditional variance of the uninformed to that of the informed, and

$$\overline{N} := \nu N_I + N_U + N_M \geq N$$

be the information weighted total population. The following theorem provides the equilibrium bid and ask prices and equilibrium security demand.[14]

[14]Because all utility functions are strictly concave, all budget constraints are linear in the amount invested in the security and both the informed and the uninformed are price takers, there is a unique solution to the problem of each informed and each uninformed given the bid and ask prices. Because the inverse demand and supply functions implied by the market clearing conditions are linear in market depths and when a market maker trades a strictly positive amount with both the informed and the uninformed, there is a unique solution to her utility maximization problem (which already takes into account the market clearing conditions). This implies that there is a unique equilibrium

**Theorem 1** *1. The equilibrium bid and ask prices are*

$$A^* := A(\alpha^*) = P_U^R + C_U\Delta + \frac{\Delta^+}{N_M+1},$$

$$B^* := B(\beta^*) = P_U^R + C_U\Delta - \frac{\Delta^-}{N_M+1},$$

*and we have $A^* > P^* > B^*$, where*

$$P^* = \frac{\nu N_I}{\overline{N}} P_I^R + \frac{N_U}{\overline{N}} P_U^R + \frac{N_M}{\overline{N}} P_M^R \tag{21}$$

*is the equilibrium price of a perfect competition equilibrium where market makers are also price takers. The bid-ask spread is*

$$A^* - B^* = \frac{|\Delta|}{N_M+1} = \frac{(1-\rho_U)\left|\left(1+\frac{\sigma_V^2}{\rho_I \sigma_\eta^2}\right)\hat{S} - \frac{\sigma_V^2}{\sigma_\eta^2}\hat{S}_s + \delta\rho_I\sigma_V^2\bar{\theta}\right|}{N_M+1}. \tag{22}$$

*2. The equilibrium quantities demanded are*

$$\theta_I^* = C_I \frac{\Delta}{\delta\, Var[\tilde{V}|\mathcal{I}_I]}, \quad \theta_U^* = -C_U \frac{\Delta}{\delta\, Var[\tilde{V}|\mathcal{I}_U]}, \quad \theta_M^* = \frac{N_M+1}{N_M}\theta_U^*; \tag{23}$$

*the equilibrium quote depths are*

$$\alpha^* = \frac{N_I}{N_M}(\theta_I^*)^+ + \frac{N_U}{N_M}(\theta_U^*)^+, \tag{24}$$

$$\beta^* = \frac{N_I}{N_M}(\theta_I^*)^- + \frac{N_U}{N_M}(\theta_U^*)^-, \tag{25}$$

when all investors trade in equilibrium. If a market maker's reservation price is different from the uninformed and the informed (e.g., due to different information, different liquidity shock, different risk aversion, etc.), some investors may not trade in equilibrium, then there would be multiple equilibria because either bid or ask would not be unique.

*which implies that the equilibrium trading volume is*

$$N_M(\alpha^* + \beta^*) = \frac{N_M N_I (N_M + 2N_U + 1)}{(N_M + 1)(\overline{N} + 1)} \left( \frac{|\Delta|}{\delta \, Var[\tilde{V}|\mathcal{I}_I]} \right). \tag{26}$$

As shown in Liu and Wang (2013), the above equilibrium can be reinterpreted as the solution to a Nash bargaining game between investors and the market maker where the market maker has all the bargaining power. In a nutshell, in the Nash bargaining game, the market maker and an investor bargain over the trading amount with the trading price determined by the trading amount and the optimal demand schedule of the investor. Therefore, the Nash bargaining game where the market maker has all the bargaining power is to choose the trading amount to maximize the market maker's expected utility given the demand schedule of the investor, and thus yields exactly the same outcome as our solution above.

Part 1 of Theorem 1 implies that in equilibrium both bid and ask prices are determined by the reservation price of the uninformed and the reservation price difference between the informed and the uninformed. In addition, given the public signal $\hat{S}_s$, all investors can indeed infer $\hat{S}$ from observing their trading prices as conjectured, because of the one-to-one mapping between the two.[15] Furthermore, Part 1 shows that the equilibrium bid-ask spread is equal to the absolute value of the reservation price difference between the informed and the uninformed, divided by $N_M + 1$.

When $\theta_I^* \leq -\bar{\theta}$, there is shortsale in equilibrium from the informed. Because the average liquidity shock of the informed is zero and the average private information is neutral, conditional on a short sale from the informed, both the private information about the payoff and the hedging premium are negative on average. We have the following result regarding to the effect of the informed's shortsale.

**Corollary 1** 1. *Conditional on the informed shorting, expected bid price, expected ask price , and expected asset payoff are on average lower than the unconditional*

[15] In our model, the market maker can infer how much informed investors are trading. However, she does not know how much of the informed investors' trades is due to information on the security's payoff or how much is due to the hedging demand. This is similar to the set-up of Glosten (1989) and Vayanos and Wang (2012).

*averages.*

2. *Conditional on observing positive short interest from the informed, the uninformed's expected payoff (of the security) is on average lower than the unconditional averages, but the difference between the uninformed's expected payoff and the ask price is on average higher than the unconditional average.*

Part 1 shows that on average current bid, ask and date 1 payoff go down because on average the private information is negative. Conditional on the informed shorting, one might expect that the expected return to go down because of the bad news and the expected spread to go up because market makers would be less willing to buy from the informed. Figure 1 shows that the opposite can be true. For example, when $\sigma_V$ is small (relative to the volatility of hedging demand and private signal), the expected return can be higher than the unconditional expected return when $\sigma_V$ is small (relative to the volatility of hedging demand) and the expected spread can be smaller when $\sigma_V$ is large. Intuitively, when $\sigma_V$ is small relative to the volatility of hedging demand, the shorting is more likely driven by hedging demand, which can drive down the current ask price by a greater amount than the expected payoff. On the other hand, the spread is proportional to the magnitude of the reservation price difference, which decreases with $\tilde{V}$ when the reservation price difference is negative. For large $\sigma_V$, the informed's shorting, implying a negative reservation price difference, is more likely caused by a decrease in $\tilde{V}$, which in turn drives down the average spread.

Part 2 shows that the uninformed's conditional expected payoff of the risky asset and the ask price should be smaller on average than the unconditional ones. It also shows that the average (dollar) return from buying the asset conditional on a shortsale of the informed is greater than the unconditional return. To understand this result, note that as can be easily verified from Theorem 1, when the uninformed buy, the difference between the uninformed's expectation of the payoff and the ask price is equal to the risk premium from carrying the inventory to date 1. Intuitively, when the informed short sell, the uninformed needs to carry more inventory than other cases where the uninformed either decrease inventory or buy less. Therefore, the uninformed require higher inventory risk premium, which drives up the average

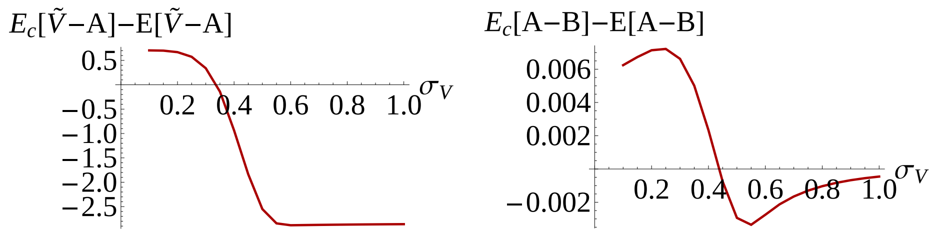


Figure 1: The difference between the conditional (on informed shorting) and unconditional expected return and expected spread against $\sigma_V$. The parameter values are: $\bar{\theta} = 1, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_\eta = 0.4, \sigma_X = 0.4, \sigma_\varepsilon = 0.4, N_I = 100, N_M = 10, N_U = 1000$.

return.

The following corollary shows that when the total initial holding of the non-informed (i.e, the uninformed and market makers) is large relative to the informed, the informed is more likely to shortsell.

