Corporate Liquidity Management under Moral Hazard*

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March 15, 2019

Abstract

We present a model of liquidity management and financing decisions under moral hazard in which a firm accumulates cash to forestall liquidity default. When the cash balance is high, a tension arises between accumulating more cash to reduce the probability of default and providing incentives for the manager. When the cash balance is low, the firm hedges against liquidity default by transferring cash flow risk to the manager via high powered incentives. Under mild moral hazard, firms with more volatile cash flows tend to transfer less risk to the manager and hold more cash. In contrast, under severe moral hazard, an increase in cash-flow volatility exacerbates agency cost, thereby reducing firm value, overall hedging and in particular precautionary cash-holdings. Agency conflicts lead to endogenous, state-dependent refinancing costs related to the severity of the moral hazard problem. Financially constrained firms pay low wages and instead promise the manager large rewards in case of successful refinancing.

*We thank Simon Board, Sebastian Gryglewicz, Julien Hugonnier and Erwan Morellec for useful comments. We also thank seminar audiences at University of Houston, EPFL, and Kellogg Northwestern. A previous version of this paper was circulated under the title "Cash and Dynamic Agency".

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1 Introduction

A firm's current owners often have limited wealth and liability, and must therefore raise financing in illiquid markets to forestall running out of cash. Consequently, firms often accumulate internal cash balances to avert such a liquidity default. At the same time, accumulating a large cash balance can cause an agency problem to occur, because such balances create a larger pool from which a firm's manager can divert cash. In the standard principal-agent model in corporate finance, for example, DeMarzo and Sannikov (2006), a firm's owners have deep pockets and can costlessly transfer cash into the firm at any moment to cover negative cash flow shocks. The presence of liquidity management and default in such a model then purely pertains to providing incentives to the firm's manager and does not directly speak to the accumulation of cash balances.

We introduce a model in which a firm's shareholders face a trade-off between accumulating cash to prevent liquidity default and optimally providing incentives to the firm's risk-averse manager. The firm's shareholders have limited liability, cannot transfer cash into the firm after inception, and have only occasional refinancing opportunities. As a consequence, they hedge against liquidity based default by optimally managing internal cash balances.

In the model, the firm requires a manager to operate. This manager can inefficiently divert from both the flow and stock of cash within the firm and therefore requires incentives. The manager has constant absolute risk averse (CARA) preferences, while the shareholders are risk neutral. Nevertheless, due to the potential for liquidity default, the shareholders are effectively risk averse over the cash stock of the firm. As such, the contracting problem between the shareholders and the manager features two forces that shape the sensitivity of the manager's pay to the performance of the firm. When the firm is far from liquidity default, the manager is more risk-averse than the shareholders, and incentive provision determines the manager's optimal exposure to cash flow shocks. When the firm is close to default, the shareholders are effectively more risk averse than the manager, and the optimal contract will give the manager high-powered incentives, that is incentives above what is required to prevent cash diversion. These high-powered incentives essentially hedge the risk of liquidity default.

Our assumption that investors cannot costlessly transfer cash into the firm introduces a novel restriction on the promise-keeping constraint in the standard dynamic principal-agent model (for example, DeMarzo and Sannikov (2006)). Specifically, only cash within the firm and incentive compatible promises of raising cash given the opportunity can be used to fulfill the promised value to the manager. Thus, the firm's cash balance is a commitment device that serves as collateral for the promise of future payments to the manager. In the extreme case where raising additional funds is impossible, only promises that are sufficiently collateralized by cash fulfill the promise-keeping constraint.

Under the optimal contract, negative cash-flow shocks not only reduce the firm's cash position but also lower the present value of compensation the firm owes to the manager. While the manager requires some minimum level of incentives to avert cash flow diversion, the firm may *hedge through labor contracts* and transfer more than this minimum level of risk by providing strong incentives. Such risk-sharing or hedging demand by the firm dominates the agency problem for low cash balances. Risk-sharing is not costless; however, as increasing the variability of the manager's pay increases risk-premium the manager requires to bear such risk. The agency problem dominates hedging needs for high cash balances, leading to labor contracts that have the minimum cash-flow sensitivity to keep the agent from stealing out of the cash-flows. Therefore our *first key finding* is that the optimal contract provides weaker incentives when the firm holds more cash and in particular incentives decrease after positive cash-flow realizations, put differently, we find that firms with low cash-holdings provide more equity-like compensation.

In addition to hedging through labor contracts, the firm can hedge liquidity risks by delaying dividend payouts and therefore accumulating more cash. Under the optimal contract, the optimal payout policy calls for a dividend whenever the firm's cash balance exceeds a threshold which we call the dividend payout boundary. Our *second key finding* is that the optimal dividend payout boundary decreases in the severity of the moral hazard problem. In particular, the manager's ability to divert from the firm's cash balance means that some of her compensation must be deferred, which leads to an endogenous *carrying cost of cash* via the risk premium that the manager applies to deferred compensation. When the moral hazard problem is more severe, that is, when the manager can divert cash with greater efficiency, the carrying cost of cash increases and the optimal dividend payout boundary decreases.

Our *third key finding* is that under moderate moral hazard firms facing high cash-flow uncertainty do not pass on this uncertainty to management via employment contracts, but instead hedge liquidity risk by holding more cash.[1] In contrast, firms with low cash-flow uncertainty hedge more via labor contracts and provide stronger incentives to management. When moral hazard is sufficiently severe, target cash-holdings are non-monotonic in cash-flow volatility. This result arises

[1] This is generally consistent with the findings of Bates et al. (2009).

because an increase in cash-flow volatility also increases the cost of incentive provision, thereby decreasing firm value and reducing the overall hedging demand.

