Equilibrium Asset Pricing in the Presence of both Liquid and Illiquid Markets*

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Abstract

I study a general equilibrium model in which investors hedge risks using two imperfectly correlated assets. The first asset is traded on a liquid market whereas the second is traded on an illiquid over-the-counter (OTC) market. The search and bargaining frictions on the OTC market combine with demand shocks to make the efficiency of the illiquid market time varying. This non-fundamental risk is priced, spills over, and increases the risk premium on the liquid asset. Furthermore, these frictions increase both the trading volume and the open interest on the liquid market. The liquid market both mitigates the frictions on the OTC market and captures some of the illiquid asset's value as a risk-sharing instrument. Each of these two effects can dominate and opening the liquid market has an ambiguous effect on the illiquid asset's expected returns. I link the model predictions to recent empirical findings.

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Today, many assets are traded increasingly, or even exclusively, in over-the-counter (OTC) markets. For example, essentially all fixed income securities and a vast majority of all existing derivatives are traded OTC. Even for liquid stocks, large block trading among financial institutions is typically done off-exchange. Trading in OTC markets requires searching for a suitable counter-party and bargaining over the exact terms of the transaction. In an important contribution, Duffie, Gârleanu, and Pedersen (2005) model the functioning of an OTC market and show how search frictions affect prices and make the equilibrium allocation of the asset inefficient. A logical question follows as to whether investors hedge the inefficiently allocated asset by trading more liquid instruments with correlated cash-flows. In this paper, I address this question and show how illiquidity due to search frictions spills over liquid markets, affecting holdings, trading volumes, and risk premia.

Prime examples of securities offering exposure to the same fundamentals but with different levels of liquidity are bonds and credit default swaps (CDSs). Both bonds and CDSs are traded OTC, but CDSs are typically much more liquid.[1] Further examples include mortgages and collateralized debt obligations (CDOs), CDSs and index CDSs, options and index options, forwards and futures contracts, and real estate assets and property total return swaps. In all of these pairs, the first asset is less liquid and traded OTC.

The possibility of trading liquid securities may have a non-trivial effect on the transactions bargained on the OTC market. At the individual level, it improves each investor's outside option by increasing the set of investment opportunities. However, it also improves the outside options of an investor's trading counter-parties, leading to equilibrium feed-back effects on the bargaining outcome. At the same time, the search friction on the OTC market creates demand for the liquid asset that is not driven by its fundamentals, leading to illiquidity spillovers. The effect of these spillovers on the equilibrium prices of both the liquid and illiquid assets is a priori unclear and can only be quantified in a general equilibrium model. In this paper, I propose and analyze such a model.

I consider an economy in which risk-averse investors share endowment risks by trading two imperfectly correlated assets. The first is traded in a frictionless way on an exchange, whereas the second is traded on an illiquid OTC market.[2] I adopt the frame-

[1] A number of features make CDSs more liquid than bonds. First, a CDS contract on a bond issuer offer a homogeneous alternative to a possibly very fragmented bond market. Second, CDSs are derivatives and, as a result, new contracts can be created when needed and there is no need to locate an existing contract. Longstaff, Mithal, and Neis (2005) further discuss the relative liquidity of CDS and bond markets. They argue that the frictions on the CDS market are negligible when compared to those on the bond market. Another important difference is that trading a CDSs is far less capital intensive than trading bonds. This difference in the margin requirements increases CDS trading volumes at the expenses of bond trading, which makes CDS trading even cheaper when compared to bond trading. The difference in margin requirements is the object of, for instance, Basak and Croitoru (2000) and Gârleanu and Pedersen (2011), and is not explicitly modeled in this paper.

[2] In my model, there is a dichotomy between the (perfectly) liquid and illiquid markets. This is for ease of exposition and my conclusions are also relevant for pairs of markets that are "unequally illiquid". Again considering the CDS-bond pair, it is true that CDS markets are not perfectly liquid. For example, Bongaerts, de Jong, and Driessen (2011) and Junge and Trolle (2013) find that illiquidity is priced on

work developed by Duffie et al. (2005) to model the OTC market. In this framework, investors are matched randomly over time and the Nash bargaining solution characterizes the bilateral trades. Illiquidity, measured by the expected search time between two meetings, affects investors both because contacting trading partners is time consuming and because prices are not competitive.

Investors on the OTC market use the liquid asset as an imperfect substitute for the illiquid asset. Specifically, investors hedge their temporary sub-optimal exposure with the liquid instrument.[3] Due to this effect, the trading volume on the liquid market is always higher in the presence of the OTC market. Interestingly, the strictly positive increase in the trading volume sometimes persists even in the limit of vanishing search frictions. I also show that the dispersion of holdings in the liquid asset increases in the illiquidity of the OTC market. This prediction is consistent with the recent empirical findings of Oehmke and Zawadowski (2013) regarding bond and CDS markets. Oehmke and Zawadowski (2013) document an average net exposure on the CDS market that increases in the illiquidity of the bond market.

I now discuss how the frictions on the OTC market affect the expected returns of the liquid asset. I explicitly characterize the equilibrium response of the liquid asset to the frictions on the OTC market, and show that the spillover effect is driven by illiquidity *risk* and not by the illiquidity *level* alone. The mechanism is as follows: An aggregate shock to investors' hedging demand changes the imbalance between buyers and sellers on the OTC market. This imbalance determines the rate at which the illiquid asset is reallocated. The endogenous relationship between the preference shocks and the search friction makes the allocative efficiency of the OTC market time-varying. In equilibrium, agents require a premium for taking exposure to this non-fundamental risk and the correlation between the efficiency of the OTC market and the returns on the liquid asset is priced. When the risk profiles of the two assets are sufficiently similar, the allocative efficiency is positively correlated to the returns of the liquid asset. This means that the liquid asset performs poorly exactly when liquidating one's illiquid portfolio becomes more difficult, and this command a positive risk premium on the liquid asset. This model prediction is consistent with the conclusions of several empirical studies. For example, both Tang and Yan (2006) and Lesplingart, Majois, and Petitjean (2012) show that CDS spreads increase with the illiquidity of the underlying bonds. Das and Hanouna (2009) show how these same CDS spreads increase with the illiquidity of the debt issuer's stock.[4] The pricing of illiquidity risk also has interesting connections with the literature

CDS markets. Still, CDS markets are usually liquid when compared to the underlying bond market, and investors typically hedge a bond portfolio with CDS contracts and not the other way around.

[3] This type of behavior is not restricted to OTC markets. For example, stock index futures may be traded instead of re-balancing a diversified stock portfolio. Even if each stock is traded on an exchange and relatively liquid, trading one liquid futures is faster, less costly, and more convenient than re-balancing a diversified stock portfolio. Also, exchange-traded funds (ETFs) give investors the opportunity to conveniently adjust their exposures to stocks, bonds, commodities, or currencies. See Greenwich (May 2012) for a survey documenting the use of ETFs as an alternative to trading the underlying market.

[4] When comparing these empirical studies to the model's predictions, the returns on the liquid asset

on long-run risk, pioneered by Bansal and Yaron (2004). Namely, as it may take a long time for the market to recover after a liquidity shock, illiquidity leads to *endogenous long run risk* in the economy. This long run risk is priced, and its price depends on the long run value of future liquidity for an investor, expressed by the corresponding certainty equivalent.

Conversely, the liquid market has two effects on the OTC market. On the one hand, it mitigates the search frictions and reduces the illiquidity discount, pushing the price on the OTC market up. On the other hand, the liquid market diverts some of the illiquid asset's value as a risk-sharing instrument, pushing the OTC price down. The strength of these two effects depends on the risk profiles of the assets and the severity of the search friction. Evaluating the overall effect of the liquid market thus raises the following question: What is a meaningful combination of risk profiles and illiquidity level?

Looking at bonds and CDSs, or at mortgages and CDOs, or at any of the pairs listed above, we always observe the same pattern: The illiquid OTC market existed first and then the liquid market was created by financial intermediaries. The financial innovation is typically initiated either by an exchange or, if the new security trades OTC as well, by the dealers that will intermediate trades on the newly created market. In both cases, the intermediaries are interested in creating an active market.

Motivated by these examples, I put more structure into my model by endogenizing the risk-profile of the liquid asset. I assume that intermediaries select the liquid asset that maximizes the trading volume. Next, I compare the prices on the OTC market with and without the liquid asset. The risk profile of the optimal liquid asset is a weighted average of two risk profiles. The first profile is that of the illiquid asset, the second is the profile that would be optimal in terms of risk-sharing. I show that the weight on the profile of the illiquid asset is monotone increasing in the liquidity of the OTC market because, with a more active OTC market, there is more trading volume to capture. Perhaps paradoxically, this also means that the search frictions are easier to mitigate when they are smaller in the first place. Overall, the endogenous liquid asset increases the price on the OTC market when the search friction is sufficiently strong, but decreases it otherwise. Thus, the mitigation of the illiquidity discount dominates when the frictions on the OTC markets are strong enough, but the diversion of risk-sharing value away from the OTC market dominates otherwise.

This ambiguous equilibrium behavior is consistent with empirical findings regarding CDS trading and bond spreads. For example, Ashcraft and Santos (2009) find that the onset of CDS trading does not decrease the bond yield of the average firm, despite the new hedging opportunities. In my model, this corresponds to the illiquid market (the bond market) being at the threshold where the mitigation and diversion effects compensate each other. Differently, Saretto and Tookes (2013) find that CDS trading may have made debt financing less costly, but did so by relaxing only the "non-price"

should be understood as the returns on a CDS contracts for a protection seller. Indeed, holding a bond or selling protection on a CDS market offer essentially the same exposure to credit risk.

terms of debt.[5] In my model, this corresponds to the parameter range for which the mitigation effect dominates, meaning the range over which the trading frictions on the illiquid (bond) market are rather severe.

Finally, Das, Kalimipalli, and Nayak (2013) study empirically the interactions between CDS and bond markets and focus on informational efficiency. They provide "[...] *evidence of a likely demographic shift by large institutional traders from trading bonds to trading CDS in order to implement their credit views, resulting in declining efficiency and quality in bond markets.*" The same mechanism drives the findings in Das, Kalimipalli, and Nayak (2013) and the equilibrium behavior of my model. Specifically, Das et al. (2013) show how a CDS market diverts some trading away from the bond market, how this reduces the value of the bond, and how this reduction dominates any other benefits brought by the new market. This mechanism coincides with the equilibrium behavior of my model when the search frictions on the OTC market are not too severe.

Literature Review My paper builds on the literature considering the general equilibrium impact of trading frictions. These frictions can be the transaction costs on centralized markets, as in Lo, Mamaysky, and Wang (2004), Acharya and Pedersen (2005), Garleanu and Pedersen (2013), and Buss and Dumas (2013), or the search and bargaining frictions on OTC markets.[6]

The analysis of the search and bargaining frictions on OTC markets started, to a large extent, with Duffie et al. (2005). Duffie et al. (2005) a model of OTC trading that shares features with job market models such as Diamond (1982).[7] The model that is most closely related to mine is Duffie, Gârleanu, and Pedersen (2007). They also study bilateral trading in OTC markets with risk-averse agents. In comparison with Duffie et al. (2007), my main contribution is to model a second, liquid market and to study the interactions between the liquid and OTC markets. On the methodological side, I provide an existence and uniqueness argument that is also valid in Duffie et al. (2007). In addition, I allow for more general aggregate shocks in the dynamic analysis of the model. Further references in asset pricing with search and bilateral trading include Weill (2008), Vayanos and Weill (2008), and Afonso and Lagos (2011). A closely related strand of research considers centralized markets to which investors have intermittent and sometimes costly access. References in this strand include Lagos and Rocheteau (2007), Weill (2007), Lagos and Rocheteau (2009), and Gârleanu (2009). In all these references, investors trade on a unique market.

My paper also relates to the literature considering the interactions of different market structures. See, for example, Pagano (1989), Rust and Hall (2003), Miao (2006), and

[5] The *non-price* terms of a bond are its maturity, notional, and its many contract details (amortization, default triggers, embedded options, etc).

[6] Further trading restriction include portfolio constraints, margin requirements, and limited market participation. References in this literature include Merton (1987), Basak and Cuoco (1998), and Hugonnier (2012).

[7] See Mortensen (1987) or Rogerson, Shimer, and Wright (2005) for surveys of search models in labor economics.

Vayanos and Wang (2007). In these models, agents must execute a single trade and balance the benefits of a better price against a costly search. In contrast, in my model, investors trade repeatedly on both markets. Vayanos (1998) and Huang (2003) also analyze models in which agents trade assets with different liquidity levels. In both cases, however, the illiquidity is modeled by exogenous transaction costs. Exogenous and constant transaction costs cannot capture the endogenous and time varying interaction between demand shocks and illiquidity. Parlour and Winton (2013) discuss the role of monitoring when a bank chooses between selling loans and buying CDS protection. Biais (1993), Yin (2005), and De Frutos and Manzano (2002) statically compare prices on centralized and fragmented markets.

My paper is also related to the literature on hedging demand as a determinant of illiquidity discounts. In the context of bond markets, this mechanism is discussed, for instance, by Duffie (1996), Duffie and Singleton (1997), Krishnamurthy (2002), and Graveline and McBrady (2011).

My discussion of illiquidity spillovers is related to the literature that investigates contagion effects across markets. References focusing on volatility contagion include Hamao, Masulis, and Ng (1990), Lin, Engle, and Ito (1994), Kyle and Xiong (2001), Kodres and Pritsker (2002), and Hasler (2013). References such as Chordia, Sarkar, and Subrahmanyam (2005) and Mancini, Ranaldo, and Wrampelmeyer (2013) document the cross-market effects of both returns and liquidity. My model explicitly describes a channel for illiquidity spillover effects.

Finally, the endogenous choice of the liquid asset in my model is similar to the security design setting in Duffie and Jackson (1989).

The outline of the paper is as follows. Section 1 introduces the model. Section 2 analyzes the investor's problem. Section 3 describes the population dynamics. Section 4 solves for an equilibrium. Section 5 considers the impact of aggregate demand shocks. Section 6 discusses the impact of the liquid market on the OTC market. Section 7 concludes.

1 Model

I study an economy in which investors share endowment risk by trading two different assets on, respectively, a liquid exchange and an OTC market. This model is an extension of Duffie, Gârleanu, and Pedersen (2007).

Assets and investors Two independent aggregate risk factors are described by the Brownian motions

$$(B_{a,t}, B_{b,t})_{t\geq 0}\,.$$

Two risky assets, c and d, are exposed to these risk factors. The cumulative dividend payouts of these assets satisfy

$$\begin{aligned} \mathrm{d}D_{c,t} &= m_c\,\mathrm{d}t + a_c\,\mathrm{d}B_{a,t} + b_c\,\mathrm{d}B_{b,t}, \\ \mathrm{d}D_{d,t} &= m_d\,\mathrm{d}t + a_d\,\mathrm{d}B_{a,t} + b_d\,\mathrm{d}B_{b,t}. \end{aligned} \tag{1}$$

These assets are available in net supplies S_c and S_d, respectively. As described below, the asset c is traded on a $c$entralized market, whereas the asset d is traded on a $d$ecentralized, OTC market. For convenience, I define the vectors

$$e_c \triangleq \begin{pmatrix} a_c \\ b_c \end{pmatrix}, \; e_d \triangleq \begin{pmatrix} a_d \\ b_d \end{pmatrix}$$

and call them the exposures of the assets c and d, respectively.There is also a risk-free asset, available in perfectly elastic supply, and paying out dividends at the constant rate $r > 0$.[8]

The economy is populated by a continuum of investors. I write μ for a normalized measure over this continuum. Each investor receives an endowment driven both by the aggregate risk factors and by idiosyncratic shocks. More specifically, the cumulative endowment of investor i satisfies

$$\mathrm{d}\eta_t = m_\eta \, \mathrm{d}t + a_{i,t} \, \mathrm{d}B_{a,t} + b_{i,t} \, \mathrm{d}B_{b,t}, \tag{2}$$

and is thus driven by the two aggregate risk factors. The vector of exposures

$$e_{i,t} \triangleq \begin{pmatrix} a_{i,t} \\ b_{i,t} \end{pmatrix} \tag{3}$$

evolves stochastically. Specifically, the stochastic vector $e_{i,t}$ is a time-homogeneous Markov chain jumping back and forth between two (two-dimensional) values.[9] These two values are

$$e_1 \triangleq \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \text{ and } e_2 \triangleq \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad (\in \mathbb{R}^2)$$

and I denote by

$$\begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{pmatrix} \tag{4}$$

the generator of the Markov chain. The Markov chains are independent across agents.

There are various ways to interpret the idiosyncratic shocks to the vectors of exposures. A shock could represent a large loss incurred by an individual investing in assets that I do not model explicitly, significant inflows or outflows experienced by a fund, a significant movement in the inventory of a dealer, or the underwriting by a bank of a new bond issue. In all of these cases, an idiosyncratic shock calls for an adjustment of the risk exposure.[10] As summarized by Cochrane (2005), no matter what the exact interpretation of these shocks is, their role is to keep investors trading with each other.

[8] The interest rate is exogenous in all the models of asset pricing with search that I am aware of.

[9] In particular, both components of the vector of exposures jump together.

[10] See Duffie et al. (2005) and Duffie et al. (2007) for discussions of these idiosyncratic preference shocks.

Trading mechanisms Investors trade the liquid asset c on a centralized market. Investors access this market without delay and trade without transaction costs. The only minor restriction is that the number of shares held by an investor must belong to the range

$$[-K, K]$$

at any time, with $K > 0$ being a fixed, large number.[11] Investors also trade the risk-free asset at any time and without costs.

The other risky asset, d, is traded OTC. Trading d thus requires searching for a counter-party and negotiating the details of the transaction. The search process is governed by a "random matching". That is, a given investor gets in touch with another investor at the jump times of an idiosyncratic Poisson process with intensity Λ. This other investor is randomly drawn from across the population of investors. The draws are independent across meetings.

As the meeting intensity Λ controls the search friction on the OTC market, I call it the *liquidity* of the OTC market. Given the dynamics of a Poisson process, the inverse

$$\xi \triangleq \frac{1}{\Lambda}$$

of the liquidity is the expected search time until the next meeting. I call this expected time the *illiquidity* of the OTC market.

Taking things together, investors from two separate subsets B and C of the population meet at the rate

$$2\Lambda\mu(B)\mu(C), \tag{5}$$

with μ being a measure on the set of investors. There is a factor 2 because the agents in B can both find an agent in C and be found by one.[12,13]

Once two agents have met, they bargain over a possible trade in the illiquid asset d. Specifically, they decide whether or not to exchange $\Theta > 0$ units of the asset and, if so, at which price. The outcome of the bargaining is given by the generalized Nash bargaining solution. Θ is an exogenous constant and the possible holdings in the illiquid asset d are restricted to two values, zero and Θ.[14]

[11] This constraint is required by the verification argument et prevents doubling strategies. The constraint never binds in equilibrium when K is chosen large enough, as seen in Proposition 10 below. Gârleanu (2009) adopts the same type of restrictions.

[12] This statement is intuitive but non-trivial. More specifically, it assumes a certain Law of Large Numbers. See Duffie and Sun (2011) for the rigorous treatment of this issue in discrete time and Footnote 34 for a similar discussion.

[13] When appropriate, statements in this paper should be understood as holding almost surely.

[14] The assumption of a fixed transaction size is restrictive but both convenient and common. With fixed transaction size, the bargaining regarding the size of a transaction is reduced to accepting the trade or not. Alternatively, Lagos and Rocheteau (2007), Lagos and Rocheteau (2009), and Gârleanu (2009) let investors choose their holdings freely, but model an intermittent access to a centralized market instead of bilateral meetings. Afonso and Lagos (2011) and Cujean and Praz (2013) model bilateral markets with endogenous portfolio holdings.

Working with a continuum of agents makes the model tractable and isolates the effect of search frictions from, say, concerns of reputation or punishment. Moreover, actual OTC markets often involve large numbers of investors and dealers, making a detailed modeling of the entire market infeasible. For example, more than 600 dealers appear in the sample of corporate bond trades analyzed by Schultz (2001). Similarly, Li and Schürhoff (2012) study a network of several hundred municipal bond dealers.[15]

My second comment regards the matching technology (5). Other specifications exist in the literature.[16] However, the matching technology (5) has two advantages. First, as argued in Weill (2008), it results from an explicitly specified search process and the existence of this random matching is, partly, justified by the discrete time results in Duffie and Sun (2011). Second, it exhibits increasing returns to scale. In the context of real assets, Gavazza (2011) argues that increasing returns to scale is an intuitively appealing and empirically important feature of search markets.

Preferences Each investor i maximizes her expected utility from consumption. Her utility function U has a constant coefficient of absolute risk aversion $\gamma > 0$ (exponential or CARA utility), meaning that

$$U : c \mapsto -e^{-\gamma x},$$

and their subjective rate of discounting is $\rho > 0$. The consumption and investment policy of i is thus dictated by the optimization

$$\sup_{\tilde{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho u} U\left(\tilde{c}_u\right) \, \mathrm{d}u \middle| \mathcal{F}_{i,0}\right], \tag{6}$$

with the admissible consumption processes $\tilde{c}$ satisfying certain conditions defined below and $\mathcal{F}_{i,0}$ being all the information available and relevant to i at time 0.

The payouts of the risky assets, defined in (1), are independent and identically distributed across time. Furthermore, the idiosyncratic exposure shocks defined by (4) offer a unique and stable stationary distribution of types 1 and 2 across the population. As a result, I expect all the aggregate quantities to be constant in the long run and I focus

[15] This hypothesis of a continuum of agents is common in the literature but sometimes criticized. The main criticism is as follows. Actual OTC market are often dominated by a limited number of dealers who form the core of the OTC market and market participants repeatedly and strategically interact with each other. Differently, in a model with a continuum of agents and random matching, investors never trade twice with the same person and this feature may prevent the model from capturing the working of actual OTC markets. This criticism is valid but the advantages of working with a continuum of investors are strong, and I decided to work with a continuum. In asset pricing with search, Duffie et al. (2005), Duffie et al. (2007), Vayanos and Wang (2007), and Weill (2007), for instance, model a continuum of investors who trade bilaterally. Alternatively, papers such as Gofman (2010), Gale and Kariv (2007), Malamud and Rostek (2012), and Babus and Kondor (2013) study decentralized trading among a finite number of investors.

[16] Weill (2008) and Inderst and Müller (2004), for instance, use more general matching technologies to study search problems in finance.

my analysis on this asymptotic, stationary case.[17,18] In a stationary equilibrium, the information set $\mathcal{F}_{i,0}$ only contains idiosyncratic quantities and the individual problem (6) becomes

$$V(w, i\theta) \triangleq \sup_{\tilde{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(\tilde{c}_u)\, \mathrm{d}u \;\middle|\; \begin{matrix} w_0 & = & w \\ e_0 & = & e_i \\ \theta_0 & = & \theta \end{matrix}\right], \tag{7}$$

with w_0 being the wealth invested by i at time zero in either the liquid risky asset c or in the risk-free asset, e_0 being i's vector of exposures at time zero, and $\theta_0 \in \{0, \Theta\}$ being i's holdings in the illiquid asset at time zero. The optimization takes place over the set of consumption processes that satisfy the budget and admissibility constraints discussed below.

Budget constraint The consumption and trading of an investor must be consistent with the dynamics of her wealth. Specifically,

$$\mathrm{d}\tilde{w}_t = r\tilde{w}_t\, \mathrm{d}t - \tilde{c}_t\, \mathrm{d}t + \mathrm{d}\eta_t + \tilde{\theta}_t\, \mathrm{d}D_{dt} + \tilde{\pi}_t\, (\mathrm{d}D_{ct} - rP_c\, \mathrm{d}t) - P_d\, \mathrm{d}\tilde{\theta}_t, \tag{8}$$

with $\tilde{\pi}_t$ being the number of shares of the liquid asset c held at time t, P_c being the price of the liquid asset c and P_d being the price of the illiquid asset d. As the asset d is traded bilaterally, defining the dynamics of the holdings $\tilde{\theta}_t$ in d and the price P_d at which it trades requires some extra care.

Two investors trade the asset d if the transaction is in their best interest. Investor a sells the illiquid asset to investor b if there exists a price $\tilde{P}_d$ that satisfies both

$$V\left(w_a + \Theta\tilde{P}_d, i_a 0\right) \geq V(w_a, i_a, \Theta), \tag{9}$$

meaning that it is rational for a to sell, and

$$V\left(w_b - \Theta\tilde{P}_d, i_b\Theta\right) \geq V(w_b, i_b 0), \tag{10}$$

meaning that it is rational for b to buy.[19] If a trade is rational for the two counterparties, the Nash bargaining solution determines the transaction price P_d. That is, P_d satisfies

$$P_d \in \arg\max_{\tilde{P}_d} \left\{ \begin{array}{c} \left(V\left(w_a + \Theta\tilde{P}_d, i_a, 0\right) - V(w_a, i_a, \Theta)\right)^{\eta_\Theta} \\ \cdot \left(V\left(w_b - \Theta\tilde{P}_d, i_b, \Theta\right) - V(w_b, i_b, 0)\right)^{\eta_0} \end{array} \right\}, \tag{11}$$

with $\eta_\Theta \in (0, 1)$ being the bargaining power of the seller and $\eta_0 = 1 - \eta_\Theta$ being the bargaining power of the buyer.

[17] I consider the impact of aggregate risk in Section 5.

[18] The cross-sectional distribution of wealth is not necessarily constant over time. However, the equilibrium policies are independent of the wealth, thanks to the CARA preferences, and this non-stationarity has no impact on the equilibrium portfolios and prices.

[19] The wealth of a prior to the trade is w_a, its vector of exposures is e_{i_a}, etc.

The bilateral trading introduces a structure of rational expectations in the investor's problem. An investor takes as given both her own value function and those of the counter-parties she will meet. This investor then deduces the prices at which she will trade the illiquid asset, and deduces her own actual value function. A solution to the investor's problem thus consists in rational expectations regarding the value functions. I must still impose certain regularity conditions on the consumption processes.

Regularity The wealth process $\tilde{w}$ satisfies

$$\lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\tilde{w}_T}\right] = 0. \tag{12}$$

The requirement (12) excludes pathological wealth processes and is needed in the verification argument for the Hamilton-Jacobi-Bellman (HJB) equations.[20] Pathological wealth processes include doubling strategies and the "financing" of consumption by an ever increasing amount of debt.

Finally, for ease of exposition, I restrict the model parameters as follows.