**Corollary 2** *The informed are more likely to shortsell than the uninformed and the market makers if and only if*

$$N_I < \frac{\lambda_U \sigma_\varepsilon^2 + (\lambda_U - \rho_I)\rho_X \sigma_\eta^2}{\lambda_I \sigma_\varepsilon^2 + +(\lambda_U + \nu\rho_I)\rho_X \sigma_\eta^2}(N_U + N_M + 1).$$

Intuitively, if the total holding of the non-informed is large (i.e., $N_U + N_M$ is large), when the informed's reservation price is slightly below that of the uninformed, the total purchasing demand from the non-informed exceeds the initial holdings of the informed and thus the informed short sell in equilibrium. On the other hand, for the non-informed to short sell, the reservation price of the informed needs to be far above that of the non-informed to induce the non-informed to sell a a large quantity because the non-informed first sell their initial holdings before going short.

Diamond and Verrecchia (1987) predict that an increase in the short interest can never be good news for the security payoff. In contrast, because on average, the hedging demand is zero, our model predicts that an increase of short interest is good news for securities of which the total initial holding of the non-informed is small and is bad news if the non-informed's holding is large.

## *B.* The equilibrium with short-sale constraints

Next, we examine the effect of short-sale constraints on both bid and ask prices. In the presence of short-sale constraints, given $A$ and $B$, the optimal demand schedule for a type $i$ investor for $i \in \{I, U\}$ becomes

$$\theta_i^*(A,B) = \begin{cases} \frac{P_i^R - A}{\delta Var[\tilde{V}|\mathcal{I}_i]} & A < P_i^R, \\ 0 & B \leq P_i^R \leq A, \\ \max\left[-\lambda_i \bar{\theta}, -\frac{B - P_i^R}{\delta Var[\tilde{V}|\mathcal{I}_i]}\right] & B > P_i^R. \end{cases} \tag{27}$$

We have

**Theorem 2** *1. If $\Delta \geq \frac{\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta}}{C_U}$, then short-sale constraints bind for $U$ investors and*

*(a) the equilibrium bid and ask prices are*

$$A_{c1}^* = P_I^R - \frac{N_M \Delta + \delta N_U Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta}}{\nu N_I + N_M + 1}, \tag{28}$$

$$B_{c1}^* = P_U^R + \delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta}, \tag{29}$$

*and bid-ask spread is*

$$A_{c1}^* - B_{c1}^* = \frac{\nu N_I + 1}{\nu N_I + N_M + 1}\Delta - \frac{\bar{N} + 1}{\nu N_I + N_M + 1}\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta}; \tag{30}$$

*(b) the equilibrium quantities demanded are*

$$\theta_{Ic1}^* = \frac{N_M \Delta + \delta N_U Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta}}{(\nu N_I + N_M + 1)\delta Var[\tilde{V}|\mathcal{I}_I]}, \quad \theta_{Uc1} = -\lambda_U \bar{\theta}, \tag{31}$$

$$\theta^*_{Mc1} = \frac{-N_M N_I \Delta + (N_M+1)\delta N_U Var[\tilde{V}|\mathcal{I}_I]\lambda_U\bar{\theta}}{N_M(\nu N_I + N_M + 1)\delta Var[\tilde{V}|\mathcal{I}_I]}, \tag{32}$$

*the equilibrium quote depths are*

$$\alpha^*_{c1} = \frac{N_M N_I \Delta + \delta N_I N_U Var[\tilde{V}|\mathcal{I}_U]\lambda_U\bar{\theta}}{N_M(\nu N_I + N_M + 1)\delta Var[\tilde{V}|\mathcal{I}_I]}, \ \beta^*_{c1} = \frac{N_U\lambda_U\bar{\theta}}{N_M}. \tag{33}$$

2. *If* $\Delta \leq -\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}}{C_I}$, *then* $I$ *investors bind in short-sale constraints and*

   (a) *the equilibrium bid and ask prices are*

$$A^*_{c2} = P^R_U - \frac{\delta N_I Var[\tilde{V}|\mathcal{I}_U]\lambda_I\bar{\theta}}{N_U + N_M + 1}, \tag{34}$$

$$B^*_{c2} = P^R_I + \delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}, \tag{35}$$

   *and bid-ask spread is*

$$A^*_{c2} - B^*_{c2} = -\Delta - \frac{\bar{N}+1}{N_U + N_M + 1}\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}; \tag{36}$$

   (b) *the equilibrium quantities demanded are*

$$\theta^*_{Ic2} = -\lambda_I\bar{\theta}, \quad \theta_{Uc2} = \frac{N_I\lambda_I\bar{\theta}}{N_U + N_M + 1}, \ \theta^*_{Mc2} = \frac{(N_M+1)N_I\lambda_I\bar{\theta}}{N_M(N_U + N_M + 1)}, \tag{37}$$

   *the equilibrium quote depths are*

$$\alpha^*_{c2} = \frac{N_I N_U \lambda_I\bar{\theta}}{(N_U + N_M + 1)N_M}, \ \beta^*_{c2} = \frac{N_I\lambda_I\bar{\theta}}{N_M}. \tag{38}$$

3. *If* $-\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}}{C_I} < \Delta < \frac{\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U\bar{\theta}}{C_U}$, *then no investor bind in short-sale constraints and the equilibrium bid and ask prices, equilibrium quantities demanded, and bid and ask depths are the same as those given in Theorem 1.*

To explain the results of Theorem 2, consider the case when $I$ investors sell and $U$

investors buy. In the presence of short-sale constraints, given bid price $B$, $I$ investors' optimal stock holding, $\theta_I^* \geq -\lambda_I \bar{\theta}$. In equilibrium, we have $\sum_{j=1}^{N_M} \beta_j^* = N_I(\theta_I^*)^- \leq N_I \lambda_I \bar{\theta}$. Given the short-sale constraints, market maker $M_j$ can only buy up to the upper bound $\frac{N_I \lambda_I \bar{\theta}}{N_M}$ shares of stock from $I$ investors. When $I$ investors would like to sell more due to negative hedging demand or private information, short-sale constraints can bind for $I$ investors for a given bid price. In these cases, the equilibrium bid price is such that the unconstrained optimal $\theta_I^*$ of $I$ investors is just equal to the lower bound $-\lambda_I \bar{\theta}$, i.e., the short-sale constraint just starts to bind. Therefore, the equilibrium bid price is $B_{c1}^*$ as defined in (29).

# 4. The effect of short-sale constraints

In this section, we study the effect of short-sale constraints on bid prices, ask prices, bid-ask spreads, liquidity risk, market participants' welfare, and investors' incentive to acquire information ex-ante.

## *A.* Bid prices, ask prices, bid-ask spreads

We first compare bid prices, ask prices, and bid-ask spreads with and without short-sale constraints. We have

**Proposition 1**

1. *The equilibrium bid price with short-sale constraints is lower than the bid price without short-sale constraints. In addition,* $\frac{\partial B_{c1}^*}{\partial \lambda_U} > 0$ *and* $\frac{\partial B_{c2}^*}{\partial \lambda_I} > 0$.

2. *The equilibrium ask price with short-sale constraints is higher than the ask price without short-sale constraints. In addition,* $\frac{\partial A_{c1}^*}{\partial \lambda_U} < 0$ *and* $\frac{\partial A_{c2}^*}{\partial \lambda_I} < 0$.

3. *Short-sale constraints increase bid and ask spread. In addition, more stringent constraints lead to a higher bid and ask spread, i.e.,*$\frac{\partial (A_{c1}^* - B_{c1}^*)}{\partial \lambda_U} < 0$ *and* $\frac{\partial (A_{c2}^* - B_{c2}^*)}{\partial \lambda_I} < 0$.