Our model questions the widely held view that firms facing more severe agency conflicts should provide stronger managerial incentives. In particular, our *fourth key finding* is that the relationship between incentive pay and the level of moral hazard is state dependent. When the firm has a large cash balance, the strength of incentives is increasing in the severity of the moral hazard problem. This relationship reverses for firms with low cash holdings. Because more severe agency conflicts decrease the value of the firm as a going concern, liquidation becomes (relatively) less costly, decreasing a firm's hedging demand decreases to liquidity default. Consequently, the firm transfers less risk to the manager when its cash-balance is low, and the moral hazard problem is severe.

Refinancing in the presence of agency conflicts imposes an endogenously-derived flotation cost to raising funds in the absence of physical refinancing costs. In our model, the firm's ability to refinance is constrained by search frictions in capital markets, as in, for example, Hugonnier et al. (2014), which lead to uncertain refinancing opportunities. Under the assumption that the firm can commit to a refinancing policy ex-ante, we find that the implied refinancing costs are state-dependent, i.e., they depend on the current cash level of the firm. Ignoring for expositional purposes possible second-order effects of different payout boundaries, our *fifth key finding* is that a firm, depending on its cash-holdings, either refinances to below the first best, or refinances to the first best but raises more money than necessary to pay the manager a lump-sum wage payment in excess of what incentive constraints would imply. In other words, the presence of agency *always* distorts the decision to raise cash away from the first-best. The key to understanding latter effect is that large promises conditional on a state in which there is unlimited access to new cash lower the required wages in states in which cash is tight without violating promise keeping, thereby lowering the likelihood of liquidity default. Furthermore, in contrast to Hugonnier et al. (2014), better refinancing opportunities do not reduce the firm's hedging of liquidity risk. On the one hand, increasing the firm's access to refinancing leads it to accumulate and raise less cash. On the other hand, it leads to increased hedging of liquidity risk through managerial incentive-pay in low cash states.

Next, we find that when moral hazard is more severe, incentive compatibility demands high-powered incentives on average. Under these circumstances, employment contracts then absorb a large part of the liquidity risk, resulting in outside equity becoming less volatile on average. We also demonstrate that a firm's stock return volatility need not be decreasing in the firm's liquidity and can follow a hump-shaped pattern since a financially constrained firm hedges cash-flow risk through labor contracts to a greater extent, which in turn reduces stock return volatility. Depending on how much risk the firm transfers to the manager, we get a different relationship between liquidity and volatility of stock returns. These model predictions are novel and contrast with the findings of related models of cash-management such as that of Décamps et al. (2011)), who find the relationship between cash and equity return volatility to be unambiguously monotonic.

Finally, the technique we use to solve our model also represents a methodological contribution. Dynamic agency problems usually introduce the manager's promised future payments as a state variable to track the agency problem. At the same time, liquidity management problems use the firm's stock of cash as a state variable to track the liquidity of the firm. Our problem thus would appear to have two state variables. While dynamic stochastic optimization problems with more than two state variables are usually hard to solve, we show how a small expansion of the allowed wage space allows for the model to collapse to a one-dimensional optimization while maintaining the liquidity-agency trade-off. The key observation is that allowing the manager to receive small negative wages, in conjunction with allowing the manager to have a savings contract that is not identically zero along the equilibrium path, relaxes the shareholders' problem. Shareholders prefer to manage liquidity, in the absence of refinancing, using costly small negative wages over holding cash-buffers in excess of the incentive constraints.[2] Cash net promised risk-adjusted future wage payments readily measure the firm's financial soundness and its distance to liquidity default

2 Model Setup

Cash-Flow & **Earnings.** We consider a 100% equity financed firm, owned by a mass of shareholders, who we also collectively refer to as the *principal*. To operate the business and produce cash-flows from assets, the firm has to hire a *manager (agent, she)*. Up to firm liquidation/termination at time τ, assets in place produce a cash-flow X that follows a controlled Arithmetic Brownian Motion with drift μ and volatility σ:

$$dX_t = \mu dt + \sigma dZ_t - db_t,$$

[2]Importantly, this dimensionality reduction goes beyond the absence of wealth effects, as studied by related papers considering a CARA-manager endowed with a savings technology (compare e.g. He (2011), He et al. (2017) or Gryglewicz et al. (2017)), which usually focus without loss of generality on zero-savings contracts.

where Z is a standard Brownian Motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. Cash-flow X is fully observable, but the agent can alter its realization through her hidden action b, where $db_t > 0$, as in DeMarzo and Sannikov (2006), corresponds to the *diversion* of funds. In our model, $dX_t > 0$ represents operating profits and $dX_t < 0$ operating cost or losses.[3]

Cash Holdings. The firm is liquidated when it is not able or willing to cover its operating cost, in which case shareholders recover a liquidation value L with $\frac{\mu}{r+\delta} \geq L$.[4] Thus, liquidation entails deadweight costs, and the firm optimally retains earnings in the form of cash-holdings M to avert liquidation. Cash-holdings M are observable to both parties. Liquidation occurs when the firm runs out of cash, i.e., at time $\tau = \inf\{t \geq 0 : M_t = 0\}$. At this point, we would like to stress that the firm could also ask the agent – who is able to maintain a savings account – to cover operating losses $dX_t < 0$ when $M_t = 0$, but optimally does not do so and indeed prefers to default at time τ. In the baseline version of our model, we assume that no refinancing is possible. Later, we introduce refinancing opportunities at Poisson times and discuss the impact of agency frictions on optimal refinancing policies.