Assumption 1. The dynamic of the exposure shocks, as described by (4), and the supply S_d of the illiquid asset d satisfy

$$\frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}} \neq S_d \neq \frac{\lambda_{21}}{\lambda_{12}+\lambda_{21}}.$$

Also, the vectors e_d, e_c, and $e_1 - e_2$ are not collinear. □

Assumption (1) prevents the lengthy treatment of non-generic cases.[21]

2 The Investor's Problem

I characterize the solution to the investor's problem (7) by the dynamic programming approach, meaning that I deduce the optimal consumption and trading policy from a HJB equation.[22]

Along an optimal path $(\pi^*, \theta^*, c^*, w^*)$, the process

$$\left(\int_0^t e^{-\rho s} U(c_s^*)\,\mathrm{d}s + e^{-\rho t} V\left(w^*, i_t\theta_t^*\right)\right)_{t\geq 0} = \left(\mathrm{E}\left[\int_0^\infty e^{-\rho s} U(c_s^*)\,\mathrm{d}s \middle| \mathcal{F}_t\right]\right)_{t\geq 0}$$

[20] The verification argument is the Appendix C.

[21] I endogenize the risk-profile e_c in Section 16. The assumption regarding the non-collinearity of the vectors of exposures will hold in this case if e_d and $e_1 - e_2$ are not collinear

[22] The idiosyncratic exposure shocks and the idiosyncratic search processes make the markets incomplete. Further, the illiquid holdings can only be adjusted at stochastic times and are restricted to two values. A martingale approach along the lines of Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989) does not seem easily applicable.

must be a martingale. Equating the expected dynamics of this process to zero and assuming that pointwise maximization characterizes the optimal consumption and investment policy yields the HJB equation

$$\begin{aligned}\rho V(w, i\theta) = \sup_{\tilde{c},\tilde{\pi}} \; & U(\tilde{c}) \\ & + \frac{\partial V}{\partial w}(w, i\theta)\left(rw - \tilde{c} + m_\eta + \theta m_d + \tilde{\pi}\left(m_c - rP_c\right)\right) \\ & + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(w, i\theta)\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}^* \\ & + \lambda_{i\bar{i}}\left(V(w, \bar{i}\theta) - V(w, i\theta)\right) \\ & + 2\Lambda\,\mathrm{E}^{\mu(b)}\left[\mathbf{1}_{\text{surplus}}\left(V(w - (\bar{\theta} - \theta)P_d, i\bar{\theta})) - V(w, i, \theta)\right)\right],\end{aligned} \tag{13}$$

with the matrix of covariations

$$\begin{aligned}\Sigma_i &\triangleq \frac{1}{\mathrm{d}t}\,\mathrm{d}\langle\begin{pmatrix}\eta_t \\ D_{d,t} \\ D_{c,t}\end{pmatrix}, \begin{pmatrix}\eta_t & D_{d,t} & D_{c,t}\end{pmatrix}\rangle \\ &= \begin{pmatrix}a_i^2 + b_i^2 & a_i a_d + b_i b_d & a_i a_c + b_i b_c \\ a_i a_d + b_i b_d & a_d^2 + b_d^2 & a_c a_d + b_c b_d \\ a_i a_c + b_i b_c & a_c a_d + b_c b_d & a_c^2 + b_c^2\end{pmatrix},\end{aligned} \tag{14}$$

for $i \in \{1, 2\}$ and $\theta \in \{0, \Theta\}$.[23,24] On the right-hand side of (13), the fifth line represents the utility gains resulting from trading on the OTC market. The indicator function

$$\mathbf{1}_{\text{surplus}}$$

appears in the equation because not every meeting results in a trade. More specifically, two investors only exchange the illiquid asset if they have a surplus to share, meaning that both (9) and (10) hold. Furthermore, the transaction price P_d may depend on the counter-party b and is characterized by (11). The other terms on the right-hand side of (13) refer to the current consumption, the drift and volatility of the wealth, and shocks to the vector of exposures. To characterize, the solution to (13) I make the following assumption.[25]

Assumption 2. The value functions satisfy

$$V(w, i\theta) = -e^{-\alpha(w + a(i\theta) + \bar{a})},$$

for a set of numbers $\alpha \in \mathbb{R}_{>0}$, $a \in \mathbb{R}^4$, and $\bar{a} \in \mathbb{R}$ to be characterized. □

Assumption 2 will be justified *ex post* by an existence result. Conditional on Assumption 2, the optimal policy of the investors is known in closed-form. First, the trading on the OTC market occurs as described below.

[23] I write "$\bar{\cdot}$" for "the other possible value". For example, if $i = 1$, then $\bar{i} = 2$.

[24] For convenience, I index the entries of Σ_i by i, d, and c.

[25] This functional form is standard for problems similar to the one at hand. It is used, among others, by Duffie, Gârleanu, and Pedersen (2007), Vayanos and Weill (2008), and Gârleanu (2009).

Proposition 3 (OTC trading)**.** *On the OTC market, investors trade as follows.*

1. *Investors with exposure type 1 sell the illiquid asset to those with type 2 exposure when*
$$a(2\Theta) - a(20) > a(1\Theta) - a(10). \tag{15}$$
In particular, the decision to trade does not depend on the wealth of the investors.

2. *Investors with exposure type 2 sell the illiquid asset to those with type 1 exposure when*
$$a(2\Theta) - a(20) < a(1\Theta) - a(10). \tag{16}$$
The decision to trade does not depend on the wealth of the investors either.

3. *If a i-investor sells the illiquid asset to a $\bar{i}$-investor, the transaction price P_d is the unique solution to*
$$(1-\eta_0)\left(1 - e^{\alpha(a(i0)+P_d\Theta - a(i\Theta))}\right) = \eta_0\left(1 - e^{\alpha(a(\bar{i}\Theta)-P_d\Theta - a(\bar{i}0))}\right). \tag{17}$$
This solution is available in closed-form.[26]

Proof. See Proof 19 in Appendix B. □

The proposition shows that the bargaining outcomes on the OTC market depend on the exposures (and holdings) of the counter-parties but not on their wealth. As investors are only interested in the rest of the population to the extent that it represents potential counter-parties, I call *type* of an agent the combination of her exposure, indexed by 1 or 2, and her illiquid holdings, 0 or Θ.

The characterization of the transactions on the OTC market in terms of inequalities (15) and (16) is intuitive. Recalling Assumption 2 regarding the value functions, the difference
$$v(i\Theta) - v(i0),\ \ i = 1, 2,$$
is the reservation value of a i-agent for the illiquid asset. Inequality (15) thus states that the reservation value of the 2-agents is higher than that of the 1-agents. In this case, the 2-investors buy the illiquid asset from the 1-investors. Inequality (16) is just the opposite.

I can also characterize the optimal consumption and investment in the liquid asset.

Proposition 4 (Consumption and Liquid Holdings)**.** *The optimal consumption is*
$$c(i\theta) = \frac{1}{\gamma}\left(\alpha\left(w + a(i\theta) + \bar{a}\right) - \log\left(\frac{\alpha}{\gamma}\right)\right)$$

[26] Equation (17) is quadratic in
$$x \triangleq \exp\left(\alpha\Theta P_d\right)\ \ (>0)$$
and admits a unique positive solution. This unique solution readily defines P_d.

and the optimal holdings in the liquid asset are

$$\pi(i\theta) = \frac{1}{\Sigma_{cc}} \left(\frac{1}{\alpha} (m_c - rP_c) - (\Sigma_{ic} + \Sigma_{cd}) \right), \tag{18}$$

for any type $i\theta$ and liquid wealth w.

Proof. See Proof 20 in Appendix B. □

Combining the HJB equation (13) with Lemma 3 and Lemma 4 provides a narrower characterization of the value function.

Proposition 5 (Value Functions)**.** *The constants in the value function*

$$V(w, i\theta) = -\exp\left(-\alpha\left(w + a(i\theta) + \bar{a}\right)\right),$$

are characterized as follows. First, $\alpha = r\gamma$. Furthermore, choosing

$$\bar{a} = \frac{1}{r\gamma}\left(-1 + \frac{\rho}{r} + \gamma m_\eta + \log(r)\right), \tag{19}$$

and taking the cross-sectional distribution of types $\mu \overset{(\Delta)}{=} \{\mu(i\theta)\}_{i\theta}$ as given, the type specific constants "$a(i\theta)$" are the unique solution to the system

$$\begin{aligned} ra(i\theta) =& \kappa(i\theta) + \lambda_{i\bar{i}} \left(\frac{e^{-r\gamma(a(\bar{i}\theta) - a(i\theta))} - 1}{-r\gamma} \right) + 2\Lambda\mu(\bar{i}\bar{\theta}) \left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}\left(a\right)\right)}{-r\gamma} \right]^+, \\ & i \in \{1,2\}, \theta \in \{0, \Theta\}, \end{aligned} \tag{20}$$

with the quantity

$$\kappa(i\theta) \overset{\Delta}{=} \theta m_d + \pi(i\theta)\left(m_c - rP_c\right) - \frac{1}{2}r\gamma\begin{pmatrix}1 & \theta & \pi(i\theta)\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta & \pi(i\theta)\end{pmatrix}^*, \tag{21}$$

measuring the mean-variance benefits of the risk-profile, the function

$$\epsilon_{i\theta} : \{a\}_{i\theta} \mapsto a(\bar{i}\theta) - a(\bar{i}\bar{\theta}) + a(i\bar{\theta}) - a(i\theta) \tag{22}$$

measuring the surplus that a $i\theta$-investor may be able to share on the OTC market, and the function

$$\begin{aligned} &\chi : (0,1) \times \mathbb{R} \to \mathbb{R} \\ &(\eta, \epsilon) \mapsto \frac{2(1-\eta)}{1 - 2\eta + \sqrt{(2\eta-1)^2 + 4\eta(1-\eta)e^{r\gamma\epsilon}}} - 1 \end{aligned} \tag{23}$$

mapping a bargaining power and a surplus to the utility change induced by a trade OTC.[27,28,29]

[27] I abuse notation and write μ both for a measure on the set of investors and for the distribution of the type of the investors under μ. μ defines the type distribution but not the other way around.

[28] I write $[x]^+ \overset{\Delta}{=} \max\{0, x\}$ for the positive part of a number.

[29] When all investors have the same bargaining powers, $\eta_\theta = 1/2$, the function χ significantly simplifies. Namely,

$$\chi\left(\frac{1}{2}, x\right) = \frac{1}{\sqrt{e^x}} - 1.$$

Proof. See Proof 21 in Appendix B. □

Equation (20) decomposes the expected utility of an agent into the sum of three terms. First, there is a flow of mean-variance benefits resulting from an investor's risk profile. Second, there are the shocks to the vector of exposures. Finally, there are the benefits resulting from trading on the OTC market. The benefits resulting from trading on the liquid market do not appear explicitly in Equation (20) but are contained in the flow "$\kappa\,(i\theta)$" of mean-variance benefits.

Intuitively, the quantity

$$\mathcal{S} \triangleq (\kappa(2\Theta) - \kappa(20)) + (\kappa(10) - \kappa(1\Theta)) \tag{24}$$

indicates whether transferring the illiquid asset from a 1Θ-investor to a 20-investor increases the overall flow of mean-variance benefits.[30] $\mathcal{S}$ should then also indicate whether a sale of the illiquid asset by a 1Θ-investor to a 20-investor is profitable in equilibrium or not. As stated in Proposition 10 below, this is indeed the case. To characterize an equilibrium, I must first characterize the distribution of types across the population.

3 Cross-Sectional Distribution of Types

The type of a given agent changes either because of a shock in her endowment correlations, or because she traded on the decentralized market. As described in Proposition 3, there are two mutually exclusive trade patterns on the decentralized market, depending on which agents have the higher valuation. Investors endogenously choose which trading pattern they follow but, for this section, I assume the following.

Assumption 6. Agents with the exposure type 2 buy the illiquid asset.

Recalling both the dynamics of the endowment correlations assumed in Equation 4 and the linear matching technology assumed in Equation 5, the type distribution μ must satisfy the stationary Kolmogorov Forward Equation

$$\begin{cases} 0 = \dot{\mu}(10) & = & 2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{12}\mu(1l) & +\lambda_{21}\mu(2l) \\ 0 = \dot{\mu}(1\Theta) & = & -2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{12}\mu(1h) & +\lambda_{21}\mu(2\Theta) \\ 0 = \dot{\mu}(2l) & = & -2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{21}\mu(2l) & +\lambda_{12}\mu(1l) \\ 0 = \dot{\mu}(2\Theta) & = & 2\Lambda\mu(1\Theta)\mu(2l) & -\lambda_{21}\mu(2\Theta) & +\lambda_{12}\mu(1\Theta) \end{cases} . \tag{25}$$

On the right hand side of each equation, the first term refers to trading, and the other ones to endowment shocks.[31] Also, μ being a density, it must satisfy both

$$\mu(10) + \mu(1\Theta) + \mu(20) + \mu(2\Theta) = 1 \tag{26}$$

[30] I write $\mathcal{S}$ like in risk *s*haring.

[31] The terms referring to trading only involves trades between agents with different endowment correlation. However, according to Proposition 3, agents with the same endowment correlations, but different holdings will also trade. However, as such agents will only swap their types, this has no impact on the distribution of types.

and

$$(\mu(1\Theta), \mu(10), \mu(2\Theta), \mu(20)) \in \mathbb{R}^4_{\geq 0}. \tag{27}$$

Finally, the OTC market has to clear, meaning that every unit of the illiquid asset d must be held by someone. This is expressed by imposing the condition

$$\Theta\left(\mu(1\Theta) + \mu(2\Theta)\right) = S_d. \tag{28}$$

As seen from (26), (27), and (28), the population can absorb at most Θ units of the asset d. This imposes the constraint

$$0 \leq \frac{S_d}{\Theta} \leq 1 \tag{29}$$

on the exogenous parameters of the model.

As shown in Duffie et al. (2005), the system defining the stationary distribution is well-behaved. I recall their result for convenience.

Proposition 7 (Duffie et al. (2005), Proposition 1)**.** *There exists a unique stationary type distribution that is reached from any initial distribution.*

Proof. For convenience, I partially recall the argument of Duffie et al. (2005) in Proof 22, in Appendix B. □

If Assumption 6 fails and 1-investors buy the illiquid asset, then all statements in this section 3 are still valid, up to a systematic swap of the indexes 1 and 2.[32]

There are thus only two possible stationary distribution. I denote the one arising under Assumption 6 by $\mu^{1h\to 2l}$ and the other one by $\mu^{2h\to 1l}$. In equilibrium, the trade surplus

$$\epsilon_{i\theta} \stackrel{(\Delta)}{=} a(i\bar{\theta}) - a(i\theta) + a(\bar{i}\theta) - a(\bar{i}\bar{\theta})$$

decides which type distribution obtains, in the sense that

$$\mu\left(a\right) = \mathbf{1}_{\{\epsilon_{1\Theta}(a)>0\}}\mu^{1\Theta\to 2l} + \mathbf{1}_{\{\epsilon_{2\Theta}(a)>0\}}\mu^{2\Theta\to 1l}. \tag{30}$$

The trade surpluses define the trading pattern on the OTC market via the Nash bargaining solution characterized in Proposition 3. In turn, the trading pattern defines the stationary distribution of types via the flow equations (25).

Finally, it may be useful to consider the behavior of the type distribution when the OTC market is relatively liquid. The exact asymptotic behavior depends on the relationship between the supply S_d of the illiquid asset and the proportion of investors

$$\mu_2 \stackrel{\Delta}{=} \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}}$$

[32] Strictly speaking, I should still consider the borderline case for which all investors have the same reservation value for the asset and are indifferent between buying and selling the illiquid asset. This case can arise but is non-generic in the exogenous parameters. Further, when the surplus to share on the OTC market is exactly zero, I should make additional assumptions regarding when a bilateral trade occurs. For these two reasons, I do not explicitly analyze this case. In proposition 10 I exactly characterize when this non-generic case arises.

having a high valuation for this illiquid asset.[33,34] For the remaining of the paper I assume

$$\mu_2 > \frac{S_d}{\Theta}. \tag{31}$$

Under this assumption, the "marginal" buyer of the asset d in a Walrasian setting would have a high valuation for the asset. By *marginal buyer* I mean the investor that would buy the additional units of the illiquid asset, should S_d be marginally increased.[35]

Proposition 8. *Under assumption* (31)*, the equilibrium density satisfies*

$$\begin{pmatrix} \mu(10) \\ \mu(1\Theta) \\ \mu(20) \\ \mu(2\Theta) \end{pmatrix} = \mu^W + \frac{1}{\Lambda}\delta_\mu \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + o\left(\frac{1}{\lambda}\right) \mathbf{1}_4, \tag{32}$$

with the limit value and sensitivities

$$\mu^W \triangleq \begin{pmatrix} 1 - \frac{S_d}{\Theta} \\ \frac{S_d}{\Theta} - \mu_2 \\ 0 \\ \mu_{2\bullet} \end{pmatrix}, \qquad \delta_\mu \triangleq \frac{\lambda_{12}}{2} \frac{\frac{S_d}{\Theta}}{\mu_2 - \frac{S_d}{\Theta}}$$

and with $\mathbf{1}_4 \in \mathbb{R}^4$ being the vector whose components are all equal to 1.

Proof. See Proof 25 in Appendix B. □

The asymptotic expressions above can be understood intuitively. The common absolute value of the four components of the first order correction reflects the functioning of the decentralized market. Namely, every time one potential buyer and one potential seller meet, a transaction occurs, and they become a satisfied holder of the asset d and one satisfied non-holder, respectively.

[33] Similarly, I define

$$\mu_1 \triangleq \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}}$$

as the proportion of investors with a low valuation for the illiquid asset.

[34] Recalling (4), the description of μ_2 as the proportion of 2-investors across the population intuitive. Whether it is correct or not, however, is far from trivial. Indeed, this statement identifies the proportion of time spent by a given agent in a given state and the proportion of agents across the population who currently are in that state. This requires the application of a certain Law of Large Numbers across the population. See Sun (2006) for a rigorous treatment of this issue.

[35] All the derivations could be done assuming the inequality opposite to 31. However, assumption 31 makes sure that the illiquidity discount is positive, meaning that the illiquidity of the OTC market decreases the value of the asset OTC. Thinking about bond markets and the well-documented positive liquidity spreads on bonds, this seems to be a desirable model feature. For this reason, presumably, assumption (31) and variants thereof recurrently appear in the literature. See, for instance, Condition 1 in Duffie et al. (2005), Equation (1) in Weill (2008), or Assumption 2 in Vayanos and Weill (2008).

4 Stationary Equilibrium

In this section I define and characterize an equilibrium of the model. In other words, I make the individual decisions consistent with the aggregate quantities in the economy.

For the centralized market, I use a classical Walrasian equilibrium concept. Namely, as seen in Proposition 4, the only aggregate quantity impacting the liquid holdings is the price P_c of the liquid asset. I thus impose the consistency between the individual and aggregate quantities by requiring the price P_c to be so that the centralized market clears.

Turning to the OTC market, the decisions to trade or not and, if so, at which price, are dictated by the parametrization

$$a = \{a(i\theta)\}_{i\theta}$$

of the value functions (see Proposition 3).

Now, on the one hand, the individual trading decisions on the OTC market yield a certain type distribution, characterized in Section 3. On the other hand, a also depends on the distribution of types across the population. This is clear both at the intuitive and at technical levels. Intuitively, because the utility of an investor searching for a counterparty on an OTC market should depend on the likelihood of finding such a counterparty. Technically, because a is a solution to the HJB equation (20), an equation in which the distribution μ appears.

I will thus impose the equilibrium condition that the type distribution assumed when writing the HJB equation (20) and the one generated by the solution to (20) are equal. I formalize this discussion as follows.

Definition 9. A *stationary equilibrium* of the model consists of a price P_c $(\in \mathbb{R})$, a collection of liquid holdings $\{\pi(i\theta)\}_{i\theta}$ $\left(\in \mathbb{R}^4\right)$ corresponding to each type, a distribution of types $\{\mu(i\theta)\}_{i\theta}$ $\left(\in \mathbb{R}^4\right)$, and the constants $\{a(i\theta)\}_{i\theta}$ $\left(\in \mathbb{R}^4\right)$ defining the value functions. The equilibrium quantities satisfy three conditions.

1. An investor of type $i\theta$ who takes the price P_c as given optimally invests the amount $\{\pi(i\theta)\}_{i\theta}$ in the liquid asset c.

2. The centralized market clears, meaning that

$$\mathrm{E}^{\mu(i\theta)}\left[\pi(i\theta)\right] = S_c.$$

3. The value functions and stationary type distribution are consistent. Namely, the vector

$$a = \{a(i\theta)\}_{i\theta}$$

solves the HJB equation (20) when the type distribution is $\mu\left(\epsilon_{1h}\left(a\right)\right)$.[36]

I now state the main result of this section.

[36] The distribution $\mu\left(\epsilon_{1h}\left(a\right)\right)$, a function of the trade surplus, is defined in (30).

Proposition 10. *There exists exactly one equilibrium of the model. In equilibrium, 2-agents have a high valuation of the illiquid asset d exactly when*

$$\det\left(\left(\begin{array}{c|c} e_d & e_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c|c} e_1 - e_2 & e_c \end{array}\right)\right) > 0, \tag{33}$$

with e_c, e_d, e_1, e_2 *being the exposures to the aggregate factors of the liquid asset, the illiquid asset, and the endowments, respectively.*

Further, the equilibrium price of the liquid asset is

$$P_c = \frac{m_c}{r} - \gamma\left(\Sigma_{cc}S_c + \Sigma_{\eta c} + \Sigma_{cd}S_d\right), \tag{34}$$

with

$$\Sigma_{\eta c} \triangleq \mu_1\Sigma_{1c} + \mu_2\Sigma_{2c}$$

being the average correlation between the endowments and the dividends of the liquid asset, and the equilibrium holdings of the four types are

$$\pi(i\theta) = S_c + \frac{1}{\Sigma_{cc}}\left(\left(\Sigma_{\eta c} - \Sigma_{ic}\right) + \Sigma_{cd}\left(S_d - \theta\right)\right), \tag{35}$$

for $i = \{1, 2\}$ *and* $\theta = \{0, \Theta\}$.

Proof. See Proof 26 in Appendix B. □

As Equation (34) shows, the liquidity Λ of the OTC market does not affect the price of the liquid asset. This result can be understood intuitively. Indeed, the search friction makes the asset allocation inefficient. Illiquidity thus increases the proportion of investors who short the liquid asset because they have not yet found a counter-party to buy their illiquid holdings. These investors reduces the aggregate demand for the liquid asset. At the same time, the search friction also increases the number of investors who buy the liquid asset while trying to increase their illiquid holdings. This second demand compensates the first one and, overall, the search friction does not affect the price of the liquid asset. It is true, however, that the two types of hedging demand exactly offset each other because there is no aggregate uncertainty. In Section 5, I consider aggregate demand shocks and, in this case, the frictions in the OTC market affect the price of the liquid asset.

Having a market friction that impacts individual policies but not prices is reminiscent of several references. Gârleanu and Pedersen (2004) documents a similar effect in a setting with adverse selection, and so does Gârleanu (2009) in a setting with search friction and a unique market. The conclusions of Rostek and Weretka (2011) are similar as well, but with illiquidity measured as a price impact.

There is also a short technical argument explaining why the equilibrium price of the liquid asset is independent of the illiquidity of the OTC market. Proposition 4 shows that the optimal holdings in the liquid asset are linear both in the covariance Σ_{ic} of the endowment with the payouts of c and in the illiquid holdings θ. Now, the cross-sectional average of the endowment correlations is independent of the illiquidity of the decentralized market. Indeed, the correlations define which holdings agents intend to hold, and this is independent of how much time it will actually take to obtain these holdings. Similarly, the cross-sectional average of the illiquid holdings is a matter a market-clearing, and not of illiquidity per se. Taking things together, as the optimal holdings in the illiquid asset are linear in the model parameters, and as the cross-sectional averages of these parameters are independent of the liquidity level, so is the aggregate demand, and so is the price of the liquid asset c.

The condition (33) characterizes which investors have the higher valuation for the illiquid asset and is rather intuitive. The first term of the product,

$$\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right),$$

measures how orthogonal the risk profiles of the liquid and illiquid assets are. Phrased differently, this first term measures how much risk sharing can only be achieved on the OTC market. The second term in the product,

$$\det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right),$$

again compares how orthogonal two vector of exposures are. The first vector, $e_1 - e_2$, is the risk-profile that 2-investors should buy to achieve an optimal risk-sharing. The second vector is again the risk profile of the liquid asset. As a result, this second term measures how much risk-sharing is left once agents have chosen their exposure on the liquid market.

Finally, the product of the two terms is positive exactly if the exposure that is specific to the OTC market and the risk-sharing that cannot be achieved on the liquid market "overlap". Figure 2 offers a visual interpretation of the condition (33).

This discussion shows that the buyers of the illiquid asset are necessarily those investors to whom the fundamentals of the illiquid asset offer high diversification benefits. In particular, even if illiquidity distorts the value functions and prices, it does not modify an agent's decision to hold an asset or not. Even in an illiquid market, the fundamentals of the asset guide this decision.

Finally, the condition (33) is equivalent to $\mathcal{S} > 0$, with $\mathcal{S}$ defined in (24).

In general, the equilibrium quantities for the decentralized market are cumbersome to deal with. Namely, I cannot characterize the parametrization a of the value functions in closed-form. Explicit expressions for a certain asymptotic case are available in Appendix A.

On the technical side, the existence and uniqueness result of Proposition 10 appears to be new. More specifically, in Duffie, Gârleanu, and Pedersen (2005), the equilibrium quantities are known in closed-form but only because agents are assumed to be risk neutral. This setting was then extended by Duffie, Gârleanu, and Pedersen (2007), Vayanos and Weill (2008), and others, to accommodate risk-averse (CARA) agents. In these cases, the authors showed how, asymptotically, the solutions to these models were formally equivalent to the ones encountered in settings with risk-neutral agents.

However, the asymptotic analyses involved either a vanishing risk-aversion, or a vanishing heterogeneity of the agents. My argument does not need these assumptions.[37]

Thanks to Proposition 10, I can characterize the equilibrium dispersion of holdings in the liquid asset.

Corollary 11. *The mean absolute deviation of the holdings in c,*

$$\mathrm{E}^{\mu(i\theta)}\left[|\pi(i\theta) - S_c|\right],$$

is increasing in the illiquidity $\frac{1}{\Lambda}$ of the OTC market.

The proof of Corollary 11 directly follows from (the proof) of Proposition 7 and from Proposition 10. Proposition 7 characterizes the equilibrium distribution of types and Proposition 10 characterizes the equilibrium holdings in c.