As we can see from Figure 3, the bid price is lower, the ask price is higher, and the bid-ask spread is wider when short-sale constraints bind. We provide the essential intuition for the results behind Proposition 1 through graphical illustrations. Suppose $P_I^R < P_U^R$ and thus $I$ investors sell and $U$ investors buy. The market clearing condition (6) implies that the inverse demand and supply functions faced by the market maker are respectively

$$A = P_U^R - k_U\alpha, \quad B = P_I^R + k_I\beta,$$

where

$$k_i = \frac{N_M\delta\mathrm{Var}[\tilde{V}|\mathcal{I}_i]}{N_i}, \quad i = I, U.$$

We plot the above inverse demand and supply functions and equilibrium spreads in Figure 2. Figure 2 shows that as a market maker buys (sells) more at the bid (ask), the bid (ask) price goes up (down). Facing the inverse demand and supply functions, market makers optimally trade off the prices and quantities. Similar to the results of classical Cournot competition models of multiple firms who compete through choosing the amount of output of a homogeneous product, the bid and ask spread is equal to the absolute value of the reservation price difference $|\Delta|$, divided by the number of market makers plus one. When the shortsale constraint binds for the informed, a market maker can only buy from the informed up to $\bar{\beta}_c^* = N_I\lambda_I\bar{\theta}/N_M$, therefore the bid price is lower. Because the informed sell less in equilibrium and the uninformed have the same reservation price as the market maker, both the uninformed and the market maker buy less in the net. This implies that the market maker sells less to the uninformed at the ask than the unconstrained case and thus the ask price is higher.

From Theorem 2, we have that the increase of bid-ask spread in the presence of short-sale constraints is:

$$(A_c^* - B_c^*) - (A^* - B^*)$$

$$= \begin{cases} \frac{\bar{N}+1}{\nu N_I + N_M + 1}\left(C_U\Delta - \delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U\bar{\theta}\right) & \Delta \geq \frac{\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U\bar{\theta}}{C_U} \\ 0 & otherwise \\ \frac{\bar{N}+1}{N_U + N_M + 1}\left(-C_I\Delta - \delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}\right) & \Delta \leq -\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}}{C_I}. \end{cases}$$

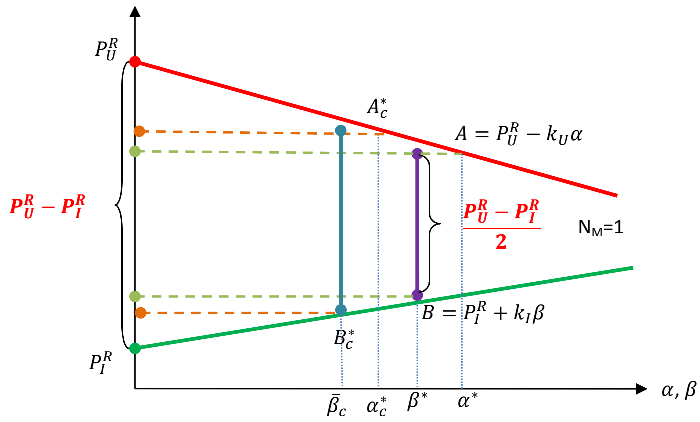


Figure 2: Inverse Demand/Supply Functions and Bid/Ask Prices With and Without Shortsale Constraints.

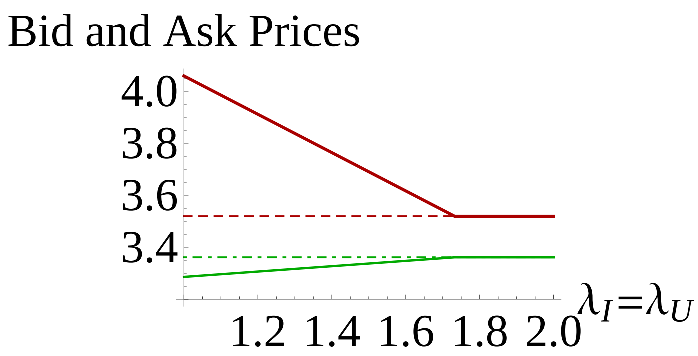


Figure 3: The bid and ask prices with and without short-sale constraints against $\lambda_I = \lambda_U$. The thick (thin) line denotes the ask (bid) price with short-sale constraints and the dashed (dot-dashed) line denotes the ask (bid) price without short-sale constraints. The parameter values are: $\bar{\theta} = 1, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_\eta = 0.4, \sigma_X = 1, \sigma_\varepsilon = 0.4, \sigma_V = 0.4, \hat{S} = 2, \hat{S}_s = 0, N_I = 100, N_M = 10, N_U = 1000$.

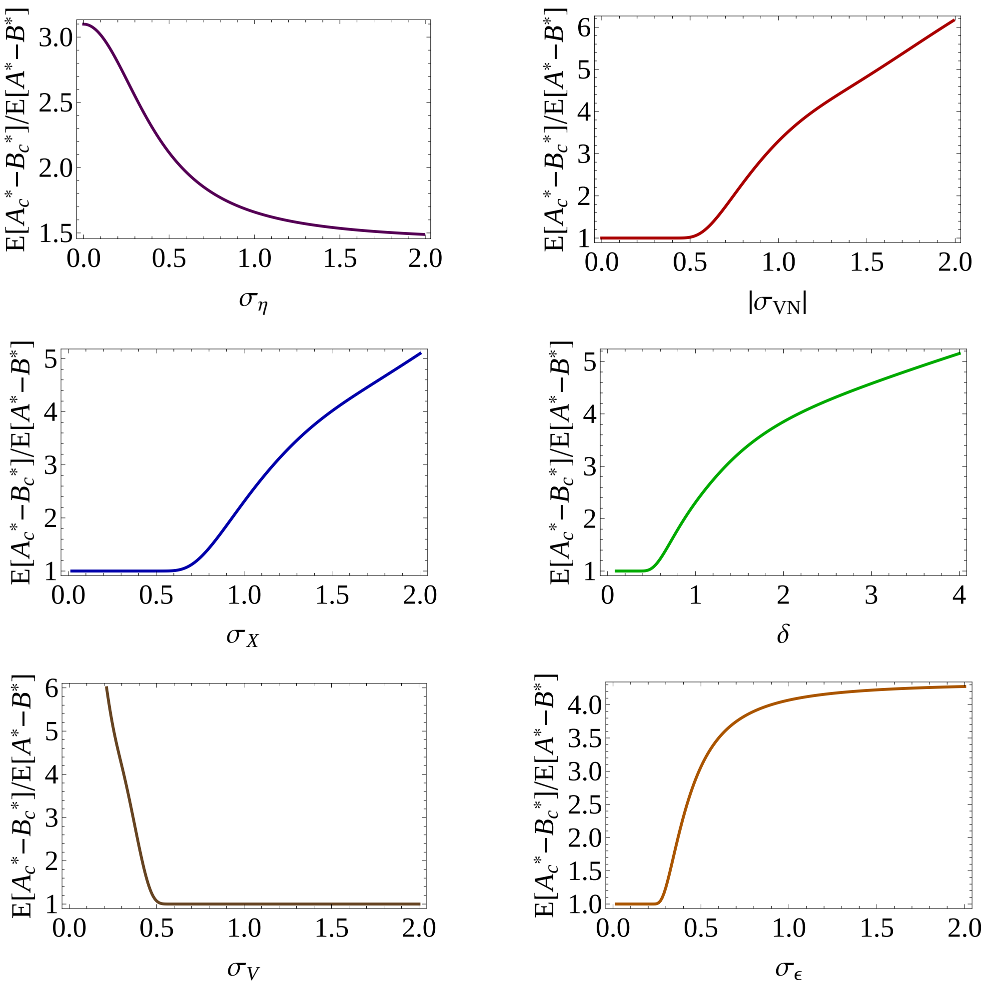


Figure 4: The expected increase of bid-ask spread due to the presence of short-sale constraints. The default parameters are: $\bar{\theta} = 1, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_\eta = 0.4, \sigma_X = 1, \sigma_\varepsilon = 0.4, \sigma_V = 0.4, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1$.