Next, we assume that the firm is subject to an exogenous shock that wipes out its entire cash-balance M and trigger immediate liquidation.[5] The shock arrives according to a Poisson process with intensity δ. This assumption is needed to ensure that the model is well-behaved, in that dividend payments are not indefinitely delayed. Without loss of generality, N is observable to both parties.[6] One example of such a shock can be a large lawsuit – for example, Purdue Pharma (the maker of OxyCotin) recently prepare to declare bankruptcy in response to a number of lawsuits related to the Opioid crisis.

The cash-stock inside the firm grows through earned interest on the balance at the market rate $r > 0$ and is directly affected by cash-flow from assets, dividend payouts to shareholders $dDiv_t$, wage payments to the manager dw_t, managerial cash-diversion db_t, and catastrophic shock $dN_t = 1$:

$$dM_t = rM_{t^-}dt - dDiv_t - dw_t + \mu dt + \sigma dZ_t - M_{t^-}dN_t - db_t. \tag{1}$$

[3]Equivalently, we could assume that cash-flow X is only observable to the agent, who then reports cash-flow $\hat{X}$ and keeps the difference $X - \hat{X}$ for her own use.

[4]We additionally impose a lower limit in the appendix that ensures that liquidation is better than having the agent run the firm, which is given by $L \geq \max\left\{\frac{\mu - \rho r\sigma^2/2}{r+\delta}, 0\right\}$

[5]The exact nature of the shock is not relevant as long as it triggers default with positive probability and some exogenous loss of cash. Instead of a fixed loss, we could equally assume that the shock size is exponentially distributed like in Hugonnier and Morellec (2017), or impose an internal cost of carrying cash.

[6]If N were only observable to the agent, the model and its solution would be entirely the same. Equivalently, we could model the disastrous shock also as cash-flow shock: $dX_t = \mu dt + \sigma dZ_t - db_t - M_{t^-}dN_t$.

Here, $M_{t^-} = \lim_{s\uparrow t} M_t$ denotes the left limit of cash M_t. More intuitively, M_{t^-} represents cash-holdings just before the catastrophic event (a jump) $dN_t \in \{0, 1\}$ realizes.

Moral Hazard. We assume that the manager can secretly divert cash for her own use by the following actions:

First, because firm performance is noisy, i.e., $\sigma > 0$, the manager can secretly steal some infinitesimal amount $db_t > 0$ with $db_t \in o_p(dt)$ from the firm's *cash-flow* dX_t. By doing so, she appropriates fraction $\lambda \leq 1$ per dollar diverted, so stealing is assumed inefficient. Conversely, the agent can also put her own money into the firm and boost cash-flow through $db_t < 0$. This transfer is not subject to efficiency losses. As long as db_t is infinitesimal and smooth – that is, $db_t = \hat{b}_t dt$ for some process $\hat{b}$ – the principal cannot detect the agent's hidden action and attributes the loss $\hat{b}_t dt$ due to the agent's cash-flow diversion mistakenly to a lower cash-flow shock dZ_t.[7]

Second, the manager can divert a lumpy amount of cash – smaller or equal than M_t – from the firm's *cash-balance* and in particular abscond with the entire *cash-balance* M_t, in which case M_t jumps down. Importantly, this *cash* diversion is immediately detected by the principal, because absent moral hazard cash-flow dX_t is not subject to jump shocks and therefore continuous, and the only exogenous jump in the model, dN_t, is publicly observable. We assume that the agent's benefit from stealing from the cash-stock is a fraction κ per dollar diverted.

Throughout the paper, we denote the amount of cash stolen by the manager up to time t by b_t and the amount received by B_t, where B_t does not necessarily equal b_t, as diversion is subject to efficiency losses.[8]

Preferences. Shareholders are risk-neutral, have zero private wealth and are protected by limited liability/commitment. That is, dividend payouts must be non-negative, that is, $dDiv_t \geq 0$ for all $t \geq 0$.[9] Shareholders discount at market rate r and maximize total firm value, which is given by discounted cumulative dividend payouts. Because shareholders cannot fully commit, they could at any time potentially pay out all cash M_{t-}, liquidate the firm and renege on the manager's promised payments. In case shareholders try to do so, we assume that the firm's cash-stock is

[7] Any stochastic, i.e., non-smooth, stealing would immediately be revealed by the quadratic variation of the process, and thus is not used by the agent.

[8] Formally, write $db_t = \hat{b}_t dt + db_t^1$, where the process $\hat{b}$ is absolutely continuous. Then, $dB_t = \max\{0, \hat{b}_t\}\lambda dt + \min\{0, \hat{b}_t\}dt + \kappa db_t^1$.

[9] The zero private wealth assumption is simply to keep the shareholders from injecting cash into the firm to keep it alive. Dispersed shareholders with positive wealth in the absence of coordination would also result in the absence of cash injections due to a free-rider problem.

divided between shareholders and manager according to the Nash-Bargaining protocol, where the shareholders possess bargaining weight $\theta > 0$. Likewise, shareholders cannot commit to any wage payments $dw_t > 0$ after liquidation for $t > \tau$ and optimally do not pay any wages after liquidation.