Corollary 11 is exactly in line with the findings of Oehmke and Zawadowski (2013) regarding CDS and bond markets. Indeed, Oehmke and Zawadowski (2013) find that the CDS market is used as a liquid alternative to an illiquid bond market, and that the dispersion of holdings in CDS contracts is increasing in the illiquidity of the bond market.

My model also has predictions regarding the trading volumes on the two markets. I define the trading volume as the number of shares of an asset that is traded per unit of time. On the illiquid market, trades occur at the rate

$$2\Lambda\mu(1\Theta)\mu(20)$$

when the inequality (33) holds, and each trade involves the exchange of Θ units of the illiquid asset. The trading volume on the illiquid market is thus

$$2\Lambda\mu(1\Theta)\mu(20)\Theta.$$

Investors trade infrequently even on the liquid market. This is a consequence of the CARA preferences and of the trade motives being driven by infrequent jumps and not by diffusions. Namely, there are six possible type changes and each of them occurs with a given intensity. These six type changes are

type change	rate	triggered by
$10 \to 20$	$\lambda_{12}\mu(10)$	correlation shock
$1\Theta \to 2\Theta$	$\lambda_{12}\mu(1\Theta)$	correlation shock
$1\Theta \to 10$	$2\Lambda\mu(1\Theta)\mu(20)$	OTC trade
$20 \to 2\Theta$	$2\Lambda\mu(1\Theta)\mu(20)$	OTC trade
$20 \to 10$	$\lambda_{21}\mu(20)$	correlation shock
$2\Theta \to 1\Theta$	$\lambda_{21}\mu(2\Theta)$	correlation shock

.

[37] Gârleanu (2009) sketches an existence argument for an alternative model of illiquid market. However, in his setting, prices are Walrasian and not bargained, which modifies the structure of the equilibrium.

The trading volume on the liquid market is then

$$\begin{aligned}
\text{Vol} =& \frac{1}{2}\left(\sum_{\text{type changes}} (\text{rate}) \times (\text{size of trade})\right) \\
=& \frac{1}{2}\left\{\begin{array}{rl}
\lambda_{12}\mu(10) & |\pi(10) - \pi(20)| \\
+\lambda_{12}\mu(1\Theta) & |\pi(1\Theta) - \pi(2\Theta)| \\
+2\Lambda\mu(1\Theta)\mu(20) & |\pi(1\Theta) - \pi(10)| \\
+2\Lambda\mu(1\Theta)\mu(20) & |\pi(20) - \pi(2\Theta)| \\
+\lambda_{21}\mu(20) & |\pi(20) - \pi(10)| \\
+\lambda_{21}\mu(2\Theta) & |\pi(2\Theta) - \pi(1\Theta)|
\end{array}\right\},
\end{aligned}$$

with the factor 1/2 recalling that each purchase must be matched by a sale.

Combining (the proofs of) Propositions 7 and Proposition 10 yields the following result.

Corollary 12. *The trading volumes on* both *markets are decreasing in the illiquidity level* $\xi = 1/\Lambda$.

Proof. See Proof 27 in Appendix B. □

Corollary 12 shows how the liquid and the illiquid assets are complements in terms of trading volumes, with the trading volumes on both markets increasing and decreasing together. Interestingly, this relationship between the trading volumes holds independently of whether the assets are complements or substitutes in terms of risk-exposure. This relationship between the trading volumes is driven by the use of the liquid asset as a hedging instrument, as detailed in Proposition 13.

Proposition 12 also deserves to be compared with a result in Longstaff (2009). Longstaff (2009) proposes a model in which, for a given period, only one of two assets can be traded. Longstaff (2009) then concludes that illiquidity increases the trading volume of the liquid asset. This is obviously in contradiction with my conclusion. The origin of this divergence is in the preferences of the agents.

In my model, investors with CARA preferences intend to keep their holdings fixed essentially all the time. A re-balancing is only triggered by a preference shock, or by a wish to adjust the liquid holdings as the result of a change in the illiquid ones. But then, making bilateral transactions more difficult makes the trade motives even less frequent, and reduces trading volumes.

Quite differently, in Longstaff (2009), investors have a constant relative risk aversion (CRRA) and would like to constantly re-balance their holdings in both assets. Now, if trading in one of them is impeded, this is compensated by trading the other one more intensively, which induces the trading volume increase.

I now provide a more detailed characterization of trading on the centralized market.

Proposition 13. *Opening the OTC market strictly increases the trading volume Vol on the liquid market.*

Furthermore, if the inequality

$$\det\left(\left(e_d \,\vdots\, e_c \right)\right) \cdot \det\left(\left(e_1 - e_2 \,\vdots\, e_c \right)\right) \left(e_c \cdot (e_1 - e_2)\right) \left(e_c \cdot e_d\right) > 0 \tag{36}$$

holds, the search frictions on the OTC market discontinuously increase the trading volume on the liquid market. In mathematical terms,

$$\lim_{\Lambda \to \infty} Vol(\Lambda) > Vol_W,$$

with Vol_W *being the trading volume in c if the asset d is traded on a competitive (Walrasian) market.*[38] *If* (36) *does not hold, the asymptotic trading volume with a vanishing friction and the Walrasian trading volume coincide.*

Proof. See Proof 28 in Appendix B. □

This last result shows how the search friction on the OTC market can generate some additional, or "excessive", trading on the centralized market. Intuitively, this arises because the investors use the liquid asset as an imperfect substitute for the illiquid one.

5 Aggregate Demand Shocks

The price impact of illiquidity can be driven both by the illiquidity level and by the illiquidity risk, that is by time variation in illiquidity.[39]

In my model, liquidity is understood as the time it takes to complete a transaction on the OTC market. Being more specific, in the steady state, an investor who is attempting to sell her illiquid holdings measures illiquidity as

$$\frac{1}{\Lambda} \frac{1}{\mathrm{P}\left[\text{sell the asset} \mid \text{contacted an investor}\right]},$$

meaning as the expected time until the sale is completed. The meeting intensity Λ represent the technology used by investors to contact each others and is unlikely to change in unpredictable ways over time. The probability of completing the trade, $\mathrm{P}\left[\text{sell} \mid \text{contacted}\right]$, however, is largely determined by the distribution of preferences across the population of investors and this distribution can reasonably be assumed to evolve stochastically, leading to illiquidity risk.

In this section, I introduce time variation in liquidity by assuming that the proportion of agents with a high valuation for the illiquid asset is driven both by aggregate and by idiosyncratic shocks. The aggregate shocks occur at the jump times of the Poisson process

$$(N_t^a)_{t \geq 0}$$

[38] I consider a Walrasian setting in which investors can trade d whenever they want, at no cost, and taking the price $P_{d,W}$ of the asset d as given, but I maintain the constraint that the holdings in d must belong to $\{0, \Theta\}$.

[39] The importance of illiquidity risk was emphasized by Pastor and Stambaugh (2003) and Acharya and Pedersen (2005), and further analyzed by Bongaerts et al. (2011), Junge and Trolle (2013), and Mancini et al. (2013).

whose intensity is λ_a. The proportion of 2-agents after such an aggregate shock is drawn from a distribution M_2 and the draws are independent across aggregate shocks.

I still assume *ex ante* that 2-agents have the high valuation and verify *ex post* this assumption. Furthermore, I assume that the support of the distribution after an aggregate shock satisfies

$$\operatorname{supp}(M_2) \subset \left(\frac{S_d}{\Theta}, 1\right].$$

This maintains a high valuation for marginal investors and any time or, equivalently, maintains a positive illiquidity discount.

In mathematical terms, these assumptions translate into the dynamics

$$\mathrm{d}\mu_2(t) = -\lambda_{21}\mu_2(t-) + \lambda_{12}\mu_1(t-) + (m_2 - \mu_2(t-))\ \mathrm{d}N_t^a,\ m_2 \sim M_2, \tag{37}$$

for the proportion $\mu_2(t)$ of 2-investors at time t. Note that, between two aggregate shocks, this proportion evolves deterministically. Using a formalism similar to the one in Duffie et al. (2005), I index the state of the system by the last aggregate shock and the time elapsed since this last shock.[40,41] For example, the proportion of 2-investors t units of time after it jumped to m_2 is

$$\mu_2(m_2, t) = e^{-(\lambda_{12}+\lambda_{21})t} m_2 + \left(1 - e^{-(\lambda_{12}+\lambda_{21})t}\right) \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}} \tag{38}$$

In particular, if no aggregate shock occurred since a long time, and independently of the last shock, the proportion of 2-investors converges toward the same level. I write

$$\mu_2(\infty) \triangleq \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}}$$

for this level.[42]

I must still specify the shocks at the individual level that will generate the aggregate dynamics (37). I do so by assuming, for a proportion of 2-investors jumping from $\mu_2(t-)$ to m_2, that each 2-investor has a probability

$$\delta(2; \mu_2(t-); m_2) \triangleq \max\left\{0; \frac{\mu_2(t-) - m_2}{\mu_2(t-)}\right\} \tag{39}$$

[40] In Duffie et al. (2005) the distribution after the shock is concentrated on one point, meaning that the state of the system can be indexed by the time since the last shock only.

[41] The last aggregate shock should be represented by the entire distribution

$$(\mu^a(1l), \mu^a(1h), \mu^a(2l), \mu^a(2h))$$

reached after the aggregate exposure shock occurred. In the asymptotic case I consider, however, only the proportion

$$\mu_2^a \overset{(\Delta)}{=} \mu^a(2l) + \mu^a(2h)$$

of high valuation investors matter. As a result, I abuse notations and index the current state of the economy by the last draw from m_a

[42] In the stationary setting of Section 4, I simply denoted the quantity $\mu_2(\infty)$ by μ_2. In this Section 5 I add "∞" as a time argument to avoid confusion.

of becoming of type 1 and that each 1-investor has a probability

$$\delta(1;\mu_2(t-);m_2) \overset{\Delta}{=} \max\left\{0;\frac{m_2-\mu_2(t-)}{1-\mu_2(t-)}\right\} \tag{40}$$

of becoming of type 2. Assuming that a suitable version of the strong law of large numbers (SLLN) holds cross-sectionally, these idiosyncratic shocks will be consistent with the aggregate dynamics (37).[43]

As far as the type distribution is concerned, the state of the economy can be described by the last aggregate valuation shock and the time elapsed since this shock. Further, the type distribution evolves continuously between two aggregate valuation shocks. I thus assume the same type of evolution for both the price of the liquid asset and the value functions.

Namely, I assume that the equilibrium price process of the liquid asset is a function

$$P_c\left(m_2,t\right) \tag{41}$$

of the last aggregate liquidity shock and of the time elapsed since this last shock. I also assume that this price is differentiable in the time. Under this assumption on the price process, the budget constraint of an investor becomes

$$\mathrm{d}w_t = rw_t\,\mathrm{d}t - c_t\,\mathrm{d}t + \,\mathrm{d}e_t + \theta_t\,\mathrm{d}D_{d,t} + \pi_t\left(\dot{P}_{c,t}\,\mathrm{d}t + \,\mathrm{d}D_{c,t} - rP_{c,t}\right) - P_{d,t}\,\mathrm{d}\theta_t, \tag{42}$$

with w being the wealth invested at the risk-free rate or in the liquid asset, c being the consumption, θ being the holdings in the illiquid asset, π being the holdings in the liquid asset, and

$$\dot{P}_{c,t} \overset{(\Delta)}{=} \frac{\mathrm{d}P_{c,t}}{\mathrm{d}t}$$

being the time derivative of the function in (41). I also denote by

$$V\left(w,i\theta;m_2,t\right) \overset{\Delta}{=} \mathrm{E}\left[\int_t^\infty e^{-\rho s}U(\hat{c}_s)\,\mathrm{d}s\middle|\,\mu_2(t)=\mu_2\left(m_2,t\right)\right]$$

the value function of a investor having a liquid wealth of w, being of type i and holding θ units of the illiquid asset when a proportion $\mu_2(t)$ of the investors have a high valuation for the illiquid asset. I assume that the value function is differentiable in the time since the last shock, which is consistent with the assumption on the price process P_c.

My analysis of the dynamic setting is similar to the static one and is based on dynamic programming. Namely, I first derive the HJB equations for the dynamic problem. Then, I assume that the value function satisfies

$$V\left(w,i\theta;h_a,t\right) = -\exp\left\{-r\gamma\left(w+a\left(i\theta;h_a,t\right)+\bar{a}\right)\right\}, \tag{43}$$

[43] This is seen by verifying that

$$\mu_2(t-)\left(1-\delta\left(2;\mu_2(t-),m_2\right)\right)+\mu_1(t-)\left(\delta\left(1;\mu_2(t-),m_2\right)\right)=m_2.$$

Indeed, the left-hand side being the proportion of agents with a high valuation given an aggregate shock h_a, the idiosyncratic shocks defined by (39) and (40), and a suitable SLLN, whereas the right-hand side is the proportion of agents with a high valuation that was assumed in the first place.

with the constant

$$\bar{a} \triangleq \frac{1}{r\gamma}\left(\frac{\rho}{r} - 1 + \log(r) + \gamma m_e\right).$$

This assumption is motivated by the static equilibrium analysis and justified *ex post*. Plugging the guess (43) into the HJB equation yields

$$\begin{aligned}
&ra\left(i\theta; m_2, t\right) \\
&\quad = \sup_{\tilde{\pi}} \dot{a}\left(i\theta; m_2, t\right) + \kappa\left(i\theta; m_2, t; \tilde{\pi}\right) \\
&\qquad + \lambda_{i\bar{i}} \frac{e^{-r\gamma(a(\bar{i}\theta; m_2, t) - a(i\theta; m_2, t))} - 1}{-r\gamma} \\
&\qquad + 2\Lambda\mu\left(\bar{i}\bar{\theta}; m_2, t\right)\left[\frac{e^{-r\gamma\left(a(i\theta; m_2, t) - P_d(m_2, t)\left(\bar{\theta} - \theta\right) - a(i\theta; m_2, t)\right)} - 1}{-r\gamma}\right]^+ \\
&\qquad + \lambda_a \operatorname{E}^{m(\tilde{m}_2)}\left[\begin{array}{l} \delta(i; m_2, t; \tilde{m}_2)\frac{e^{-r\gamma\left(a\left(\bar{i}\theta; \tilde{m}_2, 0\right) + \tilde{\pi}(P_c(\tilde{m}_2, 0) - P_c(m_2, t)) - a(i\theta; m_2, t)\right)} - 1}{-r\gamma} \\ + \left(1 - \delta(i; m_2, t; \tilde{m}_2)\right)\frac{e^{-r\gamma(a(i\theta; \tilde{m}_2, 0) + \tilde{\pi}(P_c(\tilde{m}_2, 0) - P_c(m_2, t)) - a(i\theta; m_2, t))} - 1}{-r\gamma} \end{array}\right]
\end{aligned}, \tag{44}$$

with

$$\begin{aligned}
\kappa\left(i\theta; m_2, t; \tilde{\pi}\right) \triangleq & \theta m_d + \tilde{\pi}\left(\dot{P}_c(m_2, t) + m_c - rP_{c,}(m_2, t)\right) \\
& - \frac{1}{2} r\gamma \begin{pmatrix} 1 & \theta & \tilde{\pi} \end{pmatrix} \Sigma_i \begin{pmatrix} 1 & \theta & \tilde{\pi} \end{pmatrix}^*
\end{aligned} \tag{45}$$

representing the flow of mean-variance benefits resulting from holdings a certain portfolio, conditionally on no illiquidity shock occurring.

The prices on the OTC market are still defined by the Nash bargaining solution. Adapting Proposition 4 from the static setting yields the price $P_d\left(m_2, t\right)$ bargained on the OTC market as the unique solution to the equation

$$\begin{aligned}
&\eta_0\left(1 - e^{-r\gamma(a(20; m_2, t) - (a(2\Theta; m_2, t) - P_d(m_2, t)))}\right) \\
&\quad = \eta_\Theta\left(1 - e^{-r\gamma(a(1\Theta; m_2, t) - (a(10; m_2, t) + P_d(m_2, t)))}\right).
\end{aligned} \tag{46}$$

The optimal policy and the resulting value function of the investors is characterized by the HJB equation (44). The impact of the illiquidity risk can be intuitively understood from this equation. The last line on the right-hand side represents the aggregate shocks. The random variable $\tilde{m}_2$ represent the proportion of agents with a high valuation after the shock, conditional on the occurrence of a liquidity shock.

In particular, this last line represents the utility shock expected by an investor when an aggregate shock occurs. The investor evaluate both the possibility that he may be directly affected by the shock, because her valuation may change, and the possibility of being affected by a change in the state of the economy. This change to the economy comes both from a price jump on the liquid market and from a change in the counterparties on the OTC market.

When an investor chooses her holdings, she will take into account the mean-variance properties of the liquid asset and the covariance of this asset with her endowment. With aggregate shocks, however, she will also consider how the liquid asset hedges her own preference shocks and the shocks to her trading opportunity on the OTC market. The new dimension of the individual portfolio problem is the channel by which the illiquidity of the search market spills over and affect prices on the liquid market.

The HJB equation (44) characterizes the value function, but this characterization involves both partial derivatives and integrals of the value function. The general treatment of such an equation seems challenging. Instead, I focus on a certain asymptotic case.

More specifically, I let the agents become nearly risk-neutral with respect to the jump risks. In mathematical terms, this is done by letting the risk aversion go to zero,

$$\gamma \to 0, \tag{47}$$

and by scaling up the diffusion coefficients,

$$\begin{array}{rcccl} a_i & = & a_i(\gamma) & = & \frac{a_{i0}}{\sqrt{\gamma}} \\ b_i & = & b_i(\gamma) & = & \frac{b_{i0}}{\sqrt{\gamma}} \end{array}, \tag{48}$$

for constant numbers $\{a_{i0}, b_{i0}\}_i$ and an index $i \in \{1, 2, c, d\}$. By doing so, the subjective quantity of risk

$$\gamma \Sigma_i(\gamma) = \gamma_0 \Sigma_{i0}\ ,\ i = 1, 2$$

contained in the endowments and payout remains constant, even when investors become risk-neutral with respect to the risks driven by Poisson processes. These Poisson processes drive the random matching on the OTC market and the preference shocks.

The same approach is used to obtain closed-form expressions in Biais (1993), Duffie et al. (2007), or Vayanos and Weill (2008). The procedure is particularly transparent in Gârleanu (2009). This approach is also related to Skiadas (2013) and Hugonnier, Pelgrin, and Saint-Amour (2013). In these models, there are several sources of risk and agents have a different level of risk-aversion for each risk. In my case, investors are risk-averse with respect to certain risks (diffusion risks) and risk-neutral with respect to other risks (jump risks).

Focusing on this asymptotic setting makes the analysis of aggregate demand shocks tractable.

Proposition 14. *There exists exactly one asymptotic equilibrium. In this equilibrium, the 2-investors buy the illiquid asset at all times if*

$$\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right) > 0. \tag{49}$$

Furthermore, the difference of valuations for the illiquid asset

$$(a(2\Theta; m_2, t) - a(20; m_2, t)) - (a(1\Theta; m_2, t) - a(10; m_2, t))$$

is increasing in the quantity in (49) *and decreasing in the contact rate* Λ.

Proof. See Proof 31 in the Appendix. □

The proof of Proposition 14 relies both on algebraic manipulations of the equilibrium equations and, in a second step, on an application of Blackwell's sufficient condition for a contraction.

Proposition 14 indicates that the qualitative behavior of the dynamic equilibrium is similar to the behavior in the static setting. Specifically, the inequality (49) is the same as the inequality (33) defining the trading pattern in the static setting. Furthermore, the bargained price

$$P_d(m_2,t) = v(2;m_2,t) - \eta_{2l}\left(v(2;m_2,t) - v(1;m_2,t)\right)$$

is the reservation value of a potential buyer subtracted by a share of the trade surplus. The share of the surplus is given by the bargaining power of the buyer. The trade surplus is, as stated in the last proposition, increasing in the risk-sharing made possible by the illiquid asset and decreasing in the contact rate on the OTC market. The contact rate reduces the trade surplus because it makes the search for a counter-party faster, improves the outside option of the investors, and reduces the benefits that one particular trade can bring. More generally, both Proposition 14 and its proof indicate that the intuition developed with the static model is robust to the introduction of aggregate liquidity shocks.

The aggregate demand shocks, however, creates new effects in the model. More specifically, in the dynamic setting, the illiquidity of the OTC market affect prices on the liquid market. This spillover effect and, more generally, the returns on the liquid market are the object of the next proposition.

Proposition 15. *I assume that the inequality* (49) *holds, meaning that 2-investors buy the illiquid asset. Then, equilibrium expected excess returns on the liquid asset are*

$$\begin{aligned}
&\frac{1}{\mathrm{d}t}\left(\frac{\mathrm{E}\left[P_c(m_2,t+\ \mathrm{d}t)\right|(m_2,t)]}{P_c(m_2,t)} - r\right) \\
&\quad = \frac{m_d}{r} + o\left(\gamma\right) \\
&\quad + r\gamma\left(\begin{array}{l}
\frac{1}{P_c(m_2,t)}\left(S_c\Sigma_{cc} + \lambda_a\,\mathrm{E}\left[\left(P_{c,0}-P_{c,t}\right)^2\Big|\,(m_2,t)\right]\right) \\
+\frac{1}{P_c(m_2,t)}\left(S_d\Sigma_{cd}+\Sigma_{\eta c}\right) \\
+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[\left(\frac{P_c(\tilde{m}_2,0)}{P_c(m_2,t)}-1\right)\left(W\left(\tilde{m}_2,0\right)-W\left(m_2,t\right)\right)\Big|\,(m_2,t)\right]
\end{array}\right),
\end{aligned} \tag{50}$$

with

$$W(m_2,t) \triangleq \mathrm{E}^{\mu(i\theta;m_2,t)}\left[a(i\theta;m_2,t)\right]$$

being the average certainty equivalent across the population of investors. If the illiquidity $1/\Lambda$ *is small enough, these expected returns are increasing in the illiquidity* $1/\Lambda$ *when*

$$e_c\cdot(e_1-e_2) > 0 \tag{51}$$

and decreasing otherwise.

Proof. See Proof 32 □

Proposition 15 offers a clean decomposition of the excess returns on the liquid asset into three different risk premia. The first two premia are classical. The first compensates investors for taking exposure to uncertain price movement and the dividend risk of the liquid asset. The second corrects the first by taking into account the diversification benefits against endowment risk and the dividend risk of the illiquid asset. The third premium is new and is driven by the illiquidity risk. It compensates investors for holding an asset that performs poorly exactly when trading on the OTC market becomes more difficult. To understand the underlying mechanism, let us first consider the average certainty equivalent $W(m_2, t)$. Intuitively, we can be use the average certainty equivalent $W(m_2, t)$ to measure the efficiency of the allocation on the OTC market. Indeed, whenever the illiquid asset is transferred from a low valuation agent to a high valuation agent, there is a net gain in utility across the population, and $W(m_2, t)$ precisely reflects this utility gain.[44] As a result, $W(m_2, t)$ is a measure of the efficiency of the OTC market or, equivalently, of the reallocation speed on the OTC market.

Whenever there is a negative aggregate shock, meaning that the proportion of high-valuation investors drops, the imbalance on the OTC market is reduced, the search friction becomes more acute, and the OTC market becomes slower when it comes to reallocating the illiquid asset. When the inequality (51) holds, the negative aggregate shock induces a drop in the price of the liquid asset. As this price drop occurs precisely when trading OTC becomes more difficult, it commands a positive risk premium. In addition, this risk-premium increases with the intensity of the search friction.

Interestingly, this illiquidity spillover effect increases in the *level* of illiquidity $1/\Lambda$ but is driven by illiquidity *risk*. This can be readily seen from (50), where the impact of illiquidity on the expected returns stems from the "covariance"

$$\mathrm{E}^{m(\tilde{m}_2)}\left[\left(\frac{P_c\left(\tilde{m}_2,0\right)}{P_c(m_2,t)}-1\right)\left(W\left(\tilde{m}_2,0\right)-W\left(m_2,t\right)\right)\Big|\,(m_2,t)\right]$$

between the returns of the liquid asset and the efficiency of the economy. Furthermore, this covariance is scaled by the risk-aversion γ. The role of illiquidity risk can also be directly seen by comparing Proposition 15 with the static equilibrium described in Proposition 10. Indeed, in the static version of the model, there is no illiquidity risk and the frictions of the OTC market have no impact on the price of the liquid asset.

Conceptually, the spillover effect of Proposition 15 is similar to results in Acharya and Pedersen (2005). Proposition 15, however, is based on an explicit modeling of illiquidity as a search friction and makes predictions regarding the sign of the illiquidity spillover effect. The model in Acharya and Pedersen (2005) relies on exogenous and stochastic transaction costs, and is thus more suited for illiquid but centralized markets. Furthermore, the price effect of illiquidity risk is driven by the exogenously specified covariance matrix of the transaction costs. Proposition 15 may thus be seen as a micro-foundation for the results in Acharya and Pedersen (2005). My results also show that,

[44] The proof of Proposition 15 contains a formal argument behind this statement.

unexpectedly, the measure of the illiquidity risk is determined by investors' certainty equivalent. It is interesting to compare Proposition 15 and the literature on long run risks, pioneered by Bansal and Yaron (2004). In this literature, the assumption of recursive (non time separable) preferences implies that the risk premia are determined by investors' certainty equivalent. This channel implies that, in stark contrast to the case of time separable preferences, long run risk is priced in today's returns. In my model, Proposition 15 shows that, in illiquid markets, long run risk is priced despite the fact that agents have standard, time separable preferences. This interaction between long run risk and illiquidity is an interesting and important topic for future research.

Proposition 15 can also be used to evaluate and improve empirical analysis of illiquidity. Indeed, following the example of Longstaff et al. (2005), a number of authors willing to measure the illiquidity component of bond yields have considered the so-called CDS basis, defined as the spread between bond excess returns and CDS premia. The rationale for this procedure is the relatively high liquidity of CDS markets when compared to bond markets. In particular, CDS spreads should be a clean measure of credit risk.[45] As Proposition 15 indicates, however, even the returns on a perfectly liquid market may be affected by the illiquidity of a related market.

Finally, Propostion 15 is consistent with empirical findings regarding illiquidity spillover. For example, Tang and Yan (2006) and Lesplingart, Majois, and Petitjean (2012) document how the illiquidity of the bond market increases yields on CDS contracts, which is exactly in line with Proposition 15. Das and Hanouna (2009) documents a similar effect between stock and CDS markets.