Similarly, the decrease of the trading volume due to short-sale constraints is

$$N_M(\alpha_c^* + \beta_c^*) - N_M(\alpha^* + \beta^*)$$

$$= \begin{cases} -\frac{N_U(2\nu N_I + N_M + 1)}{\delta Var[\tilde{V}|\mathcal{I}_U](\nu N_I + N_M + 1)} \left( C_U \Delta - \delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta} \right) & \Delta \geq \frac{\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta}}{C_U} \\ 0 & otherwise \\ -\frac{N_I(2N_U + N_M + 1)}{\delta Var[\tilde{V}|\mathcal{I}_I](N_U + N_M + 1)} \left( -C_I \Delta - \delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I \bar{\theta} \right) & \Delta \leq -\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I \bar{\theta}}{C_I}. \end{cases}$$

In order to understand the average size of the impact of shortsale constraints on spread and trading volume, we can compute the expected increase in the spread and the expected decrease in the trading volume due to the shortsale constraint. It can be shown that $\Delta$ is normally distributed with mean $\mu_\Delta$ and variance $\sigma_\Delta^2$ where

$$\mu_\Delta = \delta \rho_I (1 - \rho_U) \sigma_V^2 \bar{\theta}, \quad \sigma_\Delta^2 = h^2 \sigma_X^2 - \rho_I (1 - \rho_U) \sigma_V^2. \tag{39}$$

Then we have the following result.

**Proposition 2** *1. The probability of shortsale constraint binding decreases with $\sigma_\eta^2$ and $\sigma_V^2$, but increases with $\sigma_\varepsilon^2$, $\sigma_X^2$ and $|\sigma_{VN}|$.*

*2. The expected increase in the spread and the expected decrease in the trading volume due to the shortsale constraint decrease with $\lambda_U$, $\lambda_I$, and $\sigma_V^2$, but increases with $\sigma_\varepsilon^2$, $\sigma_X^2$ and $|\sigma_{VN}|$.*

Part 1 shows that the shortsale constraint is more likely to bind for lower $\sigma_\eta^2$ and $\sigma_V^2$, and for higher $\sigma_\varepsilon^2$, $\sigma_X^2$ and $|\sigma_{VN}|$. As expected, the expected increase in the spread grows when the constraint becomes more stringent. In addition, Proposition 2 implies that this expected increase in the spread increases with $\sigma_\varepsilon^2$, $\sigma_X^2$, and $|\sigma_{VN}|$. Intuitively, if $\sigma_\varepsilon^2$ or $|\sigma_{VN}|$ or $\sigma_X$ increases, then $I$ investors' trading demand has greater volatility and thus it is more likely for the constraint to bind. In addition, conditional on the constraint binding, the increase in the spread is also greater due to the increase in the volatility of the trading demand. Therefore, the expected increase of bid-ask spread due to the presence of short-sale constraints is higher. On the other hand, as the information asymmetry or the unconditional payoff uncertainty $\sigma_V^2$ increases, it is less likely that the shortsale constraints bind because the increase in the overall uncertainty tends to decrease order sizes.

Because the expected decrease in the trading volume closely follows the pattern of the expected increase in the spread, we focus on the analysis of the latter. As illustrated in Figure 4, the expected increase of bid-ask spread due to the presence of short-sale constraints increases in $\sigma_\varepsilon^2$, $|\sigma_{VN}|$ and $\sigma_X$ and decreases with $\sigma_V^2$, consistent with Proposition 2.

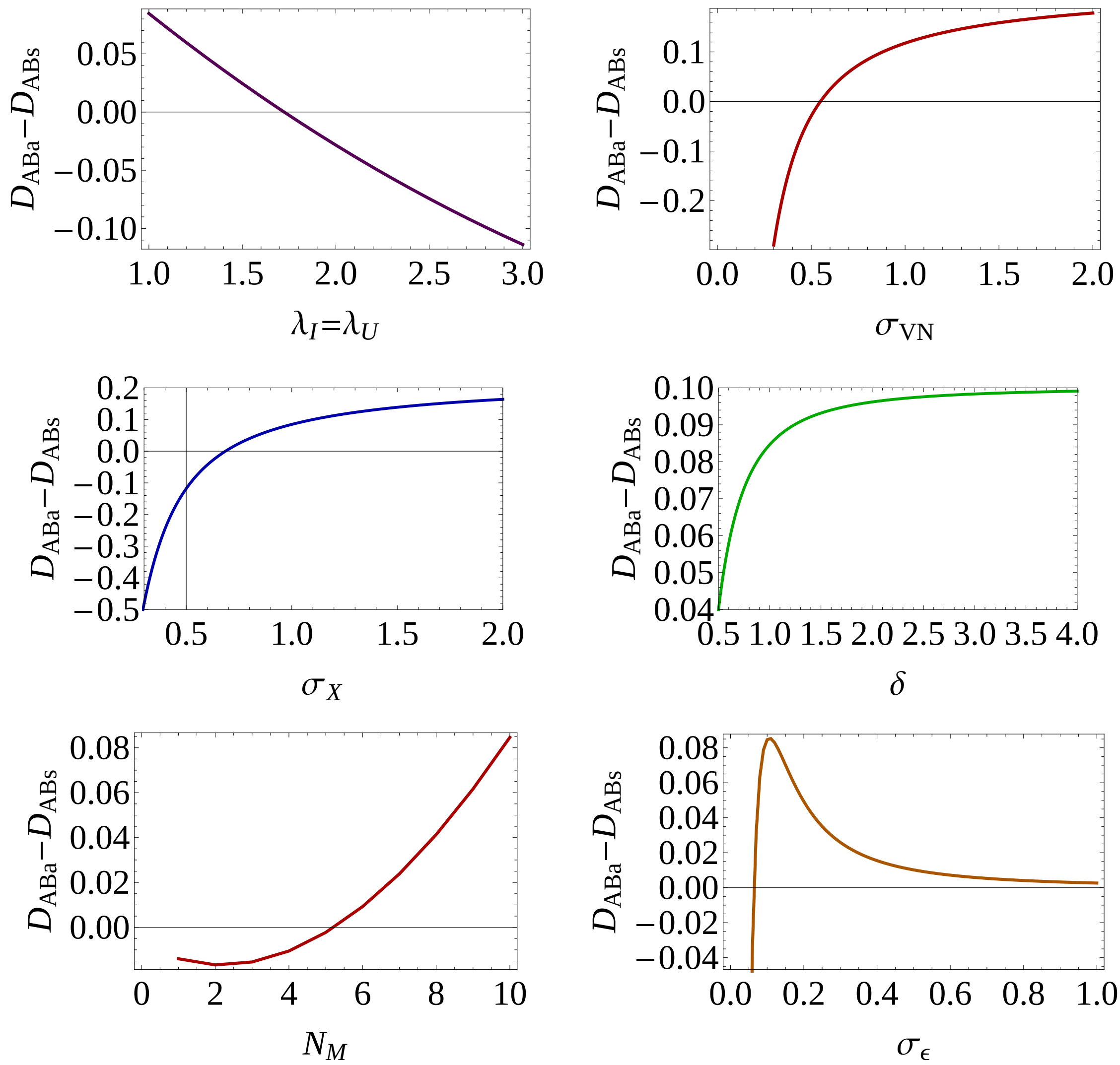


Figure 5: The difference of the expected increase of bid-ask spread due to the presence of short-sale constraints with ($\sigma_\eta \to \infty$) and without asymmetric information ($\sigma_\eta = 0$), i.e., $E[A^*_{ac} - B^*_{ac}]/E[A^*_a - B^*_a] - E[A^*_{sc} - B^*_{sc}]/E[A^*_s - B^*_s]$. The parameters are: $\bar{\theta} = 1, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_X = 1, \sigma_\varepsilon = 0.1, \sigma_V = 0.1, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1.$

The impact of information asymmetry on the expected spread increase due to shortsale constraints is ambiguous. On one hand, the probability of the constraint binding is smaller as the information asymmetry increases. On the other hand, the increase in the spread conditional on the constraint binding is greater. Therefore the net effect depends on which one dominates. In Figure 5, we plot the difference in the expected increase of bid-ask spread due to the presence of short-sale constraints with and without asymmetric information. Figure 5 shows that this difference can be positive or negative. For example, the percentage increase in the expected spread is higher with symmetric information than with asymmetric information when competition is low or the informed has more precise private information. However, the presence of information asymmetry can magnify the adverse impact of short-sale constraints. For example, when the hedging demand is volatile (e.g., liquidity shock volatility $\sigma_X$ or the covariance magnitude or the risk aversion is high), the percentage increase in the expected spread is greater with asymmetric information.