The manager discounts at market rate r and is risk-averse with CARA-utility

$$u(c_t) = -\frac{1}{\rho} \exp(-\rho c_t),$$

where $\rho > 0$ is the coefficient of absolute risk-aversion and c_t is instantaneous consumption. The manager cannot fully commit and may decide to leave the firm and abscond with the entire cash-balance or any other amount, whenever she is better off from doing so.

In addition, we assume that the manager can maintain hidden savings S, so that her consumption c is not observable to the principal. For tractability, we assume that the manager can borrow, implying that S_t need not be positive. Savings S then earn/pay interest at rate r and are subject to changes induced by wage payments dw_t, diverted cash dB_t, and consumption c_t:

$$dS_t = rS_{t^-}dt + dB_t + dw_t - c_t dt \tag{2}$$

Endowing the agent with the possibility to accumulate savings is needed to ensure consumption smoothing beyond any liquidation event. The manager maximizes expected, discounted utility and possesses an outside option u_{0^-}. We normalize initial savings $S_{0^-} = 0$ and the agent's outside option in certainty equivalence terms (properly defined in the next subsection) $W_{0^-} = 0$.

We will make the following assumption on the wage process dw:

Assumption 1. *We assume that cumulative wages must satisfy* $\lim_{\varepsilon \to 0} w_{t+\varepsilon} - w_t \geq 0$. *That is, wages have to be either continuous or exhibit upward jumps (lumpy payments to the manager), but cannot exhibit downward jumps (lumpy cash infusions from the manager).*

Note that this assumption does not preclude negative flow wages. We discuss the above and alternative assumptions in more detail in Sections 3.2.3 and 3.2.4.

The Contracting Problem. At inception $t = 0^-$, the manager is offered a contract $\mathcal{C} = (\hat{c}, w, \hat{b})$ by the shareholders, who also decide on optimal cash-holdings and the payout process Div. The contract $\mathcal{C}$ specifies the manager's recommended consumption $\hat{c}$, wage payments w and diversion $\hat{b}$. We call $\mathcal{C}$ *incentive compatible* if $c_t = \hat{c}_t$ and $\hat{b}_t = b_t = 0$, in that the manager does not (inefficiently) steal from the firm's cash-stock, and *feasible* if the principal can fully commit to it. Throughout the paper, we focus on incentive compatible and feasible contracts and denote the set of these contracts by **C**.

The agent solves

$$U_{0^-} = \max_{c,b} \mathbb{E}\left[\int_0^\infty e^{-rt} u(c_t) dt\right] \text{ s.t. (2)} \tag{3}$$

for some initial savings S_{0^-} while the shareholders' objective is it to maximize firm value:

$$V_{0^-} = \max_{Div, \mathcal{C} \in \mathbf{C}} \mathbb{E}\left[\int_0^\infty e^{-rt} dDiv_t + e^{-r\tau} L\right], \tag{4}$$

$$\text{s.t. } U_{0^-} \geq u_0, dDiv_t \geq 0, M_t \geq 0 \text{ for all } t \geq 0 \text{ and (1).} \tag{5}$$

To ensure the problem is well-behaved, we impose that the agent's savings S must satisfy the transversality condition, sometimes referred to as the *No-Ponzi condition*:

$$\lim_{t \to \infty} e^{-rt} S_t \geq 0 \text{ almost surely wrt. } \mathbb{P} \tag{6}$$

and certain other regularity conditions, which are collectively gathered in Appendix A. If ever $S_\tau < 0$, the transversality condition requires negative consumption to make up the savings shortfall.

3 Model Solution

In this section, we solve the model and derive the firm's optimal payout and executive compensation policy. First, we analyze the manager's problem and characterize conditions for the contract to be incentive compatible. In particular, we introduce the certainty equivalent W_t. Second, we focus on the principal's problem and show the restriction on the state- and strategy-space the principal faces. In particular, due to CARA, the principal faces a 2-dimensional dynamic optimization problem characterized by a PDE. Third, we show how under Assumption 1 on wages the model collapses to a 1-dimensional dynamic optimization problem characterized by an ODE while maintaining a liquidity-default trade-off.

3.1 The manager's problem

3.1.1 The Continuation Value

As is standard in the dynamic agency literature, let us define for any incentive compatible contract $\mathcal{C}$ the agent's continuation value at time t

$$U_t := \mathbb{E}_t\left[\int_0^\infty e^{-r(s-t)}u(c_s)ds\right] \tag{7}$$

and denote the agent's savings by S_t. By the martingale representation theorem, we get that dynamics of U follow

$$dU_t = rU_{t^-}dt - u(c_t)dt + \beta_t(-\rho rU_{t^-})\underbrace{(dX_t - \mu dt)}_{=\sigma dZ_t} - \alpha_t(-\rho rU_{t^-})(dN_t - \delta dt) \tag{8}$$

for some (conveniently scaled) loadings α, β defined by the contract. Here, α captures the agent's exposure to disaster risk dN_t and β the agent's exposure to cash-flow shocks dZ_t.