6 Opening the Liquid Market

In this section, I consider the effect of the liquid market on the functioning of the illiquid one. For tractability reasons, I again focus on the stationary setting of Section 4.

As Proposition 10 and Proposition 14 show, the trading pattern on the OTC market is determined by the quantity

$$\det\left(\left(e_d \;\vdots\; e_c \right)\right) \cdot \det\left(\left(e_1 - e_2 \;\vdots\; e_c \right)\right).$$

This quantity measures how much risk-sharing can be achieved on the OTC market only and, as a result, is closely linked to the competitive price of the asset traded OTC. At the same time, the Nash bargaining solution assumed for the OTC market also makes this quantity a measure of the illiquidity discount.[46] In particular, if the liquid asset mitigates the search friction and decreases the illiquidity discount on the asset traded OTC, the liquid asset will necessarily also decrease the competitive price of the asset. This *mitigation* effect occurs because the liquid asset offers an attractive alternative to the asset traded OTC. However, this alternative market also diverts some

[45]Illiquidity is also priced on CDS market, as shown by Bongaerts et al. (2011) and Junge and Trolle (2013). It is, however, true that CDS markets are typically significantly more liquid then their underlying bond markets.

[46]See Appendix A for explicit derivations supporting these claims.

of the fundamental demand for the asset traded OTC, leading to a *capture effect*. The mitigation effect tends to increase prices on the OTC market whereas the capture effect tends to decrease them. In this section I evaluate which of these effects dominates. I do so by comparing the economy with and without the liquid market.

In the real world, the decision to create a market for a new security is always endogenous and is determined by the financial intermediaries' estimates of the investors' trading needs. These intermediaries can be dealers, who will make the market in the new security, or the exchanges on which the security will be traded.[47] The revenues of these intermediaries are driven by trading volumes, and so is financial innovation. As a result, I assume that the liquid asset is designed to maximize volumes.[48]

I derive the trading volumes for the proof of Proposition 12 (see Appendix B). Shares of the liquid asset are exchanged at the rate

$$V = \frac{1}{\Sigma_{cc}} \left(|\Sigma_{cd}| \, \Theta 2 \Lambda \mu(1\Theta)\mu(20) + |\Sigma_{1c} - \Sigma_{2c}| \, \frac{\lambda_{12}\lambda_{21}}{\lambda_{12} + \lambda_{21}} \right) \tag{52}$$

when the 2-investors buy the illiquid asset.[49] Recalling the definition of the covariation matrices in (14) and defining the constants

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \triangleq \Theta 2 \Lambda \mu(1\Theta)\mu(20) \begin{pmatrix} a_d \\ b_d \end{pmatrix}$$
$$\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \triangleq \frac{\lambda_{12}\lambda_{21}}{\lambda_{12} + \lambda_{21}} \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix},$$

I rewrite Equation (52) as

$$\Sigma_{cc} V = |w_1 a_c + w_2 b_c| + |w_3 a_c + w_4 b_c| \, .$$

[47]Regarding a description of financial innovation being driven by intermediaries rather than by end-users, one may refer to Das et al. (2013):

> CDS introduction is initiated by dealer banks depending on factors such as size of outstanding debt of an issuer, underlying credit risk of the issuer, and demand for credit protection. [...] Introduction of an equity option is decided by the corresponding options exchange depending upon factors such as trading volume, market capitalization and turnover of the underlying stock.

Duffie and Jackson (1989) proposes a model of financial innovations by intermediaries who maximizes trading volumes. Rahi and Zigrand (2009) and Rahi and Zigrand (2010), for instance, propose alternative theoretical models of financial innovation driven by intermediaries.

[48]I do not model the intermediary explicitly. However, an intermediary who earns a constant bid-ask spread on transactions will attempt to maximize trading volumes. And the trading volume with a constant bid-ask spread converges toward the volume without transaction costs when the bid-ask spread decreases. See Praz (2013) for a treatment of transaction costs in a setting similar to the one of this paper.

[49]The expression (52) also describes the asymptotic trading volume in a setting with aggregate demand shocks when the risk-aversion γ goes to zero. See the characterization of the asymptotic optimal liquid holdings in the proof of Proposition 14.

When 1-investors buy the illiquid asset, the only change is that the weights w_1 and w_2 must be rescaled.[50]

It is important to understand which model parameters influence the equilibrium trading volume. First, the trading volume V is independent of the expected payout of the liquid asset, as can be seen in Equation (52). This expected payout must thus be set exogenously. Second, the number of shares exchanged can be made arbitrarily large by scaling down the risk exposures e_c of the liquid asset.[51] Without loss of generality, I normalize the overall exposure

$$\Sigma_{cc} \quad \left(= \|e_c\|_2^2 = a_c^2 + b_c^2\right)$$

to 1.

Summing up, I choose the liquid asset that maximizes the trading volume, meaning that I choose the risk-profile e_c of the liquid asset to be a point of maximum in the optimization

$$\max_{(a_c,b_c)} \{|w_1 a_c + w_2 b_c| + |w_3 a_c + w_4 b_c|\} \tag{53}$$

under the conditions

$$\begin{aligned} \|(a_c, b_c)\|_2 &= 1, \\ \det\left(\begin{pmatrix} a_d & a_c \\ b_d & b_c \end{pmatrix}\right) \cdot \det\left(\begin{pmatrix} a_1 - a_2 & \alpha_c \\ b_1 - b_2 & \beta_c \end{pmatrix}\right) &> 0. \end{aligned} \tag{54}$$

Three features of the maximization (53) should be emphasized. First, the constraint (54) ensures the consistency of the beliefs regarding the trading pattern on the OTC market. Specifically, as the objective function (53) assumes that 2-investors buy the asset traded OTC, the inequality (54) ensures that this assumption is justified *ex post*.

Second, the inequality (54) is strict. The borderline with equality corresponds to the case in which all investors have the same reservation value for the illiquid asset. In this case, the benefits resulting from any trade on the OTC market are zero, investors are indifferent on the OTC market, and I would need more assumptions to define the type dynamics and the trading volumes.[52]

Third, both the objective function and the domain of optimization in (53) are symmetric around the origin. As a result, whenever a point x one the unit circle is a point of maximum, so is its opposite $-x$.

I can characterize the solution to the optimal asset design problem (53).

[50] The rescaling factor is

$$\frac{\mu(2\Theta)\mu(10)}{\mu(1\Theta)\mu(20)}$$

and is the ratio of the type flows generated by trading when 1-investors buy the illiquid asset and when 2-investors do.

[51] Dividing e_c by $K > 0$ multiplies the trading Volume by K.

[52] The additional assumptions would require to randomize the decision to trade.

Proposition 16. *There exists a solution to the volume maximization* (53) *exactly when*

$$\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}\right) \times \left(\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}\right) < 0. \tag{55}$$

In this case, the optimal liquid asset is

$$\begin{pmatrix} a_c \\ a_d \end{pmatrix} = \pm \frac{1}{\nu} \begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}, \tag{56}$$

with the constant

$$\nu \triangleq \sqrt{(w_1 - w_3)^2 + (w_2 - w_4)^2}$$

ensuring the normalization $\Sigma_{cc} = 1$.

Proof. See Proof 29 in the Appendix. □

Proposition 16 implies that the optimal liquid asset is the weighted average of two risk profiles. The first is the profile of the illiquid asset and the second is optimal in terms of risk-sharing. This already indicates how the new liquid asset balances the attempt to capture some of the trading activity that takes place OTC and the alternative aim of being valuable to as many investors as possible. Furthermore, the weight on the profile of the illiquid asset is monotone increasing in the contact rate on the OTC market because, with a higher contact rate, there is more volume to capture. Perhaps paradoxically, this means that the search friction is easier to mitigate when it is smaller in the first place.

Importantly, the optimal security design defined by the maximization (56) does not necessarily have a solution. In particular, if inequality (55) does not hold and the liquid asset has the risk-profile in Equation (56), then 1-investors have the higher valuation for the illiquid asset. Conversely, if the trading volumes had been optimized under the assumption that 1-investor buy the illiquid asset, the resulting liquid asset would actually induced the 2-investors to buy the illiquid asset. As a result, the only way of, possibly, obtaining an optimum would be to impose the behavior of the investors on the OTC market when investors are indifferent.

Finally if the trading pattern on the OTC market is the same before and after the opening of the liquid market, the type flows across the population will not change when the liquid asset is introduced. As a result, the trading volumes will be constant at any time. However, if the opening of the liquid market inverts the trading pattern on the OTC market, the trading volume (52) only represents the asymptotic trading volume in the steady state.[53]

Intuitively, choosing a liquid asset that is very similar to the illiquid asset has two consequences. On the one hand, the liquid asset mitigates the search frictions and reduces the illiquidity discount on the OTC market. This pushes the price on the OTC

[53]See Theorem 5 (and its proof) in Duffie et al. (2005) for a similar issue. Namely, Duffie et al. (2005) show how, for a sufficiently patient intermediary, an optimal policy chosen at time zero and fixed afterwards is approximately the policy that maximizes revenues in the steady state.

market up. On the other hand, if the liquid asset is very similar to the illiquid asset, the illiquid asset has little value left as a risk-sharing instrument. This second effect pushes the price of the illiquid asset down.

Figure 4 illustrates how each of these effects can dominate. When the search friction is severe, on the right part of the plot, the mitigation of the illiquidity discount dominates the drop in the fundamental value and the price on the OTC market increases when the liquid asset starts trading. Quite differently, when the search frictions are modest, on the left part of the plot, the illiquidity discount is small and diversion of trading volume towards the new market dominates the benefits of the new hedging opportunities. To complete this section, I note that, for potential applications to bond market, it may be more natural to consider the yield on the illiquid asset. This is done in the second panel of Figure 4.

7 Conclusion

I study a general equilibrium model in which agents can trade both on an illiquid OTC market and on a liquid, centralized market. Search frictions on the OTC market increase the trading volume and open-interest on the liquid market. Furthermore, the endogenous interactions of the search frictions with the aggregate demand shocks generate a time-varying efficiency of the asset allocation on the OTC market. This liquidity risk is priced and affects the risk premium on the liquid asset. These results are consistent with a number of empirical studies.[54]

Motivated by several real-world examples in which centralized markets were created as an alternative to preexisting OTC markets, I introduce endogenous financial innovation into the model. I assume that intermediaries design the cash flows of the liquid asset that maximize the equilibrium trading volume. Then, I compare the prices on the OTC market in an economy with and without a liquid asset. I show that the risk profile of the optimal liquid asset is a weighted average of the illiquid asset's profile and of the risk profile that would lead to an efficient risk-sharing. The weight on the profile of the illiquid asset is shown to be monotone increasing in the contact rate on the OTC market because, with a more active OTC market, there is more trading volume to capture.

I show that the liquid market has two effects on the illiquid OTC market. On the one hand, it mitigates the search frictions, reduces the price discount on the illiquid asset, and increases the prices bargained on the OTC market. On the other hand, the liquid asset captures some of the illiquid asset's value as a risk-sharing instrument. I show how each of these effects can dominate, and link this equilibrium behavior with the empirical literature studying how the onset of CDS trading affects bond yields.

I believe that understanding the role of liquidity in portfolio selection and its general equilibrium feedback effects is both important and timely. Several regulatory reforms such as the Dodd-Frank Act in the US and the MiFID II proposal in the European Union propose to significantly revise the functioning of modern markets and, in particular, to

[54]Regarding the interactions between CDS and bond markets see, for example, Oehmke and Zawadowski (2013), Tang and Yan (2006), Lesplingart et al. (2012), or Das and Hanouna (2009).

move some of the OTC trading to centralized exchanges. The only way to evaluate the consequences of these reforms is to develop a general equilibrium model that accounts for the trading frictions on OTC markets and their cross-market externalities.

My model could be enriched in several directions. For example, throughout this paper, I assumed a dichotomy between an illiquid OTC market and a perfectly liquid market. In many real-world examples, however, the alternative to a costly search process will be to trade on another market immediately, but at a cost. In Praz (2013) I introduce this additional liquidity friction, and consider a general equilibrium model in which investors balance transaction costs and execution uncertainty when they select their portfolio holdings. Financial intermediaries rationally anticipate this behavior and optimally choose the bid-ask spreads on the exchange, the level of liquidity provision on the OTC market, and the form of financial innovation. Finally, introducing asymmetric information, either in terms of common value uncertainty, as in Duffie, Malamud, and Manso (2009), or in terms of private liquidity needs would also significantly enrich the structure of the model. A model of OTC market with asymmetric information would shed light on the current regulatory debates aiming at increasing the transparency of OTC markets. We take some first steps in this direction in Cujean and Praz (2013).

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Appendices

A Asymptotic Behavior of the Equilibrium

I cannot solve for the unique stationary equilibrium of the model in closed form. The technical difficulty preventing it are the exponential terms related to the jump risks in the HJB equation (20). Closed-form expressions can, however, be obtained in the asymptotic case analyzed in Section 5.[55]

I will also assume a relatively liquid OTC market. I would like to justify this assumption. An investor will only bother to enter an illiquid market if she expects to amortize the costly process of building up, and liquidating, a position over a reasonably long holding period. This intuition is formalized in Vayanos and Wang (2007) within a search model of asset pricing, and goes back to Amihud and Mendelson (1986) for a setting with exogenous transaction costs.

This suggests that I may assume the illiquidity level $\xi = 1/\Lambda$ to be small relatively to $1/\lambda_{12}$ and $1/\lambda_{21}$, which are the average times (continuously) spent with a high or a low valuation.

Proposition 17. *I assume both*

$$\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right) > 0, \tag{57}$$

which is the condition (33) *in Proposition 10, and*

$$\mu_2 > \frac{S_d}{\Theta},$$

which is condition (31) *in Section 7. Then, the price* P_d *bargained on the OTC market satisfies*

$$\begin{aligned} P_d =& P_{d,W} - \left\{\frac{1}{\Lambda}\right\}\left\{\frac{\eta_0\,(r + 2\delta_\mu) + \lambda_{21}}{2\eta_\Theta\left(\mu_2 - \frac{S_d}{\Theta}\right)}\right\}\left\{\gamma\frac{\det\left(\left(\begin{array}{c:c} e_d & e_c \end{array}\right)\right)\cdot\det\left(\left(\begin{array}{c:c} e_1 - e_2 & e_c \end{array}\right)\right)}{\Sigma_{cc}}\right\} \\ &+ o\left(\frac{1}{\Lambda}\right) + \mathcal{O}(\gamma), \end{aligned} \tag{58}$$

with the Walrasian price

$$P_{d,W} = \frac{\kappa\,(2\Theta) - \kappa\,(20)}{\Theta r}.$$

Also, the sensitivity

$$\delta_\mu \overset{(\Delta)}{=} \lim_{\frac{1}{\Lambda}\to 0} \frac{\partial\,(\mu(1h))}{\partial\left(\frac{1}{\Lambda}\right)}$$

[55] The asymptotic analysis in Section 5 amounts to letting the investors become nearly risk-neutral with respect to the jump risks but maintain their risk-aversion toward the diffusion risks. The exact definition of the asymptotic case is in Equation (47) and (48).

of the type distribution to the search friction was defined in Proposition 8.[56]

Proof 18 (Proof of Proposition 17)**.** Under the assumption (57) of the statement, and recalling Proposition 4, the only profitable type of trade on the OTC market is a sale by a 1Θ-investor to a 20-investor. As a result, the HJB equations (20) become

$$\begin{cases} ra(10) &= \kappa(10) & +\lambda_{12}\left(\frac{e^{-r\gamma(a(20)-a(10))}-1}{-r\gamma}\right) & \\ ra(1\Theta) &= \kappa(1\Theta) & +\lambda_{12}\left(\frac{e^{-r\gamma(a(2\Theta)-a(1\Theta))}-1}{-r\gamma}\right) & +2\Lambda\mu(20)\frac{\chi(\eta_\Theta,\epsilon_{1\Theta}(a))}{-r\gamma} \\ ra(20) &= \kappa(20) & +\lambda_{21}\left(\frac{e^{-r\gamma(a(10)-a(20))}-1}{-r\gamma}\right) & +2\Lambda\mu(1\Theta)\frac{\chi(\eta_0,\epsilon_{20}(a))}{-r\gamma} \\ ra(2\Theta) &= \kappa(2\Theta) & +\lambda_{21}\left(\frac{e^{-r\gamma(a(1\Theta)-a(2\Theta))}-1}{-r\gamma}\right) & \end{cases} \tag{59}$$

As

$$\chi\left(\eta_0, x\right) = -r\gamma\eta_0 x + o(\gamma),$$

this last system of equations becomes

$$\begin{cases} ra(10) &= \kappa(10) & +\lambda_{12}\left(a(20)-a(10)\right) & +\mathcal{O}(\gamma) \\ ra(1\Theta) &= \kappa(1\Theta) & +\lambda_{12}\left(a(2\Theta)-a(1\Theta)\right) & +2\Lambda\mu(20)\eta_\Theta\epsilon_{1\Theta}\left(a\right)+\mathcal{O}(\gamma) \\ ra(20) &= \kappa(20) & +\lambda_{21}\left(a(10)-a(20)\right) & +2\Lambda\mu(1\Theta)\eta_0\epsilon_{20}\left(a\right)+\mathcal{O}(\gamma) \\ ra(2\Theta) &= \kappa(2\Theta) & +\lambda_{21}\left(a(1\Theta)-a(2\Theta)\right) & +\mathcal{O}(\gamma) \end{cases} \tag{60}$$

in the asymptotic case described by Equations (47) and (48). In this same asymptotic case, Equation (17) significantly simplifies as well and the bargained price P_d is

$$\begin{aligned} P_d &= \eta_\Theta\left(a(2\Theta)-a(20)\right)+\eta_0\left(a(1\Theta)-a(10)\right)+o(\gamma) \\ &= \left(a(2\Theta)-a(20)\right)-\eta_0\begin{pmatrix} \left(a(2\Theta)-a(20)\right) \\ -\left(a(1\Theta)-a(10)\right) \end{pmatrix}+\mathcal{O}(\gamma). \end{aligned} \tag{61}$$

Using the expressions in (60) to reformulate (61) yields

$$P_d = \frac{1}{r}\left(\kappa(2\Theta)-\kappa(20)\right)-\frac{\lambda_{21}+2\Lambda\mu(1\Theta)\eta_0+\eta_0 r}{r+2\Lambda\left(\eta_\Theta\mu(20)+\eta_0\mu(1\Theta)\right)}+\mathcal{O}(\gamma). \tag{62}$$

Finally, combining the asymptotic behavior of the type distribution for a large Λ, as described in Proposition 8, with the last expression (62) yields Equation 58 in the statement. □

The terms defining the illiquidity discount in (58) are rather intuitive. The fist term in curly brackets refers to the severity of the search friction. The second term in curly brackets refers to both the respective bargaining power of the agents bargaining and to the time it would take them to find another counter-party, should the negotiation collapse. The third term in curly brackets measures the risk sharing benefits that the investors are bargaining on.

[56]Recall that these results only hold under the assumption 31 regarding the marginal buyer.

B Proofs

B.1 Proofs for Section 2

Proof 19 (Proof of Proposition 3)**.** Let a be an agent with type $i_a\theta$ and wealth w_a. She met another agent b. Clearly, no trade will be possible unless the holdings of b are $\bar{\theta}$. I denote the two other characteristics of b by i_b and w_b.

There is a surplus for a and b to share if

$$\emptyset \neq \left\{ \tilde{P} : \begin{array}{rl} V\left(w_a - \left(\bar{\theta}-\theta\right)\tilde{P}, i_a\bar{\theta}\right) \geq & V\left(w_a, i_a\theta\right) \\ V\left(w_b - \left(\theta-\bar{\theta}\right)\tilde{P}, i_b\theta\right) \geq & V\left(w_b, i_b\bar{\theta}\right) \end{array} \right\}.$$

Under Assumption (2), this is equivalent to

$$\emptyset \neq \mathcal{P} \triangleq \left\{ \tilde{P} : \quad a(i_a\bar{\theta}) - a(i_a\theta) \geq \tilde{P}\left(\bar{\theta}-\theta\right) \geq a(i_b\bar{\theta}) - a(i_b\theta) \right\},$$

or to

$$a(i\bar{\theta}) - a(i\theta) + a(j\theta) - a(j\bar{\theta}) \geq 0.$$

This proves the first two statements.

Now, if there actually is a surplus to share, the outcome of the bargaining is given by the Nash bargaining solution. Namely, a and b trade the asset at the price P_d so that

$$P_d = \arg\max_{\tilde{P}\in\mathcal{P}} \left(V(w_a - \tilde{P}(\bar{\theta}-\theta), i_a\bar{\theta}) - V(w_a, i_a\theta) \right)^{\eta_\theta} \cdot \\ \cdot \left(V(w_b - \tilde{P}(\theta-\bar{\theta}), i_b\theta) - V(w_b, i_b\bar{\theta}) \right)^{1-\eta_\theta}.$$

Unless $\mathcal{P}$ is reduced to a single point, in which case the solution of the optimization is trivial, the first order condition characterize the point of maximum P_d as the solution to

$$\begin{aligned} &\eta_\theta \frac{\partial_w \left(V(w_a - P_d(\bar{\theta}-\theta), i_a\bar{\theta}) \right)}{V(w_a + P_d(\bar{\theta}-\theta), i_a, \bar{\theta}) - V(w_a, i_a\theta)} \\ = (1-\eta_\theta) &\frac{\partial_w \left(V(w_b - P_d(\theta-\bar{\theta}), i_b\theta) \right)}{V(w_b - P_d(\theta-\bar{\theta}), i_b, \theta) - V(w_b, i_b\bar{\theta})} \end{aligned} \tag{63}$$

which, with Assumption 2, becomes (17). □

Proof 20 (Proof of Proposition 4)**.** The first order necessary condition for the maximization over the consumption rate is

$$\gamma e^{-\gamma c} - \frac{\partial V}{\partial w}(w, i\theta) = 0.$$

Recalling the Assumption 2, and solving for c yields a unique candidate $c(i\theta)$ which, by concavity of the objective function, is a point of maximum.

A similar argument yields the optimal liquid holdings $\pi(i\theta)$. □

Proof 21 (Proof of Proposition 5)**.** Starting from the HJB equation (13), picking a type $i\theta$, using Proposition (3) to transform the expected value into a deterministic quantity, Proposition 4 to express the optimal consumption, Proposition 3 to express the bagained price P_d, and normalizing by $r\gamma V(w,i,\theta)$, I obtain

$$
\begin{aligned}
0 =& r - \rho - r\log(r) + r^2\gamma\bar{a} - r\gamma m_e \\
&+ \frac{\alpha}{r\gamma}\left(\frac{\alpha}{\gamma} - r\right) w \\
&+ ra(i\theta) - \kappa(i\theta) \\
&+ \lambda_{i\bar{i}} \frac{\left(e^{-r\gamma(a(\bar{i},\theta)-a(i,\theta))} - 1\right)}{-r\gamma} \\
&+ 2\Lambda\mu\left(\bar{i}\bar{\theta}\right)\left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}(a)\right)}{-r\gamma}\right]^+ .
\end{aligned}
\tag{64}
$$

Now, I can choose the constant

$$
\bar{a} = \frac{1}{r\gamma}\left(-1 + \frac{\rho}{r} + \log(r) + \gamma m_e\right),
$$

which sets the first line of the right hand side to zero. Also, as the equation above must hold for any value of the liquid holdings w,

$$
\alpha\left(r - \frac{\alpha}{\gamma}\right) = 0.
$$

Under Assumption 2, this requires $\alpha = r\gamma$ and the second line of the right hand side equals zero. Taking these two observations into account yields the system (20).

It remains to show that this equation admits exactly one solution. I split my argument into four steps.

Step 1 I first rearrange the four equations described in (20) into two. Namely, defining the variables

$$
\Delta_\Theta \triangleq a(1\Theta) - a(2\Theta) \tag{65}
$$

and

$$
\Delta_0 \triangleq a(20) - a(10), \tag{66}
$$

and taking the corresponding differences in the HJB equations (20) ensures that

$$
\begin{aligned}
0 =& r\Delta_0 - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma\Delta_0} - 1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma\Delta_0} - 1}{-r\gamma} \\
& - 2\Lambda\left(\mu(1\Theta)\left[\frac{\chi\left(\eta_0, -\Delta_0 - \Delta_\Theta\right)}{-r\gamma}\right]^+ - \mu(2\Theta)\left[\frac{\chi\left(\eta_0, \Delta_0 + \Delta_\Theta\right)}{-r\gamma}\right]^+\right) \\
\triangleq& F_0\left(\Delta_0, \Delta_\Theta\right)
\end{aligned}
\tag{67}
$$

and

$$
\begin{aligned}
0 =& r\Delta_\Theta - \kappa(1\Theta) + \kappa(2\Theta) - \lambda_{12}\frac{e^{r\gamma\Delta_\Theta}-1}{-r\gamma} + \lambda_{21}\frac{e^{-r\gamma\Delta_\Theta}-1}{-r\gamma} \\
&- 2\Lambda\left(\mu(20)\left[\frac{\chi\left(\eta_\Theta, -\Delta_0-\Delta_\Theta\right)}{-r\gamma}\right]^+ - \mu(10)\left[\frac{\chi\left(\eta_\Theta, \Delta_0+\Delta_\Theta\right)}{-r\gamma}\right]^+\right) \\
\triangleq& F_\Theta\left(\Delta_0, \Delta_\Theta\right).
\end{aligned}
\tag{68}
$$

Inspection ensures that, for any Δ_Θ, the function $F_0\left(\cdot, \Delta_\Theta\right)$ is strictly increasing with a range equal to the entire real line. It also ensures that, for any given Δ_0, the function $F_0\left(\Delta_0, \cdot\right)$ is strictly increasing with the bounded range

$$
r\Delta_0 - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma\Delta_0}-1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma\Delta_0}-1}{-r\gamma} + \frac{2\Lambda}{r\gamma}\left[-\mu(1\Theta), \mu(2\Theta)\right]. \tag{69}
$$

Similar properties hold for F_Θ.

Step 2 Given these properties of F_0 and F_Θ, I can define the functions

$$
\Phi_0, \Phi_\Theta : \mathbb{R} \to \mathbb{R}
$$

by requiring that, for any $x \in \mathbb{R}$,

$$
0 = F_0\left(\Phi_0(x), x\right) = F_\Theta\left(x, \Phi_\Theta(x)\right). \tag{70}
$$

The monotonicity properties also ensure that both Φ_0 and Φ_Θ are decreasing. I show two more properties of these functions.