## *B.* Liquidity risk

Next, we study how short-sale constraints affect liquidity risk measured by the volatility of bid-ask spreads. We have

**Proposition 3** *Short-sales constraints increase stock's liquidity risk measured by the volatility of bid-ask spread, i.e.,* $Vol(A_c^* - B_c^*) \geq Vol(A^* - B^*)$.

Proposition 3 and Figure 6 show that short-sale constraints may significantly increase the liquidity risk measured by the volatility of bid-ask spread. This is consistent with Ho (1996) who finds an increase in stock return volatility when short sales were restricted during the Pan Electric crisis in the Singapore market in 1985-1986. Intuitively, short-sale constraints increase ask price and decrease bid price and thus tend to make the bid-ask spread more volatile. In addition, Figure 6 illustrates that the ratio of the bid-ask spread volatility with and without short-sale constraints increases in $\sigma_\varepsilon$, $|\sigma_{VN}|$ and $\sigma_X$, and decreases in $\sigma_V$. It may decrease in $\sigma_\eta$ (Figure 6) or increase

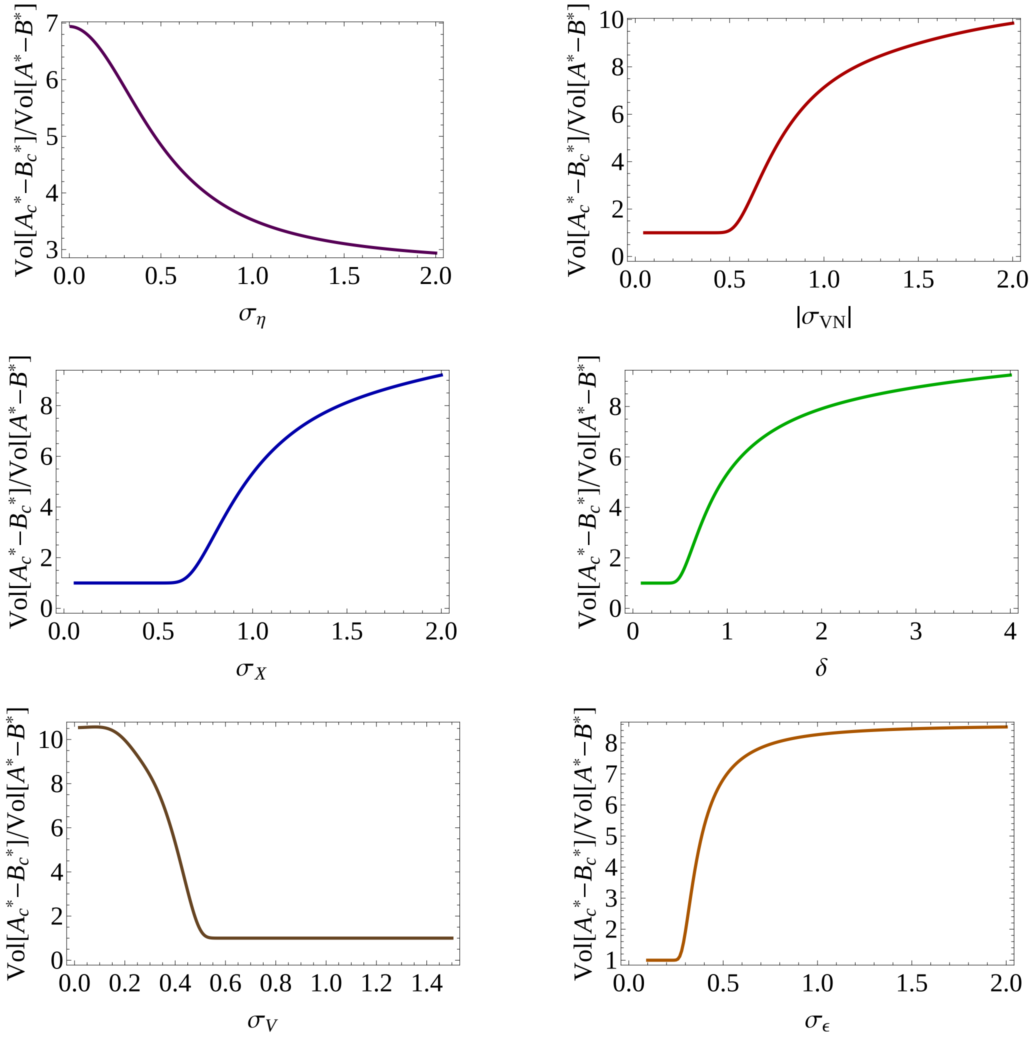


Figure 6: The ratio of the volatility of bid-ask spread with and without short-sale constraints. The default parameters are: $\bar{\theta} = 1, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_\eta = 0.4, \sigma_X = 1, \sigma_\varepsilon = 0.4, \sigma_V = 0.4, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1$.

in $\sigma_\eta$ (Figure 7). These patterns are similar to those for the expected increase in spread and the intuition is also similar.

## *C.* Welfare

We now examine the effect of short-sale constraints on investors' welfare. Let $U_{ic}$ and $U_i$ denote the expected utility of type $i$ ($i = I, U, M$) investors with and without

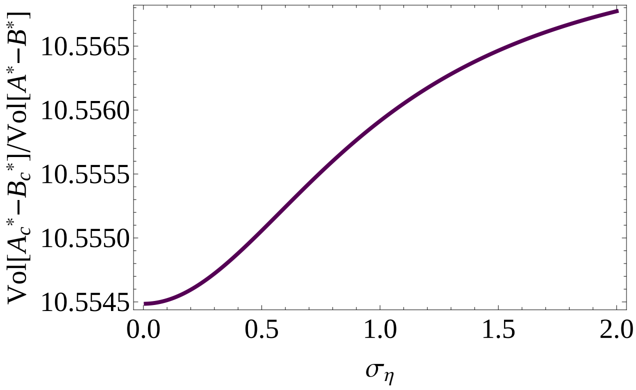


Figure 7: The ratio of the volatility of bid-ask spread with and without short-sale constraints against $\sigma_\eta$. The parameters are: $\bar{\theta} = 1, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_X = 1, \sigma_\varepsilon = 1, \sigma_V = 0.1, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1$.

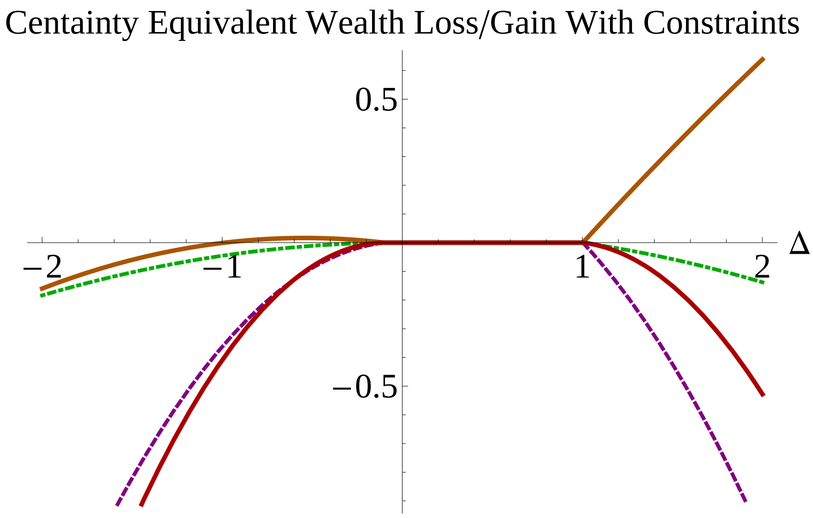


Figure 8: Certainty equivalent wealth gain/loss with short-sale constraints against $\Delta$. The thin solid curve denotes market makers' population weighted average of the certainty equivalent wealth gain/loss and the dashed (dot-dashed) curve denotes population weighted average of the certainty equivalent wealth gain/loss of $I$ ($U$) investors. The thick solid curve denotes weighted average of the certainty equivalent wealth gain/loss of both investors and market makers. The parameter values are: $\bar{\theta} = 1, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_\eta = 0.4, \sigma_X = 1, \sigma_\varepsilon = 0.4, \sigma_V = 0.4, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1$.

short-sale constraints respectively given realizations of signals on date 0 and $f_{ic}$ and $f_i$ be the corresponding certainty equivalent wealth, i.e., $U_{ic} = -\exp(-\delta f_{ic})$, and $U_i =$

$-\exp(-\delta f_i)$. The certainty equivalent wealth loss of a type $i$ investor ($i = I, U, M$) due to short-sale constraints is $f_i - f_{ic}$. We have

**Proposition 4** *When short-sale constraints bind for either I investors or U investors, both I and U investors are worse off while market makers could be better off if there is more than one market maker.*

From Proposition 4 and Figure 8, when short-sale constraints bind for some investors, all non-market makers trade at worse prices and with lower quantities and thus both of them are worse off. However, the presence of short-sale constraints may make market makers better off because they can trade at better prices. Market makers can buy at lower bid prices and sell at higher ask prices though both bid depth and ask depth are lower than the case without short-sale constraints. This positive price effect may dominate the negative quantity effect when there are more than one market makers because the constraint effectively limits competition among market makers. In the monopolistic case, market makers are also worse off regardless of the stringency of the short sale constraints.