First, note that in order to ensure that the agent does not deviate from the recommended consumption path, the optimal contract has to respect the agent's Euler equation, in that marginal utility has to follow a martingale. Next, as shown in the appendix, the first order condition with respect to consumption with the possibility of a savings account implies that $u'(c_t) = -\rho rU_{t^-} > 0$. This in turn implies that U_t is a martingale. Further, let us define the certainty equivalent W_t as the amount of wealth needed that would result in utility U_t if the agent only consumed interest rW_t, i.e.,

$$u'(rW_{t-}) = -\rho rU_{t-} \iff W(U) := \frac{-\ln(-\rho rU)}{\rho r}. \tag{9}$$

Here, W_t is the agent's continuation value in *monetary terms* while U_t is the agent's continuation value in *utility terms*.

By Ito's Lemma, we obtain

$$\begin{aligned} dW_t = & \underbrace{\frac{\rho r}{2}(\beta_t\sigma)^2}_{\text{BM Risk-Premium}} dt + \beta_t(dX_t - \mu dt) \\ & + \underbrace{\delta\left(\alpha_t - \frac{\ln(1+\rho r\alpha_t)}{\rho r}\right)}_{\text{Poisson Risk-Premium>0}} dt - \frac{\ln(1+\rho r\alpha_t)}{\rho r}(dN_t - \delta dt). \end{aligned} \tag{10}$$

Because her compensation package is exposed to cash-flow shocks dX_t and productivity shocks

dN_t, the agent demands a risk premium, so that W_t has a positive drift. In other words, as U_t is a martingale, $W_t = W(U_t)$ has a positive drift due to the convexity of $W(U)$ and Jensen's inequality. Essentially, (8) or equivalently (10) constitutes the so-called *promise-keeping* constraint. That is, shareholders promise the agent's continuation value W (resp. U) evolves according to (10) (resp. (8)).

3.1.2 Cash & Cash-Flow Diversion

In this section, we analyze the incentives the optimal contract has to provide to the manager in order to preclude diversion of cash. In principle, the agent can pursue two actions.

First, she can steal some infinitesimal amount $\varepsilon > 0$ of cash. When this amount is sufficiently small (on the order of dt), the principal mistakenly attributes the losses to an adverse cash-flow shock $dZ_t < 0$ and can accordingly not detect this misbehavior. We refer to this action as *cash-flow* diversion. Diverting and consuming amount $\varepsilon > 0$ from cash-flow increases flow utility by $u'(c_t)\lambda\varepsilon$ while dX_t falls by ε, so that on average the agent's continuation value is reduced by (recall our scaling of the loadings in the martingale representation) $\beta_t(-\rho r U_{t-})\varepsilon = \beta_t u'(c_t)\varepsilon$. Thus, stealing ε dollars is not optimal if

$$\beta_t(-\rho r U_{t-}) \geq \lambda u'(c_t) \Longleftrightarrow \beta_t \geq \lambda. \tag{11}$$

Therefore, the principal has to provide a minimum performance-pay in order to rule out the agent diverts from cash-flow. In principle, the manager can also boost cash-flow dX_t through putting in additional cash from her savings account. To prevent a violation of promise keeping, any IC contract needs to have $\beta_t \leq 1$. As we shall see, the constraint $\beta_t \leq 1$ never binds and does not affect the principal's maximization, in that the manager is optimally provided incentives $\beta_t < 1$ for all $t \geq 0$.

Second, the agent can divert any larger amount cash $0 < db_t \leq M_{t-}$, in which case the firm's cash-balance jumps down by $db_t > 0$. We refer to this action as *cash-stock* diversion, because cash-flow evolves continuously and is not subject to large shocks. Thus, absent any large shocks $dN_t = 1$, the principal immediately observes the agent's misbehavior and can accordingly punish her. It is important here to point out that the principal does not have access to the agent's savings account, so that any punishment has to arise from decreasing future wages. In order to have some leeway to punish the agent, the principal must therefore defer compensation.

Hence, whenever the firm holds a positive amount of cash $M_{t-} > 0$, i.e., for $t < \tau$, the optimal contract must provide incentives by means of deferred compensation, in order to preclude that the agent steals any amount from the cash-stock. Deferred compensation is represented by $Y_{t^-} := W_{t^-} - S_{t^-} > 0$, so that the agent's promised compensation exceeds her savings. We interpret Y_t as the *risk-adjusted value of future wages*.[10]

To see why $Y_t > 0$ discourages cash diversion, imagine that the agent considers 'just before' time t, i.e., at t^-, to abscond with the entire cash-stock M_{t^-}. Doing so, she receives κM_{t^-} dollars, the firm is liquidated and the employment contract is terminated. Hence, after stealing, the agent does not receive any future wages, but possesses the sum of her private savings and the diverted cash $S_{t^-} + \kappa M_{t^-}$. The agent refrains from stealing if the value from staying with the firm is higher than the value from stealing and leaving, that is if:[11]

$$W_{t^-} = S_{t^-} + Y_{t^-} \geq S_{t^-} + \kappa M_{t^-} \Longleftrightarrow Y_{t^-} \geq \kappa M_{t^-} \Longleftrightarrow \varphi_t := \frac{Y_{t^-}}{M_{t^-}} \geq \kappa. \tag{12}$$

By (12), high cash-holdings M_t within the firm exacerbate agency issues and tighten the IC constraint, which makes higher powered incentives by means of deferred payments Y_{t^-} necessary.