First, these functions decrease relatively slowly. Namely, for any choice of $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$, it follows from (70) that

$$
F_0\left(\Phi_0(x) - y, x + y\right) < F_0\left(\Phi_0(x), x\right) = 0 = F_0\left(\Phi_0(x+y), x+y\right),
$$

meaning that

$$
y + \Phi_0(x+y) - \Phi_0(x) > 0.
$$

As a result, the function

$$
x \mapsto x + \Phi_0(x) \tag{71}
$$

is increasing and, by a similar argument, so is

$$
x \mapsto x + \Phi_\Theta(x).
$$

Second, their range is compact. Let me first consider Φ_Θ. Recalling the bounded range described by (69), I may write, for any pair $\left(\Delta_0, \Delta_\Theta\right) \in \mathbb{R}^2$,

$$
F_0^L\left(\Delta_0\right) \leq F_0\left(\Delta_0, \Delta_\Theta\right) \leq F_0^U\left(\Delta_0\right),
$$

where I defined

$$F_0^L(x) \triangleq rx - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma x}-1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma x}-1}{-r\gamma x} - \frac{2\Lambda}{r\gamma}\mu(1\Theta)$$

and

$$F_0^U(x) \triangleq rx - \kappa(20) + \kappa(10) - \lambda_{21}\frac{e^{r\gamma x}-1}{-r\gamma} + \lambda_{12}\frac{e^{-r\gamma x}-1}{-r\gamma x} + \frac{2\Lambda}{r\gamma}\mu(2\Theta)$$

Inspection now ensures that, for $b_{U,0}$ large enough, $F_0^L(b_{U,0}) \geq 0$, and that for $b_{L,0}$ small enough, $F_0^U(b_{L,0}) \leq 0$. But then, for any Δ_Θ,

$$F_0(b_{L,0}, \Delta_\Theta) \leq F_0^U(b_{L,0}) \leq 0 \leq F_0^L(b_{U,0}) \leq F_0(b_{U,0}, \Delta_0).$$

Keeping the monotonicity and continuity of F_0 in mind, this implies that $\Phi_0(\Delta_\Theta) \in [b_{L,0}, b_{U,0}]$, and thus that

$$\Phi_0(\mathbb{R}) \subset [b_{L,0}, b_{U,0}].$$

A similar argument formulated with F_Θ would yield two other constants $b_{L,\Theta}$ and $b_{U,\Theta}$ so that

$$\Phi_\Theta(\mathbb{R}) \subset [b_{L,\Theta}, b_{U,\Theta}].$$

In particular, if I define

$$\Omega \triangleq [b_{L,\Theta} \wedge b_{L,0}, b_{U,0} \vee b_{U,0}],$$

then

$$\Phi(\Omega) \triangleq (\Phi_0, \Phi_\Theta)(\Omega) \subset \Omega \times \Omega.$$

Step 3 I now show that Φ is a contraction and, as a result, admits a unique fixed point. First, a direct verification shows the continuity of Φ_0 and Φ_Θ.

Second, if I choose $x \in \Omega$ so that

$$\Phi_0(x) + x \neq 0, \tag{72}$$

then, an application of the Implicit Function Theorem based on the relation (70) ensures that Φ_l is differentiable at x, with derivative given by

$$\Phi_0'(x) = -\frac{\frac{\partial F_0}{\partial \Delta_0}(\Phi_0(x), x)}{\frac{\partial F_0}{\partial \Delta_\Theta}(\Phi_0(x), x)}$$

$$= -\frac{2\Lambda\begin{pmatrix} \mathbf{1}_{\{-\Phi_0(x)-x>0\}}\mu(1\Theta)(-1)\frac{\partial\chi}{\partial\epsilon}(\eta_{20}, -\Phi_0(x)-x) \\ -\mathbf{1}_{\{\Phi_0(x)+x>0\}}\mu(2\Theta)\frac{\partial\chi}{\partial\epsilon}(\eta_{10}, \Phi_0(x)+x)\end{pmatrix}}{r + \lambda_{12}e^{r\gamma x} + \lambda_{21}e^{-r\gamma x} + 2\Lambda\begin{pmatrix} \mathbf{1}_{\{-\Phi_0(x)-x>0\}}\mu(1h\Theta)(-1)\frac{\partial\chi}{\partial\epsilon}(\eta_{20}, -\Phi_0(x)-x) \\ -\mathbf{1}_{\{\Phi_0(x)+x>0\}}\mu(2\Theta)\frac{\partial\chi}{\partial\epsilon}(\eta_{10}, \Phi_0(x)+x)\end{pmatrix}}.$$

Now, one checks that the numerator is positive, bounded on Ω, and also appears in the denominator. Further, the second part of the denominator,

$$r + \lambda_{12} e^{r\gamma x} + \lambda_{21} e^{-r\gamma x},$$

is also positive, and bounded on Ω. Then, there exists a constant $C \in (0,1)$, independent of the choice of x, and for which

$$\left|\Phi_0'(x)\right| < C.$$

on Ω.

Finally, remembering the monotonicity of the function in (71), there is at most one point in Ω where (72) does not hold and where, as a result, Φ_0 is not differentiable.

Summing up, the restriction of Φ_0 to Ω maps a compact into itself, is continuous, is differentiable everywhere but possibly at one point, and has a derivative whose absolute value that is bounded strictly below one.

This argument can be adapted for Φ_Θ, and classical contraction argument then ensures that Φ admits a unique fixed point $(\Delta_0^*, \Delta_\Theta^*)$ in Ω. Finally, as the range of Φ is already contained in Ω, this is the unique fixed point over $\mathbb{R}^2$.

Step 4 Finally, given the fixed point of Φ, the solution β to the HJB equations (20) can be recovered. For example,

$$a(1\Theta) = \frac{1}{r}\left(\kappa(1h) + \lambda_{12}\frac{e^{-r\gamma\Delta_h^*} - 1}{-r\gamma} + 2\Lambda\mu(20)\left[\chi\left(\eta_\Theta, -\Delta_0^* - \Delta_\Theta^*\right)\right]^-\right). \tag{73}$$

In particular, there is exactly one solution to the system of HJB equations (20). □

B.2 Proofs for Section 3

Proof 22 (Proof of Proposition 7)**.** The proposition and its proof are in Duffie et al. (2005). I only give a partial sketch to introduce some notation.

There are three linear relations linking the components of a stationary distribution μ. They follow from the stationary distribution of endowment correlation types and from the market clearing condition (28), and are

$$\begin{cases} \mu(10) + \mu(1\Theta) &= \mu_1 \\ \mu(20) + \mu(2\Theta) &= \mu_2 \\ \mu(1\Theta) + \mu(2\Theta) &= \frac{S_d}{\Theta} \end{cases}. \tag{74}$$

One can then use these equations to express one of the flow conditions (25) as an equation in, say, $\mu(20)$ only. This yields the quadratic equation

$$0 = \mu(20)^2 + b\left(\frac{1}{\Lambda}\right)\mu(20) + c\left(\frac{1}{\Lambda}\right) \triangleq Q\left(\mu(2l), \frac{1}{\Lambda}\right), \tag{75}$$

where

$$b\left(\frac{1}{\Lambda}\right) \triangleq \frac{S_d}{\Delta_\theta} - \mu_2 + \frac{1}{\Lambda}\frac{\lambda_{12}+\lambda_{21}}{2},$$
$$c\left(\frac{1}{\Lambda}\right) \triangleq -\frac{1}{\Lambda}\frac{\lambda_{12}}{2}\left(1-\frac{S_d}{\Theta}\right). \tag{76}$$

Solving this equation already characterize a unique candidate. □

I will use the following results when proving Corollary 12 and Proposition 8. They follow from the characterization (75).

Lemma 23. *The sensitivity of the stationary cross-sectional distribution of types to the illiquidity level satisfies*

$$\frac{\partial}{\partial \frac{1}{\Lambda}}\mu(1\Theta) = \frac{\partial}{\partial \frac{1}{\Lambda}}\mu(20) = -\frac{\partial}{\partial \frac{1}{\Lambda}}\mu(10) = -\frac{\partial}{\partial \frac{1}{\Lambda}}\mu(2\Theta) = \frac{-\mu(20)\frac{\lambda_{12}+\lambda_{21}}{2} + \frac{\lambda_{12}}{2}\left(1-\frac{S_d}{\Theta}\right)}{\mu(1\Theta)+\mu(20)+\frac{1}{\Lambda}\frac{\lambda_{12}+\lambda_{21}}{2}}, \tag{77}$$

which is positive. Also,

$$\frac{\partial}{\partial \frac{1}{\Lambda}}\left(\lambda\mu(1\Theta)\mu(20)\right) = -\frac{1}{\left(\frac{1}{\lambda}\right)^2}\frac{\mu(20)\mu(1\Theta)\frac{1}{2\Lambda}\left(\lambda_{12}+\lambda_{21}\right)}{\mu(2l)+\mu(1h)+\frac{1}{\lambda}\frac{\lambda_{12}+\lambda_{21}}{2}} \tag{78}$$

and is negative. Finally, for $i\theta = 1\Theta, 20$,

$$\frac{\partial}{\partial \frac{1}{\Lambda}}\left(\lambda\mu(i\theta)\right)$$

and is negative as well.

Proof 24 (Proof of Lemma 23)**.** For the first statement, the sensitivity of $\mu(20)$ on the illiquidity level follows from an application of the Implicit Function Theorem. The relation between the various sensitivities then follows from (74).

Now, recalling Equations (25) and (28), I deduce from (77) that

$$\frac{\partial}{\partial \frac{1}{\Lambda}}\mu(1\Theta) = \frac{\partial}{\partial \frac{1}{\Lambda}}\mu(20) = \frac{\lambda\mu(1\Theta)\mu(20)}{\mu(1\Theta)+\mu(20)+\frac{1}{\Lambda}\frac{\lambda_{12}+\lambda_{21}}{2}},$$

which is positive. A direct calculation then yields (78). Finally the last sensitivity follows from the elementary observation that, if the product of two positive functions is increasing, and if the first term in the product is decreasing, then the second one must be increasing. □

Proof 25 (Proof of Proposition 8)**.** From the proof of Proposition 7,

$$\mu\left(20,\frac{1}{\Lambda}\right) = \frac{1}{2}\left(-b\left(\frac{1}{\Lambda}\right)+\sqrt{\left(b\left(\frac{1}{\Lambda}\right)\right)^2-4c\left(\frac{1}{\Lambda}\right)}\right).$$

Now, as

$$\lim_{\frac{1}{\Lambda}\to 0} b\left(\frac{1}{\Lambda}\right) = \frac{S_d}{\Theta} - \mu_2,$$

which I assumed to be negative, and

$$\lim_{\frac{1}{\Lambda}\to 0} c\left(\frac{1}{\Lambda}\right) = 0,$$

it follows that

$$\lim_{\frac{1}{\Lambda}\to 0} \mu\left(20, \frac{1}{\Lambda}\right) = \mu_2 - \frac{S_d}{\Theta}.$$

Recalling the linear relationships (74), this yields the asymptotic distribution.

Now, using the previous lemma yields

$$\partial_{\frac{1}{\Lambda}}\mu(20) = \frac{-\mu(20)\frac{\lambda_{12}+\lambda_{21}}{2} + \frac{\lambda_{12}}{2}\left(1 - \frac{S_d}{\Theta}\right)}{2\mu(20) + \frac{S_d}{\Theta} - \mu_2 + \frac{1}{\Lambda}\frac{\lambda_{12}+\lambda_{21}}{2}} \overset{\frac{1}{\Lambda}\to 0}{\longrightarrow} \frac{\lambda_{21}}{2}\frac{\frac{S_d}{\Theta}}{\mu_2 - \frac{S_d}{\Theta}},$$

which concludes. □

Proof 26 (Proof of Proposition 10)**.** Keeping Proposition 4 in mind, the equilibrium condition for the centralized market becomes

$$S_c = \mathrm{E}^{\mu(i\theta)}\left[\pi(i\theta)\right] = \frac{1}{\sigma_c^2}\left(\frac{1}{r\gamma}\left(m_c - rP_c\right) - \mathrm{E}^{\mu(i\theta)}\left[\Sigma_{ic}\right] - \Sigma_{cd}\,\mathrm{E}^{\mu(i\theta)}\left[\theta\right]\right).$$

Now, realizing that

$$\mathrm{E}^{\mu(i\theta)}\left[\Sigma_{ic}\right] = \mu_1\Sigma_{1c} + \mu_2\Sigma_{2c} \overset{(\Delta)}{=} \Sigma_{\eta c}$$

is independent of the trading on the OTC market, and that, thanks to the market clearing condition (28), so is

$$\mathrm{E}^{\mu(i\theta)}\left[\theta\right] = S_d,$$

I can already solve for the equilibrium price, which yields (34). Then, combining this last result and the characterization (18) of the optimal liquid holdings yields the expression (35) in the statement.

I now turn to the OTC market. The existence and uniqueness follows from two elementary observations. First, the value function of a, say, 1h-agents is only impacted by μ via $\mu(20)$ and only as long as $\epsilon_{1\Theta}(a) > 0$. Otherwise, 1h-agents have no intention to trade and, as a result, no interest in knowing how often a counter-party may be met. In mathematical terms, this reads

$$\begin{aligned}\mu(a, 20)\left[\frac{\chi(\eta_0, \epsilon_{1\Theta}(a))}{-r\gamma}\right]^+ &= \mu(a, 20)\mathbf{1}_{\{\epsilon_{1\Theta}(a)>0\}}\frac{\chi\left(\eta_0, \epsilon_{1\Theta}(a)\right)}{-r\gamma} \\ &= \mu^{1\Theta\to 20}(20)\mathbf{1}_{\{\epsilon_{1\Theta}(a)>0\}}\frac{\chi\left(\eta_0, \epsilon_{1\Theta}(a)\right)}{-r\gamma} \\ &= \mu^{1\Theta\to 20}(20)\left[\frac{\chi\left(\eta_0, \epsilon_{1\Theta}(a)\right)}{-r\gamma}\right]^+\end{aligned}$$

In particular, this means that I can choose

$$\hat{\mu} = \begin{pmatrix} \mu^{2\Theta\to 10}(10) \\ \mu^{1\Theta\to 20}(1\Theta) \\ \mu^{1\Theta\to 20}(20) \\ \mu^{2\Theta\to 10}(2\Theta) \end{pmatrix}$$

as the "density" in (20). Note that this vector does not depend on a but does not define a density any more.

The second observation is that the proof of Proposition 5 remains valid when the components of μ are only positive numbers, and do not necessarily sum up to one. As a result, there is exactly one solution to the HJB equations defining an equilibrium, which shows the uniqueness and existence of an equilibrium.

I must still characterize the ordering of the valuations of the illiquid asset d or, equivalently, characterize the trading pattern on the OTC market. To do so I first characterize the ordering when the OTC market becomes arbitrarily liquid, and then show that this ordering is maintained at any illiquidity level. The actual argument is articulated around three claims.

Claim 1 I first show that an equilibrium a of the model can be bounded by constants that are independent of the illiquidity level.

Proof of Claim 1. Let $\{\Lambda_n\}_{n\geq 0}$ be a sequence of intensities be given, and let $\{a_n\}_{n\geq 0}$ be the corresponding sequence of equilibria. Let me assume, for the sake of contradiction, that there is an agent type $i\theta$ for which the sequence $\{a_n(i\theta)\}_{n\geq 0}$ is unbounded. I first assume it is unbounded below, meaning that, maybe up to taking a subsequence,

$$\lim_{n\to\infty} a_n(i\theta) = -\infty, \tag{79}$$

Recalling the HJB equations (20) and the first part of this proof,

$$-\lambda_{i\bar{i}}\frac{e^{-r\gamma(a_n(\bar{i}\theta)-a_n(i\theta))}-1}{-r\gamma} = -ra_n(i\theta) + \kappa(i\theta) + 2\Lambda_n\hat{\mu}\left(\Lambda_n, \bar{i}\bar{\theta}\right)\left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}(a)\right)}{r\gamma}\right]^+. \tag{80}$$

But, recalling (79), the left hand side of (80) is bounded below by a sequence that grows arbitrarily. As a result,

$$\lim_{n\to\infty} a_n(i\theta) - a_n(\bar{i}\theta) = +\infty. \tag{81}$$

and, recalling (79) one more time,

$$\lim_{n\to\infty} a_n(\bar{i}\theta) = -\infty. \tag{82}$$

Now, if (81) follows from (79), from (82) I can conclude that

$$\lim_{n\to\infty} a_n(\bar{i}\theta) - a_n(i\theta) = +\infty. \tag{83}$$

In particular, both (81) and (83) follow from (79), which is impossible. There is thus no sequence of equilibria that is unbounded below.

It remains to see whether a sequence of equilibria can be unbounded above. Let me assume that, maybe choosing a subsequence,

$$\lim_{n\to\infty} a_n(1\Theta) = +\infty. \tag{84}$$

Choosing the type 1Θ is without loss of generality. Before pursuing the argument I note that, assuming an agent of type $i\theta$ does not trade in equilibrium, it follows from (20) and the first part of the proof that

$$\begin{aligned}
0 =& ra_n(i\theta) - \kappa(i\theta) - \lambda_{i\bar{i}} \frac{e^{-r\gamma(a_n(\bar{i}\theta) - a_n(i\theta))} - 1}{-r\gamma} - 2\Lambda_n \hat{\mu}(\lambda, \bar{i}\bar{\theta}) \left[\frac{\chi\left(\eta_\theta, \epsilon_{i\theta}(a)\right)}{-r\gamma}\right]^+ \\
=& ra_n(i\theta) - \kappa(i\theta) - \lambda_{i\bar{i}} \frac{e^{-r\gamma(a_n(i\theta) - a_n(\bar{i}\theta))} - 1}{-r\gamma} \\
\geq& ra_n(i\theta) - \kappa(i\theta) - \lambda_{i\bar{i}} \frac{1}{r\gamma}.
\end{aligned}$$

In other words, I have an a priori upper bound on $a_n(i\theta)$. Namely,

$$\frac{1}{r}\left(\kappa(i\theta) + \frac{\lambda_{i\bar{i}}}{r\gamma}\right) \geq a_n(i\theta). \tag{85}$$

Now, two further cases must be distinguished, depending on whether 1Θ-agents are willing to trade or not. Maybe choosing a further subsequence, I assume that 1Θ-agents never trade. In this case, combining (85) and (84) yields

$$\frac{1}{r}\left(\kappa(1\Theta) + \frac{\lambda_{12}}{r\gamma}\right) \geq \lim_{n\to\infty} a_n(1\Theta) = +\infty,$$

which is a contradiction. The only possibility left is thus for the agents with type 1Θ are willing to trade. I can thus assume that, for any $n \geq 0$,

$$a_n(10) - a_n(1\Theta) - a_n(20) + a_n(2\Theta) \geq 0.$$

Using (85) for the two types of agent that do not trade, meaning 10 and 2Θ, then yields

$$\frac{1}{r}\left(\kappa(10) + \kappa(2\Theta) + \frac{\lambda_{12}}{r\gamma} + \frac{\lambda_{21}}{r\gamma}\right) \geq a_n(1\Theta) + a_n(20).$$

From this last inequality and (84) I deduce that

$$\lim_{n\to\infty} a_n(20) = -\infty,$$

which will, by the first part of this proof, lead to a contradiction.

To sum up, there are no circumstances under which an unbounded sequence of equilibria can be found. □

This first claim is needed when proving the second one.

Claim 2 For a sufficiently large meeting intensity Λ, the corresponding equilibrium $a(\Lambda)$ satisfies

$$\epsilon_{1\Theta}(a) > 0$$

exactly when $\mathcal{S} > 0$, with $\mathcal{S}$ defined in (24).

Proof of Claim 2. Let me choose a sequence $\{\Lambda_n\}_{n\geq 0}$ of meeting intensities so that

$$\lim_{n\to\infty} \Lambda_n = +\infty.$$

By Claim 1, there exists two constants $L < U$ so that

$$\forall n: \ a_n \in [L, U]^4 \tag{86}$$

I can thus choose a convergent subsequence, and call the limit a_∞. Maybe choosing a further subsequence, I assume that

$$\forall n: \ \epsilon_{1\Theta}(a_n) \overset{(\Delta)}{=} a_n(2\Theta) - a_n(20) + a_n(10) - a_n(1\Theta) \geq 0. \tag{87}$$

In other words, all along the sequence of intensities, and in the limit, agents with endowment correlations type 2 have the high valuation of the illiquid asset.

Under this assumption the HJB equations defining a_n become

$$\begin{cases} ra_n(10) &= \kappa(10) &+\lambda_{12}\frac{e^{-r\gamma(a_n(20)-a_n(10))}-1}{-r\gamma} & \\ ra_n(1\Theta) &= \kappa(1\Theta) &+\lambda_{12}\frac{e^{-r\gamma(a_n(2\Theta)-a_n(1\Theta))}-1}{-r\gamma} &+2\Lambda_n\hat{\mu}\left(\Lambda_n, 20\right)\frac{\chi(\eta_\Theta,\epsilon_{1\Theta}(a))}{-r\gamma} \\ ra_n(20) &= \kappa(20) &+\lambda_{21}\frac{e^{-r\gamma(a_n(10)-a_n(20))}-1}{-r\gamma} &+2\Lambda_n\hat{\mu}\left(\Lambda_n, 1\Theta\right)\frac{\chi(\eta_0,\epsilon_{20}(a))}{-r\gamma} \\ ra_n(2\Theta) &= \kappa(2\Theta) &+\lambda_{21}\frac{e^{-r\gamma(a_n(1\Theta)-a_n(2\Theta))}-1}{-r\gamma} & \end{cases}. \tag{88}$$

At this stage, I will consider the asymptotic behavior of the stationary type distribution, which requires to distinguish two cases.

I first assume

$$\mu_2 - \frac{S_d}{\Delta_\theta} > 0, \tag{89}$$

meaning that the marginal buyer of the illiquid asset has a high valuation. In this case, it is known from Lemma 8 that

$$\lim_{n\to\infty} \hat{\mu}(\Lambda_n, 20) = \mu_2 - \frac{S_d}{\Theta} > 0.$$

As a result,

$$\lim_{n\to\infty} \Lambda_n\hat{\mu}(\Lambda_n, 20) = \infty. \tag{90}$$

Now, as stated in (86) shows that the equilibria are bounded. Hence, (88) is only compatible with (90) if

$$\lim_{n\to\infty} \chi\left(\eta_\Theta, \epsilon_{1\Theta}(a)\right) = 0.$$

Recalling the definition of "χ" in (23), this is equivalent to

$$a_\infty(10) - a_\infty(1\Theta) = a_\infty(20) - a_\infty(2\Theta). \tag{91}$$

But then, as Lemma 8 ensures that

$$\lim_{n\to\infty} \Lambda_n \hat{\mu}(\Lambda_n, 1\Theta) = \frac{\lambda_{12}}{2} \frac{\frac{S_d}{\Theta}}{\mu_2 - \frac{S_d}{\Theta}}$$

letting n go to $+\infty$ in (88) yields

$$\begin{cases} ra_\infty(10) &= \kappa(10) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(20)-a_\infty(10))}-1}{-r\gamma} \\ ra_\infty(1\Theta) &= \kappa(1\Theta) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(2\Theta)-a_\infty(1\Theta))}-1}{-r\gamma} \\ & & +\lim_{n\to\infty} 2\Lambda_n\hat{\mu}\left(\Lambda_n, 20\right)\frac{\chi(\eta_\Theta, \epsilon_{1\Theta}(a))}{-r\gamma} \\ ra_\infty(20) &= \kappa(20) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(10)-a_\infty(20))}-1}{-r\gamma} \\ ra_\infty(2\Theta) &= \kappa(2\Theta) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(1\Theta)-a_\infty(2\Theta))}-1}{-r\gamma} \end{cases}. \tag{92}$$

Now, subtracting the second and third equations from the sum of the first and fourth ones in (92), and then repeatedly using (91), yields

$$\kappa(10) - \kappa(1\Theta) + \kappa(2\Theta) - \kappa(20) = \lim_{n\to\infty} 2\Lambda_n\hat{\mu}\left(\Lambda_n, 20\right)\frac{\chi\left(\eta_\Theta, \epsilon_{1\Theta}(a)\right)}{-r\gamma}. \tag{93}$$

I draw two conclusions from this last equality. First, combining it with (94) yields

$$\begin{cases} ra_\infty(10) &= \kappa(10) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(20)-a_\infty(10))}-1}{-r\gamma} \\ ra_\infty(1\Theta) &= \kappa(10)+\kappa(2\Theta)-\kappa(20) & +\lambda_{12}\frac{e^{-r\gamma(a_\infty(2\Theta)-a_\infty(1\Theta))}-1}{-r\gamma} \\ ra_\infty(20) &= \kappa(20) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(10)-a_\infty(20))}-1}{-r\gamma} \\ ra_\infty(2\Theta) &= \kappa(2\Theta) & +\lambda_{21}\frac{e^{-r\gamma(a_\infty(1\Theta)-a_\infty(2\Theta))}-1}{-r\gamma} \end{cases}. \tag{94}$$

This system defines a contraction, as Proposition 34 below formally shows, which ensures the uniqueness of the asymptotic equilibrium β_∞.

Second, (93) is only compatible with the assumption (87) as long as

$$\mathcal{S} \overset{(\Delta)}{=} \kappa(10) - \kappa(1\Theta) - \kappa(20) + \kappa(2\Theta) \geq 0. \tag{95}$$

The case of

$$\frac{S_d}{\Theta} > \mu_2$$

is handled similarly.

Assuming the reverse inequality in (87) would also give a unique candidate for a_∞, but this time require that (95) also holds with a reverse inequality.

Summing up, if (95) holds, then the sequence of equilibria converges and, for n large enough, $\epsilon_{1\Theta}(a_n) > 0$. Otherwise, the sequence converges as well but, for n large enough, $\epsilon_{2\Theta}(a_n) < 0$. $\square$

I have now characterized which trades are implemented when the meeting intensity is sufficiently large. The last step is to show that the trading pattern cannot be reverted by an increasing illiquidity level.

Claim 3 The surplus to be shared in bilateral trades is differentiable and decreasing in the meeting intensity. In other words, if $\epsilon_{i\theta}\left(a(\Lambda)\right) > 0$,

$$\frac{\partial}{\partial \Lambda}\epsilon_{i\theta}\left(a(\Lambda)\right) < 0.$$

In particular, the derivative exists.