## *D.* Information acquisition and market efficiency

When short-sale constraints bind for the uninformed, then obviously there is no loss of information efficiency due to the constraints. When short-sale constraints bind for the informed, the bid price will be such that short-sale constraints just start to bind. Because the order of $I$ investors just before the constraints bind perfectly reveal $\hat{S}$, short-sale constraints do not change information efficiency.

We now study a more interesting question: How does short-sale constraint affect information acquisition ex-ante? To study the effect of short-sale constraints on information acquisition, we assume that, on date 0, $I$-investors can acquire a costly signal $\hat{s}$ as defined in (1) with precision of $\rho_\varepsilon = \frac{1}{\sigma_\varepsilon^2}$. To get a more precise information, investors need to pay for a higher cost. The cost of the signal $\hat{s}$ is defined as $c(\rho_\varepsilon) := k\rho_\varepsilon^2$, where $k$ is a positive constant.

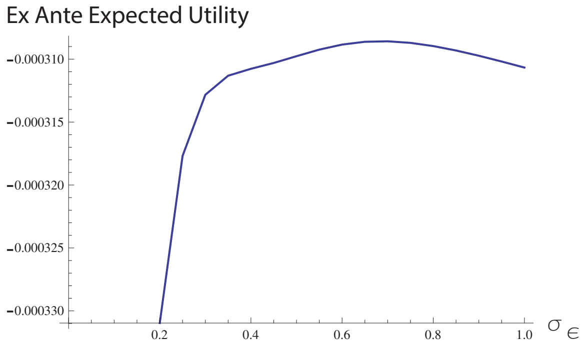


Figure 9: The ex ante expected utility of informed investors without short-sale constraints. The default parameters are: $\bar{\theta} = 3, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_\eta = 0.4, \sigma_X = 0.4, \sigma_N = 1, \sigma_V = 0.4, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1, k = 0.0065$.

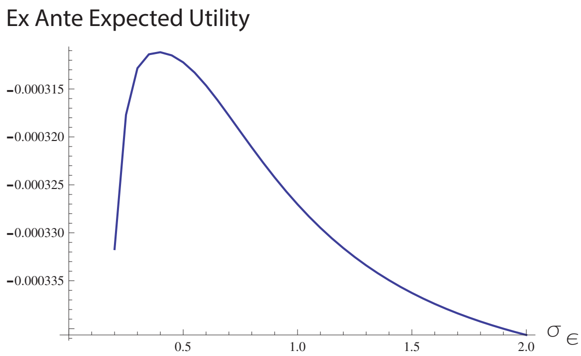


Figure 10: The ex ante expected utility of informed investors with short-sale constraints. The default parameters are: $\bar{\theta} = 3, \delta = 1, \sigma_{VN} = 0.8, \bar{V} = 3, \sigma_\eta = 0.4, \sigma_X = 0.4, \sigma_N = 1, \sigma_V = 0.4, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1, k = 0.0065$.

Figure 9 illustrates how the ex ante expected utility of the informed investors in the absence of short-sale constraints against $\sigma_\varepsilon$. As we can see that there may exist an optimal level of precision ($\rho_\varepsilon^*$) for informed investors who trade off the benefit from trading on the more precise private information and the cost of acquiring the private information. More interestingly, Figure 10 shows that the optimal precision of private information for informed investors in the presence of short-sale constraints may be higher than that in the absence of short-sale constraints.

Intuitively, the presence of short-sale constraints may reduce the incentive of investors to produce information because investors are restricted from short-selling after

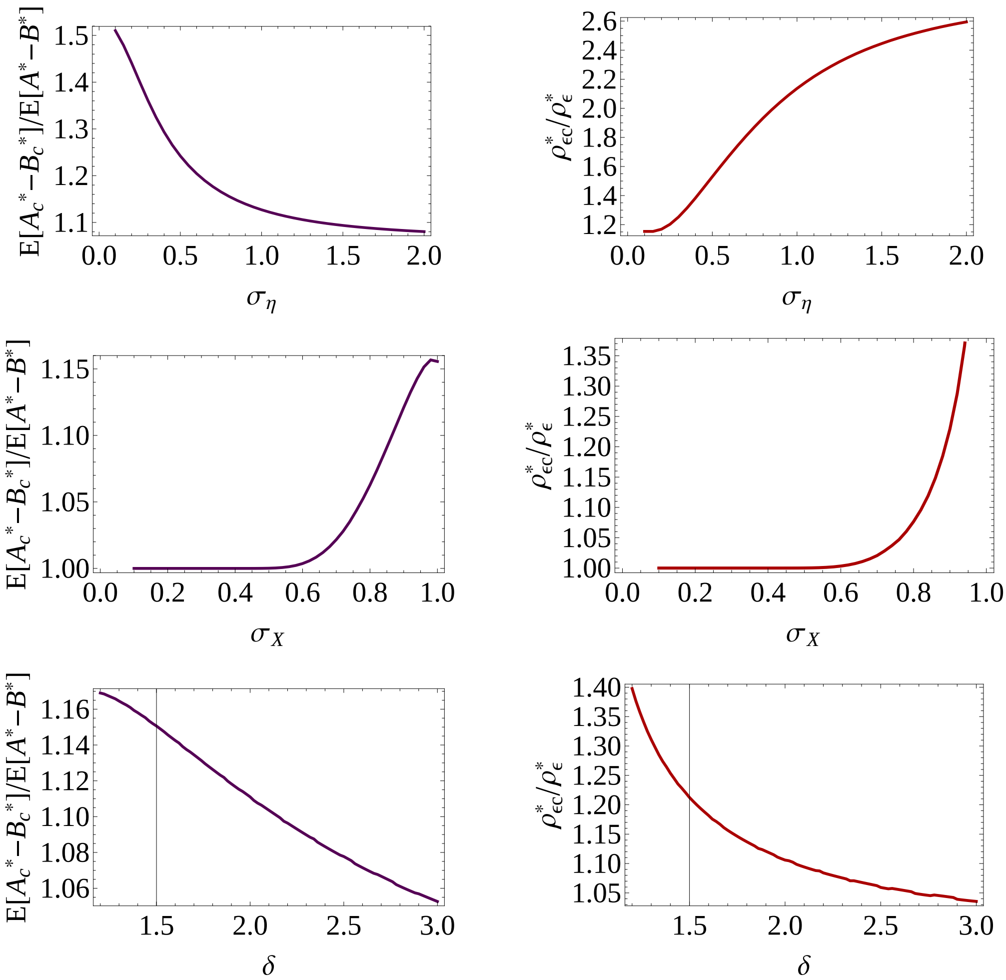


Figure 11: The ratio of the expected bid-ask spread and the ratio of the optimal precision of information with and without short-sale constraints when informed investors can optimally choose the precision of private information. The default parameters are: $\bar{\theta} = 4, \delta = 1, \sigma_{VN} = 0.6, \bar{V} = 3, \sigma_\eta = 0.8, \sigma_X = 1, \sigma_N = 0.4, \sigma_V = 0.4, N_I = 100, N_M = 10, N_U = 1000, \lambda_I = \lambda_U = 1, k = 0.02.$

observing a bad signal about stock payoffs. However, as shown in previous sections, short-sale constraints increase ask price and decrease bid price and thus make both informed and uninformed investors worse off. If the market is less informative, investors tend to trade at worse prices and thus their welfare loss due to the presence of short-sale constraints may increase. Therefore, the presence of short-sale constraints may actually increase investors' incentive to produce more precise information and increase market efficiency ex-ante.