While deferring compensation by means of $Y_{t^-} > 0$ is necessary to align the manager's incentives, it comes at a cost. This is because during any time interval $[t, t + dt]$ the firm might be hit by a disastrous shock, which fully exhausts the available cash-stock. In this case, the firm is liquidated and – due to the shareholders' limited liability – the manager looses the previously promised amount Y_{t^-}. By definition, at time of termination τ, the manager's certainty equivalent W_τ must equal her savings S_τ, i.e., $Y_\tau = 0$. Hence, upon a shock $dN_t = 1$, it follows that the manager's continuation value jumps down immediately by amount Y_{t^-}, in that $dW_t = -Y_{t^-} dN_t$. Matching coefficients in (10), this pins down the manager's exposure to disaster risk in terms of U_t:

$$\alpha_t = A(Y_{t^-}) := \frac{\exp(\rho r Y_{t^-}) - 1}{\rho r} \geq 0. \tag{13}$$

Hence, deferring compensation exposes the manager to Poisson shocks, for which she requires a risk-premium to be paid by the firm. Consequently, increasing Y_{t^-} is costly for shareholders as $A(\cdot)$ is increasing and convex in its argument.

[10] It is straightforward to show $Y_t = \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \big(dw_s - \zeta_s ds \big) \right]$ where $\zeta_t := \frac{\rho r}{2} (\beta_t \sigma)^2 + \delta \left(\alpha_t - \frac{\ln(1+\rho r \alpha_t)}{\rho r} \right)$ is the agent's required risk premium.

[11] In case the agent were able to enjoy an additional outside option $\mathcal{O}$ in monetary terms after leaving the firm, e.g., through finding a job at another firm or through extracting some of the liquidation value of the assets, the constraint (12) would change to $Y_{t^-} \geq \kappa M_{t^-} + \mathcal{O}$. Throughout our analysis, we consider without loss $\mathcal{O} = 0$ and we normalize the agent's outside option to zero.

Because higher cash-holdings M_{t^-} require by (12) more deferred compensation Y_{t^-} and therefore a higher risk-compensation $\delta A(Y_{t^-})$ and flow wage for the manager, we obtain endogenous *carry-cost* for internal cash-holdings.

To conclude this part, we summarize our findings in the following proposition.

Proposition 1. *Let $\mathcal{C}$ solve (5). Then, the following holds true:*

i) *The agent's continuation value U, defined in (7) solves the SDE (8) for some $\mathbb{F}$-progressive processes (α, β) and W solves the SDE (10).*

ii) *Given a process Y the process α satisfies (13).*

iii) *The process β satisfies $\beta_t \in [\lambda, 1]$ for all $t \geq 0$ and the process α is given through (13).*

3.2 The shareholders' problem

3.2.1 First reduction of the State Space

The problem of shareholders generally depends on three states. The agent's continuation value U_{t^-} or equivalently W_{t^-}, the agent's savings S_{t^-} and the firm's cash-holdings M_{t^-}, so that firm value at time t^- – or equivalently the shareholders' continuation value – is given by a function $\hat{V}(M_{t^-}, W_{t^-}, S_{t^-})$. Thanks to CARA-preferences and the absence of wealth effects, the exact values of W_{t^-} and S_{t^-} become irrelevant for the principal's problem, and only the difference $Y_{t^-} = W_{t^-} - S_{t^-}$ matters. Thus, we are left with the two state variables (M_{t^-}, Y_{t^-}), and the principal's value can be written in the form $\hat{V}(M_{t^-}, W_{t^-}, S_{t^-}) = V(M_{t^-}, Y_{t^-})$.

3.2.2 State constraints

Promised payments to the manager must be fully collateralized. Put differently, any uncollateralized promise $Y_{t^-} > M_{t^-}$ is an empty promise. Sufficiently negative cash-flow shocks (e.g., $dX_t = -M_{t^-}$) can wipe out the firm's cash-balance within a short amount of time $(t, t + dt)$, thereby leading to $Y_{t+dt} > M_{t+dt} = 0$. Under these circumstances, shareholders either renege on the promise Y_{t+dt} and default or ask the manager to fully absorb cash-flow risk through $\beta = 1$, in order avoid liquidation. In the first case, *promise keeping* is violated.[12] In the second case, the manager must cover consumption needs $c_t = rW_t$ and operating losses, until the firm is liquid again and financial

[12]That is, the evolution of W is inconsistent with (10). This is because default at time $t + dt$ leads to an immediate jump of payments $Y_{t+dt} > 0$, the manager expects to receive, down to zero. Equivalently, W_{t+dt} jumps down in absence of a Poisson shock, contradiction (10).

distress is resolved. Because the manager's consumption rW_t strictly exceeds the interest earned on savings, rS_t, and financial distress may prevail for an arbitrarily long time-span, she accumulates excessive debt (with positive probability), which results into a violation of the no-Ponzi condition. We conclude that the only way for promise-keeping and No-Ponzi condition to hold is to liquidate as soon as $Y_t = M_t$. Thus, the principal's optimization is subject to the following state constraint:

$$(Y, M) \in \mathcal{B} = \{(Y, M) : 0 \leq \kappa M \leq Y \leq M\}. \tag{14}$$

3.2.3 The general HJB-equation

We can now write the principal's optimization as

$$(r + \delta) V(Y, M) = \max_{\beta \geq \lambda, dw, dDiv \geq 0} \mathcal{L}^Y V(Y, M) + \mathcal{L}^M V(Y, M) + \left\langle \mathcal{L}^C, \mathcal{L}^M \right\rangle V(Y, M) \tag{15}$$

subject to the state-constraint $(Y, M) \in \mathcal{B}$. Here, $\mathcal{L}^Z$ is the linear generator of an arbitrary stochastic process Z, and $\langle \cdot, \cdot \rangle$ is the quadratic variation.