Proof of Claim 3. Without loss of generality, I assume

$$\epsilon_{1\Theta}(a) \overset{(\Delta)}{=} a(10) - a(1\Theta) - a(20) + a(2\Theta) \overset{(\Delta)}{=} -\Delta_\Theta - \Delta_0 > 0, \tag{96}$$

meaning that the 2-agents have the high valuation. From the proof of Proposition 5, I know that for any given Λ, the pair $\Delta \triangleq (\Delta_\Theta, \Delta_0)$ is the unique solution to the system

$$0 = F(\Delta;\Lambda) \overset{(\Delta)}{\Leftrightarrow} \begin{cases} 0 = & F_\Theta\left(\Delta_\Theta, \Delta_0; \Lambda\right) \\ 0 = & F_0\left(\Delta_0, \Delta_\Theta; \Lambda\right) \end{cases},$$

where the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ is implicitly defined in the last equation. Now, under the above assumption regarding the high valuation agents, I can write

$$\begin{aligned}
&\det\left(D_\Delta F\left(\Delta,\Lambda\right)\right) \\
&= \det\begin{pmatrix} \begin{matrix} r + \lambda_{12}e^{r\gamma\Delta_h} + \lambda_{21}e^{-r\gamma\Delta_h} \\ +2\Lambda\mu(20)\frac{\frac{\partial\chi}{\partial\epsilon}(\eta_{1h}, -\Delta_l - \Delta_h)}{-r\gamma}\end{matrix} & \frac{2\Lambda\mu(20)\frac{\partial\chi}{\partial\epsilon}(\eta_{1h}, -\Delta_l - \Delta_h)}{-r\gamma} \\ \frac{2\Lambda\mu(1\Theta)\frac{\partial\chi}{\partial\epsilon}(\eta_{2l}, -\Delta_l - \Delta_h)}{-r\gamma} & \begin{matrix} r + \lambda_{21}e^{r\gamma\Delta_l} + \lambda_{12}e^{-r\gamma\Delta_l} \\ +\frac{2\Lambda\mu(1\Theta)\frac{\partial\chi}{\partial\epsilon}(\eta_{2l}, -\Delta_l - \Delta_h)}{-r\gamma}\end{matrix} \end{pmatrix} \\
&= \left(r + \lambda_{12}e^{r\gamma\Delta_h} + \lambda_{21}e^{-r\gamma\Delta_h}\right)\left(r + \lambda_{21}e^{r\gamma\Delta_l} + \lambda_{12}e^{-r\gamma\Delta_l}\right) \\
&\quad + \left(r + \lambda_{12}e^{r\gamma\Delta_h} + \lambda_{21}e^{-r\gamma\Delta_h}\right)\frac{2\Lambda\mu(1\Theta)\frac{\partial\chi}{\partial\epsilon}\left(\eta_{2l}, -\Delta_l - \Delta_h\right)}{-r\gamma} \\
&\quad + \left(r + \lambda_{21}e^{r\gamma\Delta_l} + \lambda_{12}e^{-r\gamma\Delta_l}\right)\frac{2\Lambda\mu(20)\frac{\partial\chi}{\partial\epsilon}\left(\eta_{1h}, -\Delta_l - \Delta_h\right)}{-r\gamma}.
\end{aligned} \tag{97}$$

Recalling from the definition (23) that χ is decreasing in its second argument, this last quantity is positive, which justifies an application of the Implicit Function Theorem. This ensures that Δ is, locally, a differentiable function $\Delta(\lambda)$ of the meeting intensity, with derivative

$$\begin{aligned}
\partial_\lambda \Delta(\lambda) &= -\left(D_\Delta F(\Delta,\lambda)\right)^{-1} D_\lambda F(\Delta,\lambda) \\
&= \frac{-1}{\det\left(D_\Delta F\right)}\begin{pmatrix} \frac{\partial F_0}{\partial \Delta_0} & -\frac{\partial F_\Theta}{\partial \Delta_0} \\ -\frac{\partial F_0}{\partial \Delta_\Theta} & \frac{\partial F_\Theta}{\partial \Delta_\Theta}\end{pmatrix}\begin{pmatrix} \frac{\partial F_\Theta}{\partial \lambda} \\ \frac{\partial F_0}{\partial \lambda}\end{pmatrix}.
\end{aligned}$$

But then,

$$\begin{aligned}
&\frac{\partial}{\partial \Lambda}\left(\Delta_0+\Delta_\Theta\right)\\
=&\frac{-1}{\det\left(D_\Delta F\right)}\left(\left(\frac{\partial F_0}{\partial \Delta_0}-\frac{\partial F_0}{\partial \Delta_\Theta}\right)\frac{\partial F_\Theta}{\partial \lambda}+\left(\frac{\partial F_\Theta}{\partial \Delta_\Theta}-\frac{\partial F_\Theta}{\partial \Delta_0}\right)\frac{\partial F_\Theta}{\partial \Lambda}\right)\\
=&\frac{-1}{\det\left(D_\Delta F(\Delta,\Lambda)\right)}\begin{pmatrix} \left(r+\lambda_{21}e^{r\gamma\Delta_0}+\lambda_{12}e^{-r\gamma\Delta_0}\right)\frac{\chi(\eta_\Theta,-\Delta_0-\Delta_\Theta)}{-r\gamma}2\partial_\Lambda\left(\Lambda\mu(\Lambda,2l)\right)\\ +\left(r+\lambda_{12}e^{r\gamma\Delta_\Theta}+\lambda_{21}e^{-r\gamma\Delta_\Theta}\right)\frac{\chi(\eta_{2l},-\Delta_0-\Delta_\Theta)}{-r\gamma}2\partial_\Lambda\left(\Lambda\mu(\Lambda,1\Theta)\right)\end{pmatrix}.
\end{aligned}$$

With (96), both $\chi\left(\eta_\Theta,-\Delta_0-\Delta_\Theta\right)$ and $\chi\left(\eta_0,-\Delta_0-\Delta_\Theta\right)$ are negative, As a result,

$$\partial_\Lambda\left(\Delta_\Theta(\Lambda)+\Delta_0(\Lambda)\right)>0$$

or, equivalently,

$$\partial_\Lambda\left(-\Delta_\Theta(\Lambda)-\Delta_0(\Lambda)\right)<0$$

which proves the claim. □

I can finally conclude the proof of Proposition 10. Indeed, assuming that $\mathcal{S}>0$, Claim 2 ensures that, if the meeting intensity Λ is larger than a certain threshold $\bar{\Lambda}$, then, $\epsilon_{1\Theta}\left(a(\Lambda)\right)>0$, meaning that 2-agents have the high valuation. But then, Claim 3 ensures that decreasing Λ increases $\epsilon_{1\Theta}\left(a(\Lambda)\right)$. In particular, 2-agents still have the high valuation for any value of the meeting intensity. The case where $\mathcal{S}<0$ is similar.

Proof 27 (Proof of Proposition 12)**.** Without loss of generality, I assume that 2-agents have the high valuation of the illiquid asset.

Regarding the OTC market, as the transaction size is fixed, the trading volume is proportional to

$$2\Lambda\mu(1\Theta)\mu(20),$$

meaning to the meeting intensity between 1Θ and 20 agents. From Lemma 23 this quantity is increasing in the meeting intensity.

Recalling the expressions (35) for the liquid holdings in equilibrium, the volume exchanged on the centralized market per unit of time is thus

$$\begin{aligned}
\text{Vol}=&\frac{1}{2}\left\{\begin{array}{rl} \lambda_{12}\mu(10) & |\pi(10)-\pi(20)| \\ +\lambda_{12}\mu(1\Theta) & |\pi(1\Theta)-\pi(2\Theta)| \\ +2\Lambda\mu(1\Theta)\mu(20) & |\pi(1\Theta)-\pi(10)| \\ +2\Lambda\mu(1\Theta)\mu(20) & |\pi(20)-\pi(2\Theta)| \\ +\lambda_{21}\mu(20) & |\pi(20)-\pi(10)| \\ +\lambda_{21}\mu(2\Theta) & |\pi(2\Theta)-\pi(1\Theta)| \end{array}\right\}. \qquad (98)\\
=&\frac{1}{2\Sigma_{cc}}\left\{\left(\lambda_{12}\mu_1+\lambda_{21}\mu_2\right)\left|\Sigma_{1c}-\Sigma_{2c}\right|+4\Lambda\mu(1\Theta)\mu(20)\left|\Sigma_{cd}\right|\Theta\right\}\\
=&\frac{1}{\Sigma_{cc}}\left\{\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}}\left|\Sigma_{1c}-\Sigma_{2c}\right|+2\Lambda\mu(1\Theta)\mu(20)\left|\Sigma_{cd}\right|\Theta\right\}
\end{aligned}$$

Lemma 23 shows that the trading volume in increasing in Λ. □

Proof 28 (Proof of Proposition 13)**.** To obtain a model without OTC market, we can set $\Theta = 0$. In this case Equation (98) in the last proof immediately shows that the trading volume in c drops.

Further, letting the meeting intensity Λ grow arbitrarily in Equation (98), and recalling Proposition 8, I calculate the asymptotic level of trading in c as

$$\lim_{\Lambda\to\infty} \mathrm{Vol}(\Lambda) = \frac{1}{\Sigma_{cc}} \left\{ \frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}} \left|\Sigma_{1c} - \Sigma_{2c}\right| + 2\lambda_{21} S_d \left|\Sigma_{cd}\right| \right\}.$$

Now, in a Walrasian setting, the market for d can only clear if the investors with a high valuation randomize their decision to buy the asset d. If the inequality (36) holds, inspection shows that the trading volume on the market for c in a Walrasian setting is

$$\begin{aligned} \mathrm{Vol}_W &= \frac{1}{2} \left(\begin{array}{l} \mu_1 \lambda_{12} \frac{\mu_2 - \frac{S_d}{\Theta}}{\mu_2} \left|\mu(20) - \pi(10)\right| \\ + \mu_1 \lambda_{12} \frac{\frac{S_d}{\Theta}}{\mu_2} \left|\mu(2\Theta) - \pi(10)\right| \\ + \frac{S_d}{\Theta} \lambda_{21} \left|\pi(10) - \pi(2\Theta)\right| \\ + \left(\mu_2 - \frac{S_d}{\Theta}\right) \lambda_{21} \left|\pi(10) - \pi(20)\right| \end{array} \right) \\ &= \left(\begin{array}{l} \frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}} \left|\mu(20) - \pi(10)\right| \\ + \frac{S_d}{\Theta} \lambda_{21} \left(\left|\mu(2\Theta) - \pi(10)\right| - \left|\mu(20) - \pi(10)\right|\right) \end{array} \right). \end{aligned}$$

The triangular inequality ensures that

$$\left|\mu(2\Theta) - \pi(10)\right| - \left|\mu(20) - \pi(10)\right| \leq \left|\mu(2\Theta) - \pi(20)\right|$$

and, as a result, that

$$\mathrm{Vol}_W \leq \lim_{\Lambda\to\infty} \mathrm{Vol}(\Lambda).$$

Inspection finally shows that Inequality (36) defines the cases for which the triangular inequality is strict. □

Proof 29 (Proof of Proposition 16)**.** The optimization for the optimal design of the liquid asset is

$$\max_{(a_c,b_c)} f\left(a_c, b_c\right) \overset{(\Delta)}{=} \max_{(a_c,b_c)} \left\{\left|w_1 a_c + w_2 b_c\right| + \left|w_3 a_c + w_4 b_c\right|\right\} \tag{99}$$

under the conditions

$$\left\|(a_c, b_c)\right\|_2 = 1, \tag{100}$$

$$\det\left(\left(\begin{array}{c:c} a_d & a_c \\ b_d & b_c \end{array}\right)\right) \cdot \det\left(\left(\begin{array}{c:c} a_1 - a_2 & \alpha_c \\ b_1 - b_2 & \beta_c \end{array}\right)\right) > 0. \tag{101}$$

I characterize the solution to this problem by maximizing f under the constraint (100) and making sure that the constraint (101) holds *ex post*.

A solution to the maximization of f under (100) exists because f is continuous and the optimization domain is compact.

The objective function f is piecewise linear and I define

$$\mathcal{A} \triangleq \left\{(x,y) \in \mathbb{R}^2 : \; w_1 x + w_2 y \geq 0\right\}$$

and

$$\mathcal{B} \triangleq \left\{(x,y) \in \mathbb{R}^2 : \; w_3 x + w_4 y \geq 0\right\}$$

to describe this piecewise structure. Clearly, the optimal risk-profile (a_c, b_c) belongs either to $\mathcal{D}_1 \triangleq (A \cap B^c) \cup (A^c \cap B)$ or to $\mathcal{D}_2 \triangleq (A \cap B) \cup (A^c \cap B^c)$. In the first case, the method of Lagrange multipliers characterizes the optimal risk profile as

$$\begin{cases} \nabla_{(a_c,b_c)} f|_{\mathcal{D}_1} (a_c, b_c) & = & L \nabla_{(a_c,b_c)} \left(\|(a_c, b_c)\|_2\right) \\ \|(a_c, b_c)\|_2 & = & 1 \end{cases},$$

with $L \in \mathbb{R}$ being the Lagrange multiplier. Solving this system for (a_c, b_c) yields the unique candidate

$$\begin{pmatrix} a_c \\ a_d \end{pmatrix} = \pm \frac{1}{\nu} \begin{pmatrix} w_1 - w_3 \\ w_2 - w_4 \end{pmatrix}, \tag{102}$$

with the constant

$$\nu \triangleq \sqrt{(w_1 - w_3)^2 + (w_2 - w_4)^2}$$

ensuring the normalization $\Sigma_{cc} = 1$.[57] I must still make two checks to ensure the validity of this candidate as a solution to the original problem. First, does the candidate satisfy the constraint (101)? Plugging (102) into (101) yields

$$\frac{2\Theta\Lambda\lambda_{12}\lambda_{21}\mu(20)\mu(1\Theta) \left((b_2 - b_1)\, a_d + (a_1 - a_2)\, b_d\right)^2}{\nu^2 \left(\lambda_{1,2} + \lambda_{2,1}\right)} > 0$$

and the consistency constraint is necessarily satisfied. Second, does the candidate actually belongs to $\mathcal{D}_1$? A vector $\left(\tilde{a}_c, \tilde{b}_c\right)$ belongs to $\mathcal{D}_1$ exactly when

$$(w_1 a_c + w_2 b_c)\,(w_3 a_c + w_4 b_c) < 0.$$

For the choice $(a_c, b_c) = \left(\tilde{a}_c, \tilde{b}_c\right)$, this inequality becomes the assumption (55) in the statement, and is thus satisfied.

Finally, we can consider the second case $(a_c, b_c) \in \mathcal{D}_2$ and follow the same steps as in the first case. In this second case, however, the unique candidate does not satisfy the condition (101) and there is no solution to the original problem (99). □

Lemma 30. *The equilibrium price P_c of the liquid asset and the corresponding holdings $\{\pi(i\theta; m_2, t\}_{i\theta}$ are uniquely defined in the asymptotic case characterized by Equation* (47)

[57] The $\pm$ follows from the symmetry of the optimization, as discussed in the main text.

and Equation (48). *In case of an aggregate shock from the state* (m_2, t) *to* $\tilde{m}_2, 0$, *the price* P_c *jumps up when*

$$(\Sigma_{2c} - \Sigma_{1c})\,(\tilde{m}_2 - \mu_2(m_2,t)) < 0$$

and down when the other inequality holds. Finally, the "$\kappa(i\theta; m_2, t)$*" defined in* (45) *satisfy*

$$\begin{aligned}\mathcal{S} &\triangleq \kappa(10; m_2,t) - \kappa(1\Theta; m_2,t) - \kappa(20; m_2,t) + \kappa(2\Theta; m_2,t)\\ &= \frac{r\gamma\Theta}{\Sigma_{cc}} \det\left(\begin{pmatrix} e_d & e_c \end{pmatrix}\right) \cdot \det\left(\begin{pmatrix} e_1 - e_2 & e_c \end{pmatrix}\right) + o(\gamma)\end{aligned}$$

for any state (m_2, t) *of the economy.*

Proof. I start from the optimization over the liquid holdings $\tilde{\pi}$ in the HJB equation 44. The first-order condition for this optimization characterizes the optimal holdings $\pi(i\theta; m_2, t)$ as the unique solution to the equation

$$\begin{aligned}0 = \dot{P}_c(m_2,t) + m_c - rP_c(m_2,t) - r\gamma\begin{pmatrix}\Sigma_{ic} & \Sigma_{cd} & \Sigma_{cc}\end{pmatrix}\begin{pmatrix}1\\ \theta\\ \pi(i\theta;m_2,t)\end{pmatrix}\\ + \lambda_a \,\mathrm{E}^{M(\tilde{m}_2)}\left[\begin{array}{l}(P_c(\tilde{m}_2,0) - P_c(m_2,t))\cdot\\ \cdot\left(\begin{array}{l}\delta(i;m_2,t;\tilde{m}_2)\cdot\\ \cdot e^{-r\gamma(a(\bar{i}\theta;\tilde{m}_2,0)+\pi(i\theta;m_2,t)(P_c(\tilde{m}_2,0)-P_c(m_2,t))-a(i\theta;m_2,t))}\\ +(1-\delta(i;m_2,t;\tilde{m}_2))\cdot\\ \cdot e^{-r\gamma(a(i\theta;\tilde{m}_2,0)+\pi(i\theta;m_2,t)(P_c(\tilde{m}_2,0)-P_c(m_2,t))-a(i\theta;m_2,t))}\end{array}\right)\end{array}\right]\end{aligned} \tag{103}$$

In the asymptotic case of a small risk-aversion to the jump risks, as characterized by the equations (47) and (48), this first-order condition becomes

$$\begin{aligned}0 = \dot{P}_c(m_2,t) + m_c - rP_c(m_2,t) - r\gamma\begin{pmatrix}\Sigma_{ic} & \Sigma_{cd} & \Sigma_{cc}\end{pmatrix}\begin{pmatrix}1\\ \theta\\ \pi(i\theta;m_2,t)\end{pmatrix}\\ + \lambda_a\left(\mathrm{E}^{M(\tilde{m}_2)}\left[P_c(\tilde{m}_2,0)\right] - P_c(m_2,t)\right) + \mathcal{O}(\gamma).\end{aligned} \tag{104}$$

Equation (104) can be solved for $\pi(i\theta; m_2, t)$ in closed-form. A direct calculation then shows

$$\begin{aligned}\mathcal{S} &\triangleq \kappa(10; m_2,t) - \kappa(1\Theta; m_2,t) - \kappa(20; m_2,t) + \kappa(2\Theta; m_2,t)\\ &= \frac{r\gamma\Theta}{\Sigma_{cc}} \det\left(\begin{pmatrix} e_d & e_c \end{pmatrix}\right) \cdot \det\left(\begin{pmatrix} e_1 - e_2 & e_c \end{pmatrix}\right) + \mathcal{O}(\gamma)\end{aligned}$$

for any state (m_2, t). In particular, even if the "κ" in the dynamic setting are different from their counterparts in the stationary setting, they generate the same flow of surplus.[58]

[58] The "κ"s are defined in Equation (21) in the stationary setting and in Equation (45) in the dynamic setting.

Alternatively, aggregating (104) across the population and recalling the market-clearing condition

$$\mathrm{E}^{\mu(i\theta;m_2,t)}\left[\pi(i\theta;m_2,t)\right]=S_c$$

that holds for any state (m_2,t), yields the ODE

$$\begin{aligned}&\dot{P}_c(m_2,t)-(r+\lambda_a)\,P_c(m_2,t)\\&\quad=-\Big(m_c-r\gamma\left(\Sigma_{\eta c}(m_2,t)+\Theta\Sigma_{cd}+S_c\Sigma_{cc}\right)+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[P_c\left(\tilde{m}_2,0\right)\right]\Big)+\mathcal{O}(\gamma).\end{aligned} \tag{105}$$

for the price of the liquid asset. Deriving

$$\Sigma_{\eta c}(m_2,t)=\Sigma_{1c}+\Big(\mu_2+(m_2-\mu_2)\,e^{-(\lambda_{12}+\lambda_{21})t}\Big)\left(\Sigma_{2c}-\Sigma_{1c}\right)$$

from the type distribution (38) and taking as given the value

$$k_0\triangleq\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[P_c\left(\tilde{m}_2,0\right)\right],$$

I can solve the ODE (105) in closed form under the no-bubble condition

$$\lim_{T\to\infty}e^{-rT}P_c(m_2,t)=0.$$

The solution is

$$P_c(h_a,t)=\frac{k_0+k_1}{r+\lambda_a}+\frac{k_2(m_2)}{r+\lambda_a+\lambda_{12}+\lambda_{21}}e^{-(\lambda_{12}+\lambda_{21})t}, \tag{106}$$

with the constants

$$\begin{aligned}k_1&\triangleq m_c-r\gamma\left(\Sigma_{1c}+\mu_2\left(\Sigma_{2c}-\Sigma_{1c}\right)+\Theta\Sigma_{cd}+S_c\Sigma_{cc}\right),\\k_2(m_2)&\triangleq\ -r\gamma\left(m_2-\mu_2\right)\left(\Sigma_{2c}-\Sigma_{1c}\right).\end{aligned}$$

I must still find the constant k_0. This is done by solving the linear equation

$$\begin{aligned}k_0&=\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[P_c\left(\tilde{m}_2,0\right)\right]\\\Leftrightarrow\quad k_0&=\lambda_a\left(\tfrac{k_0+k_1}{r+\lambda_a}-r\gamma\tfrac{\mathrm{E}^{M(\tilde{m}_2)}[\tilde{m}_2]-\mu_2}{r+\lambda_a+\lambda_{12}+\lambda_{21}}\left(\Sigma_{2c}-\Sigma_{1c}\right)\right)\end{aligned}$$

for k_0.

Finally, I characterize how the price P_c of the liquid asset reacts to an aggregate shock that moves the economy from the state (μ_2,t) to the state $(\tilde{m}_2,0)$. Namely a direct calculation based on Equation (106) shows how P_c jumps up when

$$\left(\Sigma_{2c}-\Sigma_{1c}\right)\left(\tilde{m}_2-\mu_2(m_2,t)\right)<0$$

and down when the other inequality holds. □

Proof 31 (Proof of Proposition 14)**.** In the asymptotic case described by the equations (47) and (48), the HJB equations (44) become

$$\begin{aligned}
&ra\left(i\theta;m_2,t\right) \\
&= \dot{a}\left(i\theta;m_2,t\right)+\kappa\left(i\theta;m_2,t;\pi(i\theta;m_2,t)\right) \\
&\quad+\lambda_{i\bar{i}}\left(a\left(\bar{i}\theta;m_2,t\right)-a\left(i\theta;m_2,t\right)\right) \\
&\quad+2\Lambda\mu\left(\bar{i}\bar{\theta};m_2,t\right)\left[a\left(i\theta;m_2,t\right)-P_d\left(m_2,t\right)\left(\bar{\theta}-\theta\right)-a\left(i\theta;m_2,t\right)\right]^{+} \\
&\quad+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[\begin{array}{l}\delta(i;m_2,t;\tilde{m}_2)\cdot \\ \cdot\left(\begin{array}{c}a\left(\bar{i}\theta;\tilde{m}_2,0\right)+\pi(i\theta;m_2,t)\left(P_c\left(\tilde{m}_2,0\right)-P_c\left(m_2,t\right)\right)\\ -a\left(i\theta;m_2,t\right)\end{array}\right)\\ +\left(1-\delta(i;m_2,t;\tilde{m}_2)\right)\cdot \\ \cdot\left(\begin{array}{c}a\left(i\theta;\tilde{m}_2,0\right)+\pi(i\theta;m_2,t)\left(P_c\left(\tilde{m}_2,0\right)-P_c\left(m_2,t\right)\right)\\ -a\left(i\theta;m_2,t\right)\end{array}\right)\end{array}\right], \\
&\quad+o(\gamma)
\end{aligned} \tag{107}$$

with the optimal holdings "$\pi(i\theta;m_2,t)$" being defined in Lemma 30 for any type $i\theta$ and state (m_2,t). Further, the asymptotic behavior of Equation (46) characterizes the bargained price P_d as

$$P_d\left(m_2,t\right)=\left(a(2\Theta;m_2,t)-a(20;m_2,t)\right)-\eta_\Theta\left(\begin{array}{c}\left(a(2\Theta;m_2,t)-a(20;m_2,t)\right)\\ -\left(a(1\Theta;m_2,t)-a(10;m_2,t)\right)\end{array}\right).$$

Just like in the stationary setting, it is convenient to first work with value function *differences*. I thus define

$$\Delta_\Theta(m_2,t)\triangleq a(1\Theta;m_2,t)-a(2\Theta;m_2,t) \tag{108}$$

and

$$\Delta_0(m_2,t)\triangleq a(20;m_2,t)-a(10;m_2,t). \tag{109}$$

Using the HJB equations (120) on the right-hand side of the definition (108) and rearranging yields

$$\begin{aligned}
&\left(r+\lambda_{12}+\lambda_{21}+\lambda_a\right)\Delta_\Theta\left(m_2,t\right)-\dot{\Delta}_\Theta\left(m_2,t\right) \\
&=\kappa\left(1\Theta;m_2,t;\pi(i\theta;m_2,t)\right)-\kappa\left(2\Theta;m_2,t;\pi(i\theta;m_2,t)\right) \\
&\quad-2\Lambda\mu\left(\bar{i}\bar{\theta};m_2,t\right)\left(\Delta_\Theta(m_2,t)+\Delta_0(m_2,t)\right) \\
&\quad+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[\pi(i\theta;m_2,t)\left(P_c\left(\tilde{m}_2,0\right)-P_c\left(m_2,t\right)\right)\right] \\
&\quad+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta(1;m_2,t;\tilde{m}_2)-\delta(2;m_2,t;\tilde{m}_2)\,\Delta_\Theta(\tilde{m}_2,t)\right] \\
&\quad+o(\gamma)
\end{aligned}. \tag{110}$$

The same procedure applied to the definition (108) yields

$$\begin{aligned}&(r+\lambda_{12}+\lambda_{21}+\lambda_a)\,\Delta_0\,(m_2,t)-\dot{\Delta}_0\,(m_2,t)\\=\;&\kappa\,(20;m_2,t;\pi(i\theta;m_2,t))-\kappa\,(10;m_2,t;\pi(i\theta;m_2,t))\\&-2\Lambda\mu\left(\bar{i}\bar{\theta};m_2,t\right)(\Delta_\Theta(m_2,t)+\Delta_0(m_2,t))\\&+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[\pi(i\theta;m_2,t)\,(P_c\,(\tilde{m}_2,0)-P_c\,(m_2,t))\right]\\&+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta(1;m_2,t;\tilde{m}_2)-\delta(2;m_2,t;\tilde{m}_2)\,\Delta_\Theta(\tilde{m}_2,t)\right]\\&+o(\gamma)\end{aligned}\quad. \tag{111}$$

Finally, taking the difference of (110) and (111) and recalling the characterization of $\mathcal{S}$ in Lemma 30 yields the ODE

$$\begin{aligned}&(r+\lambda_{12}+\lambda_{21}+\lambda_a+2\Lambda\,(\eta_0\mu(1\Theta;m_2,t)+\eta_\Theta\mu(20;m_2,t)))\,\epsilon_{1\Theta}(m_2,t)-\dot{\epsilon}_{1\Theta}(m_2,t)\\=\;&\mathcal{S}+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta\,(1;m_2,t;\tilde{m}_2)-\delta\,(2;m_2,t;\tilde{m}_2))\,\epsilon_{1\Theta}\,(\tilde{m}_2,t)\right]+o(\gamma)\end{aligned} \tag{112}$$

for the surplus

$$\begin{aligned}\epsilon_{1\Theta}(m_2,t)&\overset{\Delta}{=}-\Delta_\Theta(m_2,t)-\Delta_0(m_2,t)\\&\overset{(\Delta)}{=}a(10;m_2,t)-a(1\Theta;m_2,t)-a(20;m_2,t)+a(2\Theta;m_2,t).\end{aligned}$$

Defining

$$R(m_2,t)\overset{\Delta}{=}r+\lambda_{12}+\lambda_{21}+\lambda_a+2\Lambda\,(\eta_0\mu(1\Theta;m_2,t)+\eta_\Theta\mu(20;m_2,t))$$

and taking as given the function

$$F(m_2,t)\overset{\Delta}{=}\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta\,(1;m_2,t;\tilde{m}_2)-\delta\,(2;m_2,t;\tilde{m}_2))\,\epsilon_{1\Theta}\,(\tilde{m}_2,t)\right],$$

the solution to the ODE (112) under the “no-bubble” condition

$$\lim_{T\to\infty}e^{-rT}\epsilon_{1\Theta}(m_2,t)=0$$

is

$$\epsilon(m_2,t)=\mathcal{S}\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}\,\mathrm{d}u+\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}F(m_2,u)\,\mathrm{d}u. \tag{113}$$

Finally, I must still check the existence of $F(m_2,t)$. $F(m_2,t)$ must satisfy

$$\begin{aligned}&F(m_2,t)=\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta\,(1;m_2,t;\tilde{m}_2)-\delta\,(2;m_2,t;\tilde{m}_2))\,\epsilon_{1\Theta}\,(\tilde{m}_2,t)\right]\\\Leftrightarrow\;&F(m_2,t)=\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[(1-\delta\,(1;m_2,t;\tilde{m}_2)-\delta\,(2;m_2,t;\tilde{m}_2))\right]\cdot\\&\qquad\cdot\left(\mathcal{S}\textstyle\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}\,\mathrm{d}u+\int_t^\infty e^{-\int_t^u R(m_2,s)\,\mathrm{d}s}F(m_2,u)\,\mathrm{d}u\right).\end{aligned} \tag{114}$$

One checks that the right-hand side of this last equality, seen as the image of the function $u\mapsto F(m_2,u)$ by an operator, satisfies the Blackwell’s sufficient conditions for a contraction (monotonicity and “discounting”). In particular, there is exactly one solution to the equality (114), and this solution must be positive when $\mathcal{S}$ is. Furthermore, this solution is increasing in $\mathcal{S}$ and decreasing in Λ because the right-hand side of (114) is.