# 5. Conclusion

In this paper, we develop an equilibrium model to study how short-sale constraints affect market prices, market liquidity and information acquisition in the presence of information asymmetry, inventory risk, and market power. In contrast to Diamond and Verrecchia (1987), we show that if competition among market makers is imperfect and market makers can make offsetting trades, then short-sale prohibition decreases the equilibrium bid price, increases the equilibrium ask price and also increases the bid-ask spread. In addition, short-sale constraints increase liquidity risk measured by the volatility of bid-ask spreads. The presence of asymmetry information can further magnify the adverse impact of short-sale constraints on market liquidity. On the other hand, the presence of short-sale constraints may increase investors' incentive to produce more precise information and thus increase market efficiency ex-ante.

# Appendix

**Proof of Theorem 1:** We prove the case when $\Delta < 0$. In this case, we conjecture that $I$ investors sell at the bid and $U$ investors buy at the ask. Given bid price $B$ and ask price $A$, the optimal demand of $I$ and $U$ are:

$$\theta_I^* = \frac{E[\tilde{V}|\mathcal{I}_I] + h\hat{X}_I - B}{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_I]} - \bar{\theta} \quad \textit{and} \quad \theta_U^* = \frac{E[\tilde{V}|\mathcal{I}_U] - A}{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_U]} - \bar{\theta}. \tag{40}$$

Substituting (40) into the market clearing conditions (6), we get that the market clearing bid and ask prices are:

$$\begin{aligned} A &= E[\tilde{V}|\mathcal{I}_U] - \delta \mathrm{Var}[\tilde{V}|\mathcal{I}_U]\bar{\theta} - \frac{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_U]}{N_U}\sum_{j=1}^{N_M}\alpha_j, \\ B &= E[\tilde{V}|\mathcal{I}_I] + h\hat{X}_I - \delta \mathrm{Var}[\tilde{V}|\mathcal{I}_I]\bar{\theta} + \frac{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_I]}{N_I}\sum_{j=1}^{N_M}\beta_j, \end{aligned} \tag{41}$$

where $\beta_j$ and $\alpha_j$ are the optimal shares of security $M_j$ choose to buy from $I$ investors and sell to $U$ investors respectively. Market maker $M_j$'s problem is:

$$\min_{\alpha_j,\beta_j} -\delta(\alpha_j A - \beta_j B) - \delta(\bar{\theta} + \beta_j - \alpha_j)E[\tilde{V}|\mathcal{I}_U] + \frac{1}{2}\delta^2 \mathrm{Var}[\tilde{V}|\mathcal{I}_U](\bar{\theta} + \beta_j - \alpha_j)^2, \tag{42}$$

where $A$ and $B$ are the market clearing prices given in (41). F.O.C with respect to $\beta_j$ gives us:

$$\begin{aligned} & E[\tilde{V}|\mathcal{I}_I] + h\hat{X}_I - E[\tilde{V}|\mathcal{I}_U] + \delta\left(\mathrm{Var}[\tilde{V}|\mathcal{I}_U] - \mathrm{Var}[\tilde{V}|\mathcal{I}_I]\right)\bar{\theta} \\ & + \frac{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_I]}{N_I}\sum_{j=1}^{N_M}\beta_j + \left(\frac{\mathrm{Var}[\tilde{V}|\mathcal{I}_I]}{N_I} + \mathrm{Var}[\tilde{V}|\mathcal{I}_U]\right)\delta\beta_j - \delta \mathrm{Var}[\tilde{V}|\mathcal{I}_U]\alpha_j = 0. \end{aligned} \tag{43}$$

Summing all, we get:

$$N_M\left(E[\tilde{V}|\mathcal{I}_I] + h\hat{X}_I - E[\tilde{V}|\mathcal{I}_U] + \delta\left(\mathrm{Var}[\tilde{V}|\mathcal{I}_U] - \mathrm{Var}[\tilde{V}|\mathcal{I}_I]\right)\bar{\theta}\right)$$

$$+\frac{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_I] N_M}{N_I} \sum_{j=1}^{N_M} \beta_j + \left( \frac{\mathrm{Var}[\tilde{V}|\mathcal{I}_I]}{N_I} + \mathrm{Var}[\tilde{V}|\mathcal{I}_U] \right) \delta \sum_{j=1}^{N_M} \beta_j - \delta \mathrm{Var}[\tilde{V}|\mathcal{I}_U] \sum_{j=1}^{N_M} \alpha_j = 0. \tag{44}$$

Using the F.O.C with respect to $\alpha_j$, we get:

$$\frac{\delta}{N_U} \sum_{j=1}^{N_M} \alpha_j - \delta(\beta_j - \alpha_j) + \frac{\delta}{N_U} \alpha_j = 0. \tag{45}$$

Summing all, we get:

$$\sum_{j=1}^{N_M} \beta_j = \frac{N_U + N_M + 1}{N_U} \sum_{j=1}^{N_M} \alpha_j. \tag{46}$$

Substituting (46) into (44), we get

$$\sum_{j=1}^{N_M} \alpha_j = -\frac{N_M N_I N_U}{(N_M + 1)\left(\overline{N} + 1\right)} \frac{\Delta}{\delta \mathrm{Var}[\tilde{V}|\mathcal{I}_I]}. \tag{47}$$

Substituting (47) into (41), we can get the equilibrium ask and bid price $A^*$ and $B^*$. And then substituting $A^*$ and $B^*$ into (40), we can get the optimal security holdings of $I$ and $U$ investors as stated in (23). In addition, $A^* < P_U^R$ and $B^* > P_I^R$ are equivalent to $\Delta < 0$ which is exactly the condition we conjecture for $I$ investors to sell and $U$ investors to buy. Similarly, we can prove the other case of this Theorem when $I$ investors buy and $U$ investors sell. *Q.E.D.*

**Proof of Corollary 1:** Part 1 follows from the fact that bid, ask , and $\tilde{V}$ all increase with $\tilde{V}$, $\hat{X}I$, and $\hat{\varepsilon}$. Conditional on the informed shorting, $\tilde{V}$, $\hat{X}I$, and $\hat{\varepsilon}$ are all on average lower than the unconditional expected values. Part 2 follows from direct computation using the expressions in Theorem 1. *Q.E.D.*

**Proof of Corollary 2:** The informed short if and only if $\Delta \leq -\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I \bar{\theta}}{C_I}$ and the non-informed short if and only if $\Delta > \frac{\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U \bar{\theta}}{C_U}$. The result then follows from the distribution of $\Delta$ and direct comparison with simplification. *Q.E.D.*

**Proof of Theorem 2:** We prove this theorem for when it is optimal for $I$ investors sell and $U$ investors buy. The proof for the other case is very similar. We thus skip it here. In the presence of short-sale constraints, $I$ investors' stock holding becomes

$$\theta_I^* = \max\left[\frac{P_I^R - B}{\delta Var[\tilde{V}|\mathcal{I}_I]}, -\lambda_I\bar{\theta}\right]. \tag{48}$$

In equilibrium, if short-sale constraints bind for $I$ investors, then we have

$$\theta_I^* = \frac{P_I^R - (P_U^R + C_U\Delta + \frac{\Delta}{N_M+1})}{\delta Var[\tilde{V}|\mathcal{I}_I]} \leq -\lambda_I\bar{\theta}, \tag{49}$$