Note that to respect the state-constraint $(Y, M) \in \mathcal{B}$, the principal has to engage in certain strategies when hitting the boundaries of $\mathcal{B}$ to keep the (Y, M) from exiting $\mathcal{B}$. First, at $Y = M$ the principal has to pays out all cash to the agent and subsequently liquidates the firm to ensure promise keeping. Second, at $Y = \kappa M$, the principal has to pick wage and dividend payments in such a way as to not have Y drop below κM as response to shocks or drift exposure of Y and M.

For this discussion, briefly ignore Assumption 1. Panel A in Figure 1 gives a graphical representation of the problem of controlling the process to stay in $\mathcal{B}$. Consider point A strictly inside $\mathcal{B}$. The principal has two strategies at his disposal: (1) Paying a dividend $dDiv > 0$, which shifts A straight left, and (2) paying a wage $dw \neq 0$ which shifts A along the 45-degree line, for example to point A', as the continuation value Y shifts 1-for-1 with the current wage payment.

Let us now discuss two natural restrictions one would consider imposing on the control problem:

- Consider restricting the agent savings to be non-negative, i.e., $S \geq 0$. This, destroys the first reduction in the state space, as S now has to be separably tracked. In other words, the problem with $(S, W, M) = (0, W_0, M_0)$ is now different from the problem $(S, W, M) = (Z, W_0 + Z, M_0)$ for any $Z > 0$. Consequently, the principal now faces a true 3-D optimization in the (S, W, M) with an additional state-constraint.

- Consider restricting wages to be non-negative, i.e., $dw \geq 0$, to keep the first dimensionality

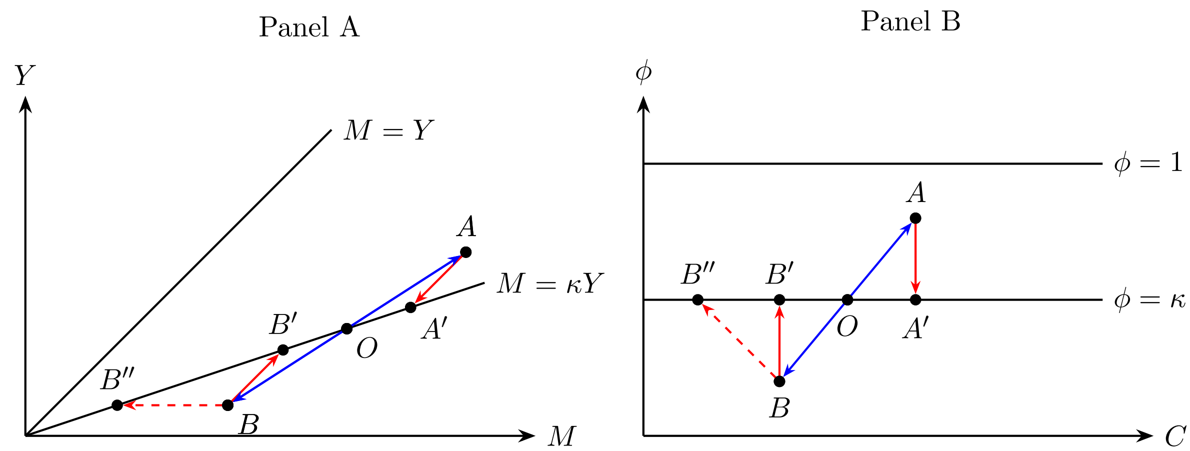

Figure 1: **Schematic Representation** of the state- and strategy space

reduction intact. This requires dividend payments after any shocks push (M, Y) below the $Y = \kappa M$ ray. This can be seen in Panel A in Figure 1 as moving from point B to point B'' – after a negative shift pushes O below the $Y = \kappa M$ ray to B, only dividend payments are effective in returning (Y, M) to within the wedge $\mathcal{B}$. Such a dividend payout magnifies cash outflows, amplifying the specter of liquidity-based default. The firm will therefore want to consider building up a cash-buffer to stay away from the $Y = \kappa M$ ray. Consequently, the optimization is taking place on the full 2-D space (M, Y) with a non-standard, as non-perpendicular, reflection at $Y = \kappa M$.

Thus, either of these restrictions leads to a relatively intractable problem requiring a numerical solution. We will next show how Assumption 1 makes the problem tractable while maintaining the key economic mechanism between liquidity and agency that we are after.

3.2.4 Second reduction of the State Space

Let us return to Panel A in Figure 1, and let us consider a shift of point O to below $Y = \kappa M$ brought about by a negative cash-flow shock, to say point B. Recall from the discussion of non-negative wages that a dividend payout at such a point, to point B'' say, magnifies the cash-outflows, leading to a heavy reliance on precautionary buffer cash and a full-fledged 2-D problem. Consider instead paying a negative wage $dw < 0$, i.e., requiring the agent to contribute a small amount of her own cash to the firm, in return for a higher Y. Importantly, this payment does not violate Assumption 1, as the shock is driven by a continuous process, a Brownian motion. In essence, we are shifting the problem up along the 45-degree line to satisfy the state-constraint $(Y, M) \in \mathcal{B}$, here to point B'. Importantly, this strategy does not change the firm's net-cash position $C = M - Y \geq 0$ which measures the firms distance from default. Such negative wage contributions of course are not free, in that they entail higher deferred earnings, which in turn will require a larger risk-premium, i.e., a higher drift of Y. Thus, we are replacing a hard constraint on cash with a weaker constraint that essentially implies an increasing cost of cash the more the agent is made to contribute.