Given F, Equation (113) gives the surplus $\epsilon_{1\Theta}$, and inspection shows that the surplus is also positive when $\mathcal{S}$ is, increasing in $\mathcal{S}$, and decreasing in Λ.

Then, Equations (110) and (111) uniquely characterize Δ_Θ and Δ_0, respectively. This can be shown by an argument similar to the one characterizing P_c in Lemma 30. Finally, with $\epsilon_{1\Theta}$, Δ_Θ, and Δ_0, four more arguments similar to the one in Lemma 30 uniquely characterize the "$a(i\theta; m_2, t)$"s. □

Proof 32 (Proof of Proposition 15)**.** I assume that I can write

$$\begin{aligned} P_c(m_2,t) &= P_{c,0}(m_2,t) + r\gamma P_{c,1}(m_2,t) + o(\gamma), \\ \pi(i\theta;m_2,t) &= \pi_{c,0}(i\theta;m_2,t) + r\gamma\pi_{c,1}(i\theta;m_2,t) + o(\gamma), \end{aligned} \tag{115}$$

for differentiable functions $P_{c,1}$ and $\{\pi_1(i\theta;m_2,t\}_{i\theta}$. Injecting (115) into the first order condition (103) for the optimal liquid holdings $\pi(i\theta,m_2,t)$ and recalling the characterization of $P_{c,0}$ and $\{\pi_0(i\theta;m_2,t\}_{i\theta}$ in Proposition 30 yields the equation

$$\begin{aligned} 0 = {} & \dot{P}_{c,1}(m_2,t) - rP_{c,1}(m_2,t) - r\Sigma_{cc}\pi_1(i\theta;m_2,t) \\ & + \lambda_a \operatorname{E}^{M(\tilde{m})}\left[\begin{array}{l} P_{c,1}\left(\tilde{m}_2,0\right) - P_{c,1}\left(m_2,0\right) \\ -r\left(\begin{array}{l} \pi_0(i\theta;m_2,t)\left(P_{c,0}\left(\tilde{m}_2,0\right) - P_{c,0}\left(m_2,0\right)\right) \\ +\delta(i;m_2,t)a_0\left(\bar{i}\theta;\tilde{m}_2,0\right) \\ +\left(1-\delta(i;m_2,t)\right)a_0\left(i\theta;\tilde{m}_2,0\right) \\ -a_0\left(i\theta;m_2,0\right)) \end{array}\right) \\ \quad\cdot\left(P_{c,0}\left(\tilde{m}_2,0\right) - P_{c,0}\left(m_2,0\right)\right) \end{array}\right]. \end{aligned} \tag{116}$$

for $P_{c,1}$ and $\{\pi_1(i\theta;m_2,t\}_{i\theta}$. Now, as

$$S_c = \operatorname{E}^{\mu(i\theta)}\left[\pi(i\theta;m_2,t\right] = \operatorname{E}^{\mu(i\theta)}\left[\pi_0(i\theta;m_2,t\right]$$

it follows that

$$0 = \operatorname{E}^{\mu(i\theta)}\left[\pi_1(i\theta;m_2,t\right].$$

Aggregating Equation (116) across the population then yields

$$\begin{aligned} 0 = {} & \dot{P}_{c,1}(m_2,t) - rP_{c,1}(m_2,t) \\ & + \lambda_a \operatorname{E}^{M(\tilde{m})}\left[\begin{array}{l} P_{c,1}\left(\tilde{m}_2,0\right) - P_{c,1}\left(m_2,0\right) \\ -r\left(\begin{array}{l} S_c\left(P_{c,0}\left(\tilde{m}_2,0\right) - P_{c,0}\left(m_2,0\right)\right)^2 \\ +\left(W\left(\tilde{m}_2,0\right) - W\left(m_2,t\right)\right)\left(P_{c,0}\left(\tilde{m}_2,0\right) - P_{c,0}\left(m_2,0\right)\right) \end{array}\right) \end{array}\right], \end{aligned} \tag{117}$$

with the notation

$$W_0\left(m_2,t\right) \triangleq \operatorname{E}^{\mu(i\theta;m_2,t)}\left[a\left(i\theta;m_2,t\right)\right] \ \left(\overset{(\Delta)}{=} \mu(m_2,t)\cdot a\left(m_2,t\right)\right). \tag{118}$$

for the average certainty equivalent across the population. Combining Equation (105) with Equation (117) then yields

$$\frac{1}{\mathrm{d}t}\left(\frac{\mathrm{E}\left[P_c(m_2,t+\ \mathrm{d}t)\right|(m_2,t)]}{P_c(m_2,t)}-r\right)$$
$$=r\gamma\left(\begin{array}{l}\frac{1}{P_{c,t}}\left(S_c\Sigma_{cc}+\lambda_a\,\mathrm{E}\left[\left(P_{c,0}-P_{c,t}\right)^2\Big|\,(m_2,t)\right]\right)\\+\frac{1}{P_{c,t}}\left(S_d\Sigma_{cd}+\Sigma_{\eta c}\right)\\+\lambda_a\,\mathrm{E}^{m\left(\tilde{h}\right)}\left[\left(\frac{P_c(\tilde{m}_2,0)}{P_c(m_2,t)}-1\right)\left(W\left(\tilde{m}_2,0\right)-W\left(m_2,t\right)\right)\Big|\,(m_2,t)\right]\end{array}\right)+o\left(\gamma\right), \tag{119}$$

which is Expression 50 in the statement.

I still have to characterize the sensitivity of the expected returns

$$\frac{1}{\mathrm{d}t}\left(\frac{\mathrm{E}\left[P_c(m_2,t+\ \mathrm{d}t)\right|(m_2,t)]}{P_c(m_2,t)}-r\right)$$

on the meeting rate Λ. On the right-hand side of (119), only the difference

$$W\left(\tilde{m}_2,0\right)-W\left(m_2,t\right)$$

asymptotically depends on Λ. Hence, I will first looks more carefully at $W(m_2,t)$. It follows from the definition (118) of the average certainty equivalent W and the asymptotic HJB equations (120) that

$$\begin{aligned}&rW(m_2,t)-\mu(m_2,t)\cdot\dot{a}(m_2,t)\\=\;&\mu(m_2,t)\cdot\kappa\left(m_2,t;\pi(m_2,t)\right)\\&+\left(\lambda_{12}\mu(1l;m_2,t)-\lambda_{21}\mu(2l;m_2,t)\right)\Delta_0(m_2,t)\\&+\left(\lambda_{21}\mu(2\Theta;m_2,t)-\lambda_{12}\mu(1\Theta;m_2,t)\right)\Delta_\Theta(m_2,t)\\&+2\Lambda\mu\left(20;m_2,t\right)\mu\left(1\Theta;m_2,t\right)\\&+\lambda_a S_c\left(\mathrm{E}^{M(\tilde{m}_2)}\left[P_c\left(\tilde{m}_2,0\right)\right]-P_c\left(m_2,t\right)\right)\\&+\lambda_a\left(\mathrm{E}^{M(\tilde{m}_2)}\left[W\left(\tilde{m}_2,0\right)\right]-W\left(m_2,0\right)\right)\\&+o(\gamma)\end{aligned}\quad. \tag{120}$$

Rearranging then yields the ODE

$$\left(r+\lambda_a\right)W(m_2,t)-\dot{W}(m_2,t)=\mu(m_2,t)\cdot\kappa(m_2,t)+\lambda_a\,\mathrm{E}^{M(\tilde{m}_2)}\left[W\left(\tilde{m}_2,0\right)\right]$$

for $W(m_2,t)$. Under a "no bubble" condition, the unique solution to this ODE is

$$\begin{aligned}W(m_2,t)=\;&\int_t^{+\infty}e^{-(r+\lambda_a)(u-t))}\mu\cdot\kappa\,\mathrm{d}u\\&+\frac{r+\lambda_a}{r}\int_0^{+\infty}e^{-(r+\lambda_a)u}\,\mathrm{E}^{M(\tilde{m}_2)}\left[\mu\left(\tilde{m}_2,u\right)\cdot\kappa\left(\tilde{m}_2,u\right)\right]\,\mathrm{d}u,\end{aligned}$$

and I can characterize the quantity of interest as

$$\begin{aligned} W(\tilde{m}_2, 0) - W(m_2, t) = & \int_0^{+\infty} e^{-(r+\lambda_a)u} \mu(\tilde{m}_2, u) \cdot \kappa(\tilde{m}_2, u) \, \mathrm{d}u \\ & - \int_t^{+\infty} e^{-(r+\lambda_a)(u-t))} \mu(m_2, u) \cdot \kappa(m_2, u) \, \mathrm{d}u. \end{aligned} \tag{121}$$

Finally, combining Equation (121) with the result of Lemma 30 regarding the "$\kappa(i\theta; m_2, t)$"s, and those of Lemma 8 regarding the asymptotic type distribution yields

$$\begin{aligned} & \partial_{\frac{1}{\Lambda}} \left(W(\tilde{m}_2, 0) - W(m_2, t) \right) + o\left(\frac{1}{\Lambda}\right) + \mathcal{O}(\gamma) \\ = & \int_0^{+\infty} e^{-(r+\lambda_a)u} \delta_\mu(\tilde{m}_2, u) \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \kappa(\tilde{m}_2, u) \, \mathrm{d}u \\ & - \int_t^{+\infty} e^{-(r+\lambda_a)(u-t))} \delta_\mu(m_2, u) \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \kappa(m_2, u) \, \mathrm{d}u \\ = & -\mathcal{S} \int_0^{+\infty} e^{-(r+\lambda_a)u} \left(\delta_\mu(\tilde{m}_2, u) - \delta_\mu(m_2, u) \right) \mathrm{d}u. \end{aligned}$$

Finally, recalling that

$$\delta_\mu(m_2, u) = \frac{\lambda_{12}}{2} \frac{\frac{S_d}{\Theta}}{\mu_2(m_2, t) - \frac{S_d}{\Theta}}$$

is decreasing in $\mu_2(m_2, t)$ over the support of M_2, I conclude that

$$\partial_{\frac{1}{\Lambda}} \left(W(\tilde{m}_2, 0) - W(m_2, t) \right) < 0$$

exactly when

$$\tilde{m}_2 < \mu_2(m_2, t).$$

Combining this last result with Lemma 30 completes the argument. □

B.3 Technical results

Lemma 33. *Consider a smooth map $H : \Omega \to \Omega$ for some $\Omega \subset \mathbb{R}^d$. If for any $i = 1, \ldots, d$, there exists a $\eta < 1$ so that*

$$\sum_{j=1}^{d} \left| \frac{\partial H_i}{\partial x_j} \right| \leq \eta,$$

then H is a contraction in l_∞ and has a unique fixed point.

Proof of Lemma 33. Fix $x_1, x_2 \in \Omega$ and define, for $t \in [0,1]$,

$$x(t) \triangleq x_1 + t(x_2 - x_1).$$

Then, for any $i \in \{1, \ldots, d\}$,

$$\begin{aligned}
|H_i(x_2) - H_i(x_1)| &= \left| \int_0^1 \sum_j \frac{\partial H_i}{\partial x_j}(x(t))(x_j^2 - x_j^1)\, \mathrm{d}t \right| \\
&\leq \int_0^1 \sum_j \left| \frac{\partial H_i}{\partial x_j}(x(t)) \right| \left| x_j^2 - x_j^1 \right| \, \mathrm{d}t \\
&\leq \sum_j \left(\partial_{x_j} H\right) \max_j |x_{2j} - x_{1j}| \int_0^1 \, \mathrm{d}t \\
&\leq \eta \|x^2 - x^1\|_{l_\infty}.
\end{aligned}$$

The last claim follows from the Contraction Mapping Theorem (see (Stokey and Lucas, 1989, Theorem 3.2, p.50)). □

Proposition 34. *Let us consider the system of equations*

$$0 = r\beta_k + \sum_{j \neq k} \kappa_{kj} e^{\beta_k - \beta_j} + c_k \triangleq F_k(\beta), \;\; k \in \{1, \ldots, d\} \tag{122}$$

with the unknowns $\beta \equiv (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d$. Then, this system admits a unique solution and this solution is monotone decreasing in the components of K and c.

Proof of Proposition 34. I write β_{-k} for the vector of β without β_k.

First note that there exists a unique smooth function

$$G_k \; = G_k(\beta_{-k}, K_k, c_k)$$

for which $\beta = G_k(\beta_{-k}, K_k, c_k)$ is the unique solution to

$$r\beta + \sum_{j \neq k} \kappa_{kj}\, e^{\beta - \beta_j} + c_k = 0.$$

Furthermore, G_k is monotone increasing in the components of β_{-k}, and monotone decreasing in κ_{kj} and c_k for all $j \neq k$.

Then, I show that the functions G_k define a contraction by applying Lemma 33. Namely, I first show that G maps a compact set into itself. Let me choose two real numbers $L < U$, and assume that for any $k \in \{1, \ldots, d\}$,

$$\beta_k \in [L, U]^{d-1}.$$

For a given k, let me further define two functions, F_k^L and F_k^U, that bound the function F_k defined in (122). Namely,

$$\begin{aligned} r\beta + \sum_{j\neq k} \kappa_{kj} e^{\beta - U} + c_k \triangleq & F_k^L(\beta) \\ \leq & F_k(\beta) \\ \leq & F_k^U(\beta) \triangleq r\beta + \sum_{j\neq k} \kappa_{kj} e^{\beta - L} + c_k. \end{aligned}$$

Now, due to the monotonicity of $F_k(\cdot, \beta_{-k})$, if

$$0 \leq F_k^L(U) = rU + \sum_{j\neq k} \kappa_{jk} + c_k \tag{123}$$

and

$$0 \geq F_k^U(L) = rL + \sum_{j\neq k} \kappa_{jk} + c_k \tag{124}$$

then

$$G_k(\beta_{-k}) \in [L, U].$$

But both (123) and (124) will hold for all $k \in \{1, \ldots, d\}$ as soon as

$$U \geq \max_{k\in\{1,\ldots,d\}} \frac{-1}{r} \left(\sum_{j\neq k} \kappa_{jk} + c_k \right)$$

and

$$L \leq \min_{k\in\{1,\ldots,d\}} \frac{-1}{r} \left(\sum_{j\neq k} \kappa_{jk} + c_k \right).$$

Now, by the Implicit Function Theorem,

$$\frac{\partial G_k(\beta_{-k})}{\partial \beta_j} = \frac{\kappa_{kj} e^{G_k(\beta_{-k}) - \beta_j}}{r + \sum_{j\neq k} \kappa_{kj} e^{G_k(\beta_{-k}) - \beta_j}},$$

which can be bounded strictly below 1, uniformly in $\beta_{-k} \in [L, U]^{d-1}$, for L and U chosen as above. But then, Lemma 33 ensures the existence and uniqueness of a fixed point on $[L, U]^d$. Finally, as $-L$ and U can be chosen arbitrarily large, the existence and uniqueness on $\mathbb{R}^d$ hold.

Monotonicity follows because

$$\beta^* = \lim_{n\to\infty} G^n(\beta_0)$$

for any fixed β_0 and G is monotone. □

C Verification argument

I intend to show that the HJB equations (13) actually describe an optimal behaviour.

Being more specific, on the one hand, a given agent with wealth w and type $i\theta$ maximizes

$$V(w, i\theta) \triangleq \max_{\tilde{c}} \mathrm{E}\left[\int_0^\infty e^{-\rho t}\left(-e^{-\gamma \tilde{c}_s}\right)\, \mathrm{d}s \middle| w_0 = w, i_0\theta_0 = i\theta\right], \tag{125}$$

under the conditions that follow.

- The budget constraint

$$\mathrm{d}w_t = rw_t\, \mathrm{d}t - \bar{c}_t\, \mathrm{d}t + \mathrm{d}\eta_t + \theta_t\, \mathrm{d}D_{dt} + \pi_t\left(\mathrm{d}D_{ct} - rP_c\, \mathrm{d}t\right) - P_d\, \mathrm{d}\theta_t$$

  holds for a liquid holding process taking values in $[-K, K]$, with K positive and large.[59]

- The price P_d is the outcome of a bargaining with another agent;
- For any $T > 0$,

$$\mathrm{E}^{w,i\theta}\left[\int_0^T \left(e^{-\rho u} e^{-r\gamma w_u}\right)^2\, \mathrm{d}u\right] < +\infty \tag{126}$$

  and

$$\lim_{T\to\infty} e^{-\rho T}\, \mathrm{E}^{w,i\theta}\left[e^{-\gamma W_T}\right] = 0. \tag{127}$$

On the other hand, the HJB equation for the problem above is

$$\begin{aligned}
\rho V(w, i\theta) = \sup_{\tilde{c},\tilde{\pi}}\, & U(\tilde{c}) \\
& + \frac{\partial V}{\partial w}(w, i\theta)\left(rw - \tilde{c} + m_\eta + \theta m_d + \tilde{\pi}\left(m_c - rP_c\right)\right) \\
& + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(w, i\theta)\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}^* \\
& + \lambda_{i\bar{i}}\left(V(w, \bar{i}\theta) - V(w, i\theta)\right) \\
& + 2\Lambda\, \mathrm{E}^{\mu(b)}\left[\mathbf{1}_{\text{surplus}}\left(V(w - (\bar{\theta} - \theta)P_d, i\bar{\theta})) - V(w, i, \theta)\right)\right],
\end{aligned} \tag{128}$$

and Proposition 5 shows that there exists a unique solution of the form

$$\tilde{V}(w, i\theta) = -\exp\left(-r\gamma\left(w + a(i\theta) + \bar{a}\right)\right)$$

to (128). It remains to show that the candidate $\tilde{V}$ is the solution to the problem (125). This is the object of the next proposition.

Proposition 35. *If the risk aversion γ is small enough, the function $\tilde{V}$ is the solution to the HJB equations* (128) *and the associated consumption and investment strategies are optimal.*

[59] See footnote 11 for conditions on K.

Proof of Proposition 35. My argument comprises four steps.

- Lemma 36 shows that no admissible strategy can achieve an expected utility higher than $\tilde{V}$.
- Lemma 37 shows that the strategy dictated by $\tilde{V}$ is admissible when the risk aversion γ is small enough.
- Lemma 38 shows the strategy dictated by the HJB equations yields an expected utility equal to $\tilde{V}$.

I first show that $\tilde{V}$ represents an upper bound on the attainable expected utilities.

Lemma 36. *If all the agents believe that their value function is given by $\tilde{V}$, then, for any admissible consumption strategy $\tilde{c}$ financed by the trading strategy $\tilde{\pi}$,*

$$\tilde{V}(w, i\theta) \geq \sup_{\tilde{c}} \mathrm{E}^{w,i\theta}\left[\int_0^\infty e^{-\rho u} U(c_u)\, \mathrm{d}u\right].$$

Proof. First note that the beliefs regarding the value functions will already fix the outcome of the Nash bargaining, meaning that both the price P_d of the illiquid asset and the cross-sectional distribution of types μ are fixed.

Let me choose an admissible consumption strategy c financed by the trading strategy π, and a time $T > 0$. Recalling the budget constraint,

$$\begin{aligned}
&\mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\, \mathrm{d}u + e^{-\rho T}\tilde{V}(w_T, i_T\theta_T)\right] \\
=&\mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\, \mathrm{d}u + \tilde{V}(w_0, i_0\theta_0) + \int_0^T \mathrm{d}\left(e^{-\rho u}\tilde{V}(w_u, i_u\theta_u)\right)\right] \\
=&\mathrm{E}\left[\begin{array}{l} \tilde{V}(w_0, i_0\theta_0) + \int_0^T e^{-\rho u} U(c_u)\, \mathrm{d}u \\ + \int_0^T \left(-\rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u)\right)\, \mathrm{d}u + \int_0^T e^{-\rho u}\, \mathrm{d}\left(\tilde{V}(w_u, i_u\theta_u)\right) \end{array}\right] \\
=&\mathrm{E}\left[\begin{array}{l} \tilde{V}(w_0, i_0\theta_0) \\ + \int_0^T e^{-\rho u} \left(\begin{array}{ll} & U(c_u)\, \mathrm{d}u \\ - & \rho\tilde{V}(w_u, i_u\theta_u)\, \mathrm{d}u \\ + & \frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\left(\begin{array}{l} (rw_u - c_u)\, \mathrm{d}u \\ + \mathrm{d}e_u \\ +\theta_u\, \mathrm{d}D_{du} \\ +\pi_u\left(\mathrm{d}D_{cu} - rP_c\, \mathrm{d}u\right) \end{array}\right) \\ + & \frac{1}{2}\frac{\partial^2\tilde{V}}{\partial w^2}(w_u, i_u\theta_u)\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}^*\, \mathrm{d}u \\ + & \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta)\right)\, \mathrm{d}N^i_u \\ + & \max\left\{0, \begin{array}{c} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{array}\right\}\, \mathrm{d}N^m_u \end{array}\right) \end{array}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left[\begin{array}{l}
\tilde{V}(w_0, i_0\theta_0) \\
+\int_0^T e^{-\rho u}\left(\begin{array}{ll}
 & U(c_u)\,\mathrm{d}u \\
- & \rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u)\,\mathrm{d}u \\
+ & \frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\begin{pmatrix} rw_u - c_u \\ +m_e \\ +\theta_u m_d \\ +\pi_u\,(m_c - rP_c)\end{pmatrix}\mathrm{d}u \\
+ & \frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\begin{pmatrix}\alpha_\eta(i_u) \\ \alpha_d(i_u) + \theta_u\sigma_d \\ \alpha_d(i_u) + \pi_u\sigma_c\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}Z_u \\ \mathrm{d}B_{d,u} \\ \mathrm{d}B_{c,u}\end{pmatrix} \\
+ & \frac{1}{2}\frac{\partial^2\tilde{V}}{\partial w^2}(w, i\theta)\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}^*\,\mathrm{d}u \\
+ & \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right)\,\mathrm{d}N_u^i \\
+ & \max\left\{0, \begin{array}{c}\tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u)\end{array}\right\}\,\mathrm{d}N_u^m
\end{array}\right)
\end{array}\right] \\
&\triangleq (*),
\end{aligned}
$$

with N^i being the idiosyncratic jump process driving the exposure changes and N^m the jump process defining the meeting times on the OTC market.

Now, defining

$$
K_1 \triangleq (r\gamma)^2 \sup_{\substack{i\theta \\ \tilde{\pi}\in[-K,K]}} e^{-2(a(i\theta)+\bar{a})}\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta & \tilde{\pi}\end{pmatrix}^* \in \mathbb{R},
$$

and recalling the admissibility condition (127) on (c, π), I may write

$$
\begin{aligned}
&\mathrm{E}\left[\left(\int_0^t e^{-\rho u}\frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\begin{pmatrix}\alpha_\eta(i_u) \\ \alpha_d(i_u) + \theta_u\sigma_d \\ \alpha_d(i_u) + \pi_u\sigma_c\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}Z_u \\ \mathrm{d}B_{d,u} \\ \mathrm{d}B_{c,u}\end{pmatrix}\right)^2\right] \\
&= \mathrm{E}\left[\int_0^t \left(e^{-\rho u}\frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u)\right)^2\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}\Sigma_i\begin{pmatrix}1 & \theta_u & \pi_u\end{pmatrix}^*\,\mathrm{d}u\right] \\
&\le K_1\,\mathrm{E}\left[\int_0^t e^{-2(\rho u + r\gamma w_u)}\,\mathrm{d}u\right] \\
&< \infty.
\end{aligned}
$$

In particular, in $(*)$, the stochastic integrals against the Brownian motions are true martingales, and their expected values equal zero.