*i.e.*,

$$\Delta \leq -\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}}{C_I}, \tag{50}$$

from (46) and (50), we know that market makers cannot buy their optimal number of shares in the absence of short-sale constraints from $I$ investors when $I$ investors are binding in short-sale constraints because $\sum_{j=1}^{N_M}\beta_j^* \geq N_I\lambda_I\bar{\theta}$, while we know that $\theta_I^* \geq -\lambda_I\bar{\theta}$ in the presence of short-sale constraints, so $\sum_{j=1}^{N_M}\beta_j \leq N_I\lambda_I\bar{\theta}$. Therefore, when $I$ investors bind in short-sale constraints, $\beta_j^* = \frac{N_I\lambda_I\bar{\theta}}{N_M}$. From (46), we get:

$$\sum_{j=1}^{N_M}\alpha_j^* = \frac{N_I N_U}{N_U + N_M + 1}\lambda_I\bar{\theta},$$

from (41), we get the equilibrium bid price $B_{c2}^*$ and ask price $A_{c2}^*$ when $I$ investors bind in short-sale constraints. *Q.E.D.*

**Proof of Proposition 1:** We prove this Proposition for the case when $I$ investors sell, the proof of the other case is very similar and we thus skip it here. If $I$ investors sell, from Theorem 1 and Theorem 2, we have

$$B_c^* - B^* = C_I\Delta + \delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}, \tag{51}$$

$$A_c^* - A^* = -C_U\Delta - \frac{N_I}{N_U + N_M + 1}\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_I\bar{\theta}, \tag{52}$$

and

$$(A_c^* - B_c^*) - (A^* - B^*) = -\frac{N_M}{N_M+1}\Delta - \frac{\bar{N}+1}{N_U+N_M+1}\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}. \tag{53}$$

From (50), we get that

$$A_c^* \geq A^*, \quad B_c^* \leq B^* \quad and \quad A_c^* - B_c^* \geq A^* - B^*.$$

In addition, from Theorem 2, we have $A_c^*$ decreases in $\lambda_I$, $B_c^*$ increases in $\lambda_I$ and $(A_c^* - B_c^*) - (A^* - B^*)$ decreases in $\lambda_I$. *Q.E.D.*

**Proof of Proposition 2:** The results follow directly from (tediously) taking the derivatives of the expected spread with respect to the corresponding variables. *Q.E.D.*

**Lemma 1** *Let $f(x) := |x|$ and*

$$g(x) := \begin{cases} c_1(k_1x - h_1) & x > \frac{h_1}{k_1} \\ 0 & otherwise \\ c_2(-k_2x - h_2) & x < -\frac{h_2}{k_2}, \end{cases}$$

*where $k_1$, $k_2$, $h_1$, $h_2$, $c_1$ and $c_2$ are positive constants and $x$ is randomly distributed in $(-\infty, +\infty)$ with probability density function $p(x)$ which is an even function. Then we have $Cov(f(x), g(x)) > 0$.*

*Proof :*

$$Cov(f(x), g(x)) = E(f(x)g(x)) - E(f(x))E(g(x))$$

$$= \int_{-\infty}^{+\infty} f(x)g(x)p(x)dx - \int_{-\infty}^{+\infty} f(x)p(x)dx \int_{-\infty}^{+\infty} g(x)p(x)dx$$

$$= \int_{-\infty}^{+\infty} p(y)dy \int_{-\infty}^{+\infty} f(x)g(x)p(x)dx - \int_{-\infty}^{+\infty} f(y)p(y)dy \int_{-\infty}^{+\infty} g(x)p(x)dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x)g(x) - f(y)g(x))\, p(x)p(y)dxdy$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))\, (g(x) - g(y))\, p(x)p(y)dxdy. \tag{54}$$

Since $p(-x) = p(x)$ and $p(-y) = p(y)$, we have

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(-x) - f(-y))\, (g(-x) - g(-y))\, p(x)p(y)dxdy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))\, (g(x) - g(y))\, p(x)p(y)dxdy. \tag{55}$$

From (54) and (55), we have $Cov(f(x), g(x)) =$

$$\frac{1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x) - f(y))\, (g(x) + g(-x) - g(y) - g(-y))\, p(x)p(y)dxdy. \tag{56}$$

(1) If $x$ and $y$ have the same sign, the term inside the integral can be written as $(f(x) - f(y))\,(g(x) - g(y)) + (f(-x) - f(-y))\,(g(-x) - g(-y))$, which is non-negative.
(2) If $x < 0$ and $y > 0$, the term inside the integral can be written as
$(f(-x) - f(y))\,(g(-x) - g(y)) + (f(x) - f(-y))\,(g(x) - g(-y))$, which is non-negative.
(3) If $x > 0$ and $y < 0$, the term inside the integral can be written as
$(f(x) - f(-y))\,(g(x) - g(-y)) + (f(-x) - f(y))\,(g(-x) - g(y))$, which is non-negative.
In addition, at least for some $x$ and $y$, the term inside the integral is non-zero. Therefore, $Cov(f(x), g(x)) > 0$. *Q.E.D.*

**Proof of Proposition 3:** Since $A_c^* - B_c^* = A^* - B^* +$

$$\begin{cases} \frac{\nu N_I N_M}{(\nu N_I + N_M + 1)(N_M + 1)}\Delta - \frac{\bar{N}+1}{\nu N_I + N_M + 1}\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U\bar{\theta} & \Delta \geq \frac{\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U\bar{\theta}}{C_U} \\ 0 & otherwise \\ -\frac{N_M}{N_M+1}\Delta - \frac{\bar{N}+1}{N_U + N_M + 1}\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta} & \Delta \leq -\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}}{C_I}, \end{cases}$$

from Lemma 1, the two terms in the right hand side of the above equation are positively correlated, it follows that $Var(A^*_c - B^*_c) > Var(A^* - B^*)$. *Q.E.D.*

**Proof of Proposition 4:** It can be shown that

$$f_I - f_{Ic} = \frac{1}{2}\delta Var[\tilde{V}|\mathcal{I}_I]\left[(\theta^*_I)^2 - (\theta^*_{Ic})^2\right],\; f_U - f_{Uc} = \frac{1}{2}\delta Var[\tilde{V}|\mathcal{I}_U]\left[(\theta^*_U)^2 - (\theta^*_{Uc})^2\right],$$

where $(\theta^*_I)^2 \geq (\theta^*_{Ic})^2$ and $(\theta^*_U)^2 \geq (\theta^*_{Uc})^2$. Therefore, both $I$ and $U$ investors are worse off when short-sale constraints bind for some investors. In addition,

$$\begin{aligned} f_M - f_{Mc} = {} & \alpha^*_j(A^* - P^R_M) - \alpha^*_{jc}(A^*_c - P^R_M) - \beta^*_j(B^* - P^R_M) + \beta^*_{jc}(B^*_c - P^R_M) \\ & -\frac{1}{2}\delta Var[\tilde{V}|\mathcal{I}_U]\left((\beta^*_j - \alpha^*_j)^2 - (\beta^*_{jc} - \alpha^*_{jc})^2\right), \end{aligned} \tag{57}$$

(57) implies that $f_M - f_{Mc} = 0$ is a quadratic function about $\Delta$ which has two real roots, one of the roots is $\Delta = \frac{\delta Var[\tilde{V}|\mathcal{I}_U]\lambda_U\bar{\theta}}{C_U}$ (resp. $\Delta = -\frac{\delta Var[\tilde{V}|\mathcal{I}_I]\lambda_I\bar{\theta}}{C_I}$) for the case when short-sale constraints bind for $U$ (resp. $I$) investors. In addition, if $N_M = 1$, then the two roots are the same, more specifically,

$$f_{Mc1} - f_{M1} = -\frac{\nu N_U\left(\Delta \times N_I - 2\lambda_U(\bar{N}+1)\delta Var[\tilde{V}|\mathcal{I}_I]\bar{\theta}\right)^2}{4(2+\nu N_I)(\bar{N}+1)\delta Var[\tilde{V}|\mathcal{I}_I]},$$

$$f_{Mc2} - f_{M2} = -\frac{N_I\left(\Delta(2+N_U) + 2\lambda_I(\bar{N}+1)\delta Var[\tilde{V}|\mathcal{I}_I]\bar{\theta}\right)^2}{4(2+N_U)(\bar{N}+1)\delta Var[\tilde{V}|\mathcal{I}_I]}.$$

Therefore, a monopolistic market maker is also worse off with short-sale constraints, however, multiple market makers can be better off in the presence of short-sale constraints.

*Q.E.D.*

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