To reduce the state-space further, we (1) rotate the state-space and (2) relax the problem:

1. Rotate the state space from (Y, M) to (C, φ), so $1 \geq \varphi = Y/M \geq \kappa$ and $C = M - Y \geq 0$ are now representing $\mathcal{B}$. In Figure 1, this rotation is represented by Panel B. Again, consider point A, which has a slack constraint $Y \geq \kappa M$. In the (C, φ) space, we see that paying out wages simply results in a vertical shift down, to point A'. Similarly, consider point B. A dividend payment would shift the point in a north-west direction, to say point B'', whereas a negative wage payment simply results in a vertical shift up to say point B'.

2. Relax Assumption 1 by allowing lumpy wage payments of any sign. With unconstrained wages, φ can be freely adjusted up and down without affecting the firms default outlook C as discussed in point 1. Consequently, φ has now become a *control.* Assumption 1 is satisfied in the relaxed specification if the optimal control $\varphi(C)$ is continuous (absent refinancing).

Thus, we recast the principal's problem as a maximization over *controls* β and φ with the one-dimensional *state* C. Throughout the remainder of the paper, we refer to C also as *net cash* or *liquidity (reserves)*. Utilizing (10), (2), $dB_t = 0$, $c_t = rW_t$, and $Y = \frac{\varphi}{1-\varphi}C$, for any IC and implementable contract we have

$$dC_t = rC_{t^-}dt - \frac{\rho r}{2}(\beta_t \sigma)^2 dt - \delta A \left(\frac{\varphi_{t^-}}{1 - \varphi_{t^-}} C_{t^-} \right) dt + \mu dt + (1 - \beta_t)\sigma dZ_t - dDiv_t - C_{t^-} dN_t. \quad (16)$$

Note that so far dw has not been explicitly specified – it will be defined as the *residual* that implements the optimal choice of φ.

3.2.5 Optimization and the HJB-equation

To derive the HJB, let us write the value function $v(C) = v(M - (W - S)) = V(M, W - S) = \hat{V}(M, W, S)$. Next, we conjecture that dividend payouts only occur at an upper boundary $\overline{C}$. On the conjectured continuation region $C \in (0, \overline{C})$, we have the following HJB:

$$(r+\delta)v(C) = \max_{\beta \geq \lambda, 1 \geq \varphi \geq \kappa} \left\{ v'(C) \left(rC - \frac{\rho r}{2}(\beta\sigma)^2 - \delta A \left(\frac{\varphi C}{1-\varphi} \right) + \mu \right) + \frac{\sigma^2(1-\beta)^2}{2} v''(C) \right\}. \quad (17)$$

First, maximizing w.r.t. φ, as $v'(C) > 0$ in equilibrium, and $\frac{\partial A\left(\frac{\varphi C}{1-\varphi}\right)}{\partial \varphi} > 0$, we optimally set

$$\varphi(C) = \kappa, \quad (18)$$

i.e., we pick the minimum level of cash that implements no-stealing from cash stock. With φ continuous, the solution to the relaxed problem is indeed the solution to the full problem.

Second, maximizing w.r.t. β, the first-order conditions and the IC constraint imply that

$$\beta(C) = \max\{\lambda, \beta^*(C)\} \text{ with } \beta^*(C) := \frac{-v''(C)}{\rho r v'(C) - v''(C)} < 1. \quad (19)$$

Raising incentives β transfers risk to the agent and reduces the volatility of C, thereby lowering the likelihood of liquidation. Consequently, it can be optimal to provide more incentives β than required by incentive compatibility when C is low, as discussed in more detail in the next subsection.

Note that the solutions to φ and β imply that the firm never experiences *agency-based* default, i.e., default triggered by $C = 0$ with $M = Y > 0$.

Boundary Conditions. The standard boundary conditions[13] of value-matching at default $C = 0$ and smooth-pasting at the dividend payout boundary $C = \overline{C}$ are given by

$$v(0) = L \text{ and } v'(\overline{C}) = 1. \quad (20)$$

Recall that shareholders are not able to fully commit to their promises, and may decide to trigger liquidation if it is beneficial to them. Liquidating yields a cash payout – due to the Nash-bargaining assumption – of $\theta M = \frac{\theta}{1-\kappa} C$ in addition to the liquidation value L to the principal, while paying $(1-\theta)M = \frac{1-\theta}{1-\kappa} C$ to the agent. Thus, for any feasible contract, we must have[14]

$$v(\overline{C}) \geq \frac{\theta}{1-\kappa}\overline{C} + L. \quad (21)$$

[13] Observe that a positive unit cash-flow shock to M at $C = \overline{C}$ leads to an increase in C of $(1-\beta)$, and unit payouts of $(1-\beta)$ as dividends and β as wages. Re-norming, a unit shock to C then leads to a unit dividend payout.

[14] Strictly speaking, we must have $v(C) \geq \frac{\theta}{1-\kappa} C + L$ for all $C \in [0, \overline{C}]$, but from $v'(C) \geq 1 \geq 0 \geq v''(C)$ it is sufficient to check this condition at $C = \overline{C}$.

If constraint (21) is slack, the payout boundary satisfies the optimality or *super-contact condition*

$$v''(\overline{C}) = 0 \tag{22}$$

In other words, if payouts are optimally