I now turn to the stochastic integrals against the Poisson processes. Keeping in mind the admissibility condition (126),

$$
\int_0^t \left|\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right|\,\mathrm{d}u
$$

$$
\begin{aligned}
&\leq \sup_{i\theta} \left| e^{-r\gamma(a(i_u\theta_u)+\bar{a})} - e^{-r\gamma(a(i_u\theta_u)+\bar{a})} \right| \int_0^t e^{-r\gamma w_u} \, \mathrm{d}u \\
&< \infty.
\end{aligned}
$$

But then, using a classical result (see, for example, Brémaud (1981)[Lemma C4, p.235]),

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^T e^{-\rho u} \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right) \, \mathrm{d}N_u^i\right] \\
=&\mathrm{E}\left[\int_0^T e^{-\rho u} \lambda_{i_u\bar{i}_u} \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right) \, \mathrm{d}u\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^T e^{-\rho u} \max\left\{0, \begin{array}{c} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{array}\right\} \, \mathrm{d}N_u^\theta\right] \\
=&\mathrm{E}\left[\int_0^T e^{-\rho u} 2\lambda\mu(\bar{i}_u\bar{\theta}_u) \max\left\{0, \begin{array}{c} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{array}\right\} \, \mathrm{d}u\right].
\end{aligned}
$$

I may thus write

$$
(*) = \mathrm{E}\left[\begin{array}{l}
\tilde{V}(w_0, i_0\theta_0) \\
+\int_0^T e^{-\rho u} \left(\begin{array}{ll}
 & U(c_u) \\
- & \rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u) \\
+ & \frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u) \left(\begin{array}{l} rw_u - c_u \\ +m_\eta \\ +\theta_u m_d \\ +\pi_u (m_c - rP_c) \end{array}\right) \\
+ & \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial w^2}(w, i\theta) \begin{pmatrix} 1 & \theta_u & \pi_u \end{pmatrix} \Sigma_i \begin{pmatrix} 1 & \theta_u & \pi_u \end{pmatrix}^* \\
+ & \lambda_{i_u\bar{i}_u} \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right) \\
+ & 2\Lambda\mu\left(\bar{i}_u\bar{\theta}_u\right) \max\left\{0, \begin{array}{c} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{array}\right\}
\end{array}\right) \mathrm{d}u
\end{array}\right]
$$

$$
\leq \mathrm{E}\left[\begin{array}{l}
\tilde{V}(w_0, i_0\theta_0) \\
+\int_0^T e^{-\rho u} \sup_{\tilde{c},\tilde{\pi}} \left(\begin{array}{ll}
 & U(\tilde{c}) \\
- & \rho e^{-\rho u}\tilde{V}(w_u, i_u\theta_u) \\
+ & \frac{\partial \tilde{V}}{\partial w}(w_u, i_u\theta_u) \left(\begin{array}{l} rw_u - \tilde{c} \\ +m_\eta \\ +\theta_u m_d \\ +\tilde{\pi} (m_c - rP_c) \end{array}\right) \\
+ & \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial w^2}(w_u, i_u\theta_u) \begin{pmatrix} 1 & \theta_u & \tilde{\pi} \end{pmatrix} \Sigma_i \begin{pmatrix} 1 & \theta_u & \tilde{\pi} \end{pmatrix}^* \\
+ & \lambda_{i_u\bar{i}_u} \left(\tilde{V}(w_u, \bar{i}_u\theta_u) - \tilde{V}(w_u, i_u\theta_u)\right) \\
+ & 2\Lambda\mu\left(\bar{i}_u\bar{\theta}_u\right) \max\left\{0, \begin{array}{c} \tilde{V}(w_u - (\bar{\theta}_u - \theta_u)P_d, i_u\theta_u) \\ -\tilde{V}(w_u, i_u\theta_u) \end{array}\right\}
\end{array}\right) \mathrm{d}u
\end{array}\right]
$$

$$=\tilde{V}(w_0, i_0\theta_0).$$

Taking things together, this means that, for any $T > 0$,

$$\tilde{V}(w_0, i_0\theta_0) \geq \mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\,\mathrm{d}u\right] + e^{-\rho T}\,\mathrm{E}\left[\tilde{V}(w_T, i_T\theta_T)\right].$$

Letting T become arbitrarily large in this last expression, recalling the admissibility condition (127) satisfied by the strategy (c, π), and realizing that the process

$$(a(i_t\theta_t))_{t\geq 0}$$

can only take one of four finite values, yields

$$\begin{aligned}\tilde{V}(w_0, i_0\theta_0) \geq & \lim_{T\to\infty} \mathrm{E}\left[\int_0^T e^{-\rho u} U(c_u)\,\mathrm{d}u\right] + e^{-\rho T}\,\mathrm{E}\left[\tilde{V}(w_T, i_T\theta_T)\right] \\ \geq & \,\mathrm{E}\left[\int_0^\infty e^{-\rho u} U(c_u)\,\mathrm{d}u\right] + \lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[-e^{-r\gamma w_T}\right] \sup_{i\theta} e^{-r\gamma(a(i\theta)+\bar{a})} \\ = & \,\mathrm{E}\left[\int_0^\infty e^{-\rho u} U(c_u)\,\mathrm{d}u\right].\end{aligned}$$

As the consumption and trading strategies were arbitrary, this concludes. □

I now propose a condition under which the strategy dictated by the HJB equations is admissible.

Lemma 37. *For γ small enough, the strategy $(\hat{c}_t, \hat{\pi}_t)$ dictated by the optimization in the HJB equation is admissible.*[60]

Proof. The candidate strategy must satisfy two admissibility properties. The first one is (126), meaning

$$\mathrm{E}\left[\int_0^T \left(e^{-\rho u} e^{-r\gamma \hat{w}_u}\right)^2\,\mathrm{d}u\right] < \infty.$$

Now, from Proposition 4, the optimal consumption policy is

$$\hat{c}(i\theta, w) = r\,(w + a(i\theta) + \bar{a}) - \frac{1}{\gamma}\log(r),$$

and the resulting wealth dynamics are

$$\mathrm{d}\hat{w}_t = \left(-r\,(a(i\theta) + \bar{a}) + \frac{1}{\gamma}\log(r)\right)\,\mathrm{d}t + \,\mathrm{d}\eta_t + \theta_t\,\mathrm{d}D_{d,t} + \hat{\pi}_t\,(\,\mathrm{d}D_{d,t} - rP_d\,\mathrm{d}t) - P_d\,\mathrm{d}\theta_t.$$

I may thus write

$$\hat{w}_t - w_0$$

[60]I let the risk aversion coefficient go to zero, $\gamma \to 0$, and simultaneously scale up the diffusion coefficients, as described in Equation (48).

$$= \int_0^t \left(-r\left(a(i\theta) + \bar{a}\right) + \frac{1}{\gamma}\log(r) + m_\eta + \theta_u m_d + \hat{\pi}_u \left(m_c - rP_c\right) \right) \,\mathrm{d}u - P_d\left(\theta_T - \theta_0\right)$$
$$+ \int_0^t \begin{pmatrix} \alpha_d(i_u) + \theta_u \sigma_d \\ \alpha_c(i_u) + \pi_u \sigma_c \\ \alpha_\eta(i_u) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}B_{d,u} \\ \mathrm{d}B_{c,u} \\ \mathrm{d}Z_u \end{pmatrix}.$$

In particular, recalling that the Brownian motions and Poisson processes are independent, and defining, for $t \geq 0$,

$$m_t \triangleq \int_0^t \left(-r\left(a(i\theta) + \bar{a}\right) + \frac{1}{\gamma}\log(r) + m_\eta + \theta_u m_d + \hat{\pi}_u \left(m_c - rP_c\right) \right) \,\mathrm{d}u - P_d\left(\theta_t - \theta_0\right),$$

and

$$s_t^2 \triangleq \int_0^t \begin{pmatrix} 1 & \theta_u & \hat{\pi}(i_u\theta_u) \end{pmatrix} \Sigma_i \begin{pmatrix} 1 \\ \theta_u \\ \hat{\pi}(i_u\theta_u) \end{pmatrix} \,\mathrm{d}u,$$

I know that the distribution of the wealth conditional on the history of the correlation shocks and OTC trades is

$$\mathcal{L}\left(\hat{w}_t \middle| (i_u\theta_u)_{0 \leq u \leq t} \right) = \mathcal{N}\left(m_t, s_t^2\right).$$

Further, for $t \geq 0$, and defining the two constants

$$K_2 \triangleq \min_{\substack{i\theta \\ \pi \in [-K, K]}} \left\{ -ra(i\theta) + \theta m_d + \pi\left(m_c - rP_c\right) \right\}$$

and

$$K_3 \triangleq \sup_{i\theta} \begin{pmatrix} 1 & \theta & \pi(i\theta) \end{pmatrix} \Sigma_i \begin{pmatrix} 1 \\ \theta \\ \pi(i\theta) \end{pmatrix},$$

I can write both

$$m_t \geq t\left(K_2 + \frac{1}{\gamma}\log(r) - r\bar{a} + m_\eta \right) - |P_d|\,\Delta_\theta,$$

and

$$s_t^2 \leq tK_3.$$

As a result,

$$\begin{aligned} \mathrm{E}\left[\int_0^T \left(e^{-\rho u} e^{-r\gamma\hat{w}_u}\right)^2 \,\mathrm{d}u \right] &= \int_0^T e^{-2\rho u}\, \mathrm{E}\left[e^{-2r\gamma\hat{w}_u} \right] \,\mathrm{d}u \\ &= \int_0^T e^{-2\rho u}\, \mathrm{E}\left[\mathrm{E}\left[e^{-2r\gamma\hat{w}_u} \middle| (i_v\theta_u)_{0 \leq v \leq u} \right] \right] \,\mathrm{d}u \\ &= \int_0^T e^{-2\rho u}\, \mathrm{E}\left[e^{-2r\gamma m_u + 2(r\gamma)^2 s_u^2} \right] \,\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T e^{-2\rho u-2r\gamma u\left(K_2+\frac{1}{\gamma}\log(r)-r\bar{a}+m_\eta\right)+2r\gamma|P_d|\Delta_\theta+2u(r\gamma)^2K_3}\,\mathrm{d}u\\
&\leq e^{2r\gamma|P_d|\Delta_\theta}\int_0^T e^{-2u\left(\rho+r\gamma\left(K_2+\frac{1}{\gamma}\log(r)-r\bar{a}+m_\eta\right)-(r\gamma)^2K_3\right)}\,\mathrm{d}u\\
&<\infty.
\end{aligned}
$$

I must still show that the candidate policy satisfies the transversality condition (127), meaning that

$$
\lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\right]=0.
$$

The argument is similar to the one in the first part. Namely, for a given $T>0$,

$$
\begin{aligned}
e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\right] &= e^{-\rho T}\,\mathrm{E}\left[\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\,\middle|\,(i_s\theta_s)_{0\leq s\leq t}\right]\right]\\
&= \mathrm{E}\left[e^{-\rho T-r\gamma m_T+\frac{1}{2}(r\gamma)^2 s_T^2}\right].
\end{aligned}
\tag{129}
$$

Now, recalling the choice

$$
\bar{a}\overset{(\Delta)}{=}\frac{1}{r\gamma}\left(-1+\frac{\rho}{r}+\gamma m_e+\log(r)\right)
$$

in (20), the exponent on the right-hand side of (129) is

$$
\begin{aligned}
&-\rho T-r\gamma m_T+\frac{1}{2}(r\gamma)^2 s_T^2\\
=&\int_0^T \left(\begin{array}{c}-\rho-r\gamma\left(\frac{1}{\gamma}\log(r)-r\left(a(i_u\theta_u)+\bar{a}\right)+m_\eta+\theta_u m_d+\hat{\pi}_u\left(m_c-rP_c\right)\right)\\ +\frac{1}{2}(r\gamma)^2\begin{pmatrix}1&\theta_u&\hat{\pi}(i_u\theta_u)\end{pmatrix}\Sigma_i\begin{pmatrix}1&\theta_u&\hat{\pi}(i_u\theta_u)^*\end{pmatrix}\end{array}\right)\mathrm{d}u\\
&-r\gamma P_d\left(\theta_T-\theta_0\right)\\
=&\int_0^T r\gamma\left(-\frac{1}{\gamma}+ra\left(i_u\theta_u\right)-\kappa(i_u\theta_u)\right)\mathrm{d}u-r\gamma P_d\left(\theta_T-\theta_0\right).
\end{aligned}
\tag{130}
$$

Now, recall that the "$\kappa(i\theta)$"s are independent of γ in the asymptotic case defined by the equations (47) and (48). Hence, for a small enough γ, there exists an $\epsilon>0$ for which

$$
\int_0^T\left(-\frac{1}{\gamma}+ra(i_u\theta_u)-\kappa(i_u\theta_u)\right)\mathrm{d}u\leq\int_0^T-\epsilon\,\mathrm{d}u=-\epsilon T.
\tag{131}
$$

Finally, combining (129), (130), and (131),

$$
0\leq\lim_{T\to\infty}e^{-\rho T}\,\mathrm{E}\left[e^{-r\gamma\hat{w}_T}\right]\leq\lim_{T\to\infty}e^{r\gamma|P_d|\Delta_\theta}e^{-r\gamma\epsilon T}=0,
$$

as stated. □

The last step is to show that the beliefs are rational. In other words, I must show that the strategy dictated by $\tilde{V}$ and the HJB equations indeed generates an expected utility from consumption equal to $\tilde{V}$.

Lemma 38. *Assuming that γ is small enough, in the sense of Lemma 37, and writing $(\hat{c}, \hat{\pi})$ for the strategy dictated by the HJB equations, then*

$$\tilde{V}(w, i\theta) = \mathrm{E}\left[\int_0^\infty e^{-\rho u} U(\hat{c}_u) \middle| w_0 = w, i_0\theta_0 = i\theta\right].$$

Proof. Thanks to the admissibility of the candidate policy, first, the process

$$\left(\int_0^t e^{-\rho u} U(c_u)\,\mathrm{d}u + e^{-\rho u}\tilde{V}(w_u, i_u\theta_u)\right)_{t\geq 0}$$

is a martingale and, second,

$$\lim_{T\to\infty} e^{-\rho T}\,\mathrm{E}\left[-e^{-r\gamma w_T}\right] = 0.$$

One may then conclude that

$$\tilde{V}(w_0, i_0\theta_0) = \mathrm{E}\left[\int_0^T e^{-\rho u} U(\hat{c}_u)\,\mathrm{d}u\right],$$

by an argument similar to the one in the proof of Lemma 36. □

This concludes the proof of Proposition 35. □

Table 1: Baseline parameter values.

notation	parameter	value
S_c	supply of the liquid asset	0
S_d	supply of the asset traded OTC	0.8
η_Θ, η_0	bargaining powers	$\frac{1}{2}$
λ_{21}	arrival rate of idiosyncratic liquidity shocks	$\frac{1}{5}$
λ_{12}	recovery rate from a liquidity shocks	5
Λ	meeting rate	50
r	risk-free rate	0.037
m_c	expected payouts of the liquid asset	0.05
m_d	expected payouts of the asset traded OTC	0.05
(a_c, b_c)	exposures of the liquid asset	$(1.0000, -0.0016)$
(a_d, b_d)	exposures of the asset traded OTC	$(0.1022, -0.0002)$
(a_1, b_1)	exposures of the endowment for the investors of type 1	$(9.4718, -0.0150)$
(a_2, b_2)	exposures of the endowment for the investors of type 2	$(-0.5017, 0.0008)$
γ	coefficient of absolute risk aversion	2
Θ	holdings in the asset traded OTC	1

Choice of the parameters The supply of the illiquid asset, the holdings size Θ, and the dynamics of the idiosyncratic shocks are taken from Duffie et al. (2007). The liquid asset is understood to be a derivative and its net supply is zero. The risk-free rate and expected payouts of the assets are the same as in Gârleanu (2009) (the calibration in Gârleanu (2009) is itself based on Campbell and Kyle (1993) and Lo et al. (2004)). The baseline meeting intensity is within the standard range and corresponds an average of one meeting per week. The risk-aversion is chosen within the standard range. The exposures of the assets and endowments are chosen to satisfy the following conditions.

1. The profile of the illiquid asset maximizes the reservation value of the illiquid asset in a setting without liquid asset, conditionally on the exposures of the endowments. This maximization captures, in a reduced form, the strong clientèle effects on OTC markets.

2. The two risky assets should have an expected return of approximately 5%.

3. The 2-investors buy the illiquid asset both before and after the opening of the liquid market. See the discussion after Proposition 16.

4. The two risky assets should have the same price for the baseline parameter values. This is for ease of visualization only.

The values of the asset and endowment exposures are then found numerically.

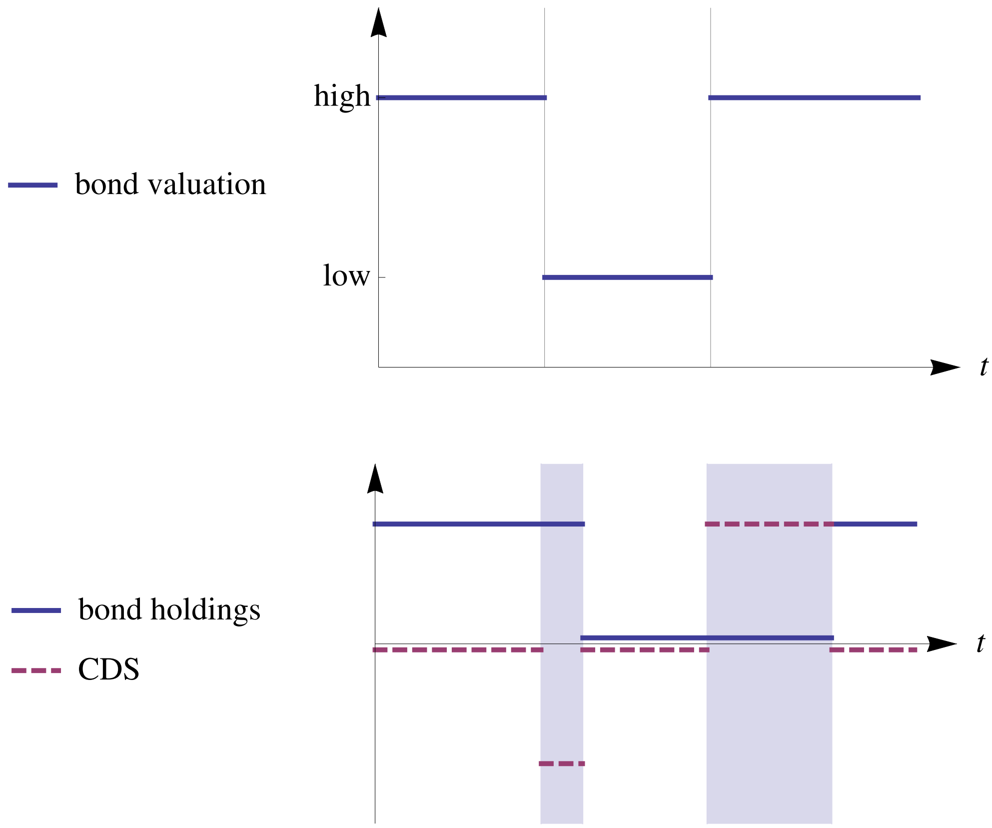

Figure 1: The first panel represents the subjective valuation of an asset traded OTC, say a bond, by a given investor and how this valuation changes over time. The second panel represents this investor's holdings in the illiquid asset (solid line) and a more liquid security offering a similar exposure (dashed line). If the illiquid asset is a bond, the liquid security could be a CDS (as a protection seller). The shaded areas corresponds to the periods during which the investor is searching for a counter-party on the OTC market. During these periods, the investor hedges her sub-optimal exposure to the illiquid asset by trading the liquid asset. These plots are illustrative and not based on the parameters in Table 1.

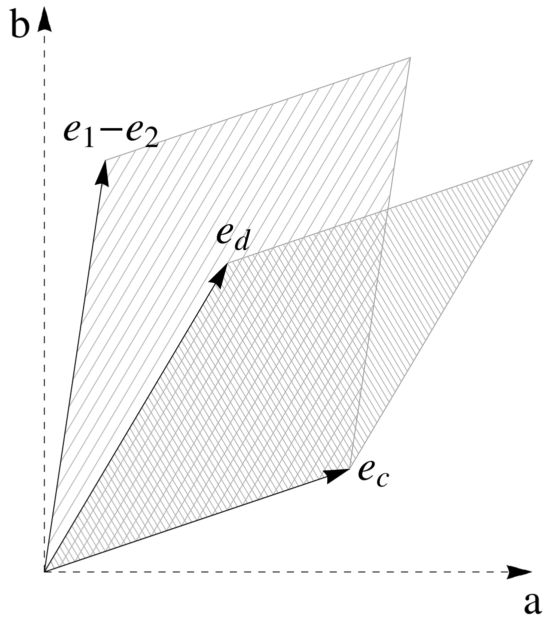

Figure 2: This plot represents the vector of exposures of the liquid asset (e_c) and of the illiquid asset (e_d), along with the differences of the exposures between the two types of investors ($e_1 - e_2$). The horizontal axis measures the exposure to the aggregate risk a and the vertical axis measures the exposure to the aggregate risk b. The surface of the quadrangle with the narrow dash is $|\det(e_d : e_c)|$ and measures how orthogonal the exposures of the two assets are. The surface of the quadrangle with the broad dash is $|\det(e_1 - e_2 : e_c)|$ and measures how orthogonal the risk profile of the liquid asset is to the profile that would be optimal in terms of risk-sharing. The two quadrangles intersect when $\det(e_d : e_c)\det(e_1 - e_2 : e_c) > 0$, which is the condition appearing in Proposition 10 and defining the trading pattern on the OTC market.

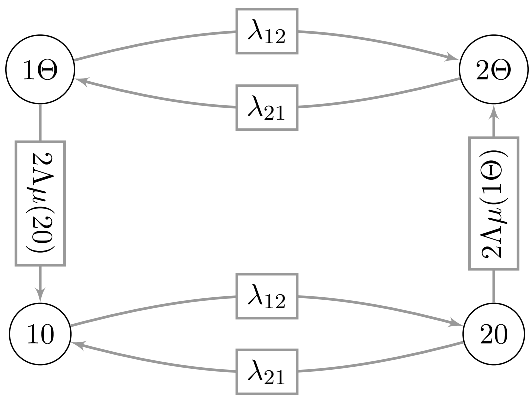

Figure 3: Each vertex is a type of investors (for a type $i\theta$, $i \in \{1, 2\}$ is the type of exposure and $\theta \in \{0, \Theta\}$ are the holdings in the illiquid asset). Each arrow indicates a flow between types and the number on each arrow is the corresponding transition intensity for a given investor. These flow are valid under Assumption 6 and corresponds to the flow equations (25).

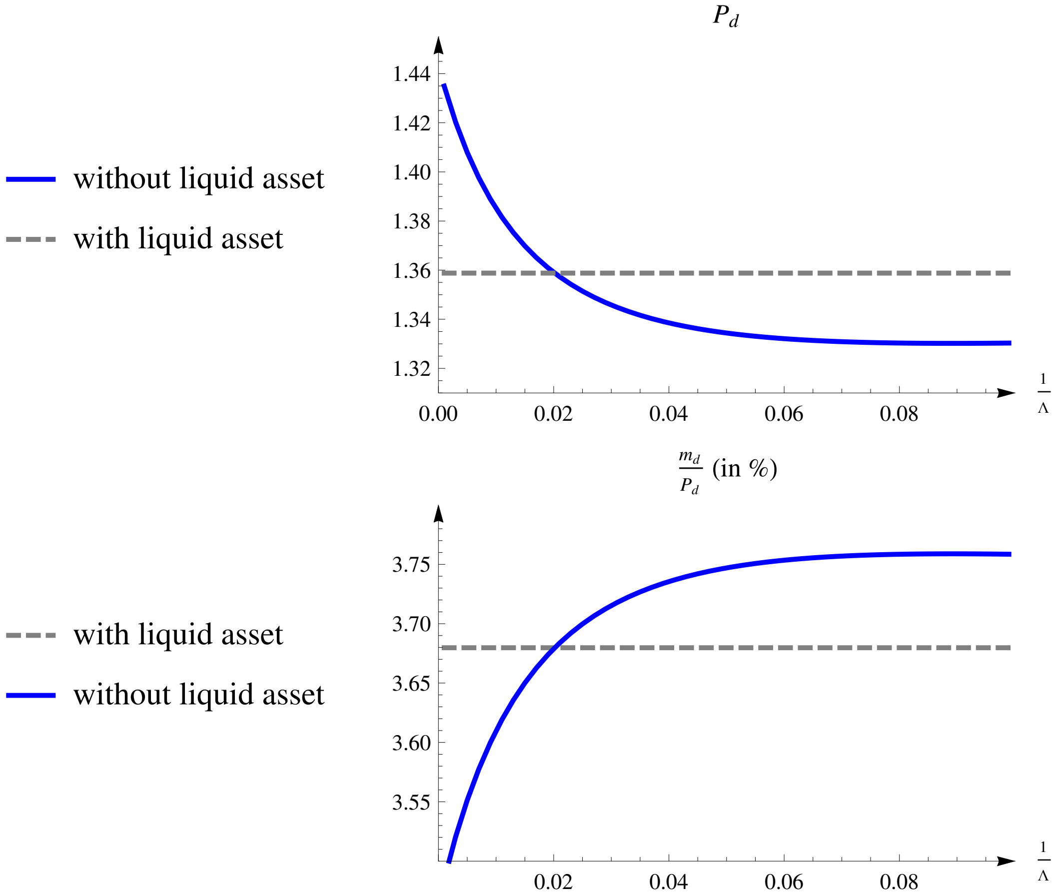

Figure 4: The upper panel is a plot of the price bargained on the OTC market as a function of the meeting intensity on the OTC market. The continuous line is the price when the OTC market is the only market in the economy, whereas the dashed line is the price when investors can trade both on the liquid and on the OTC market. The lower panel plots the expected returns on the illiquid assets, but is otherwise similar to the upper one. The parameter values for this plot are in Table 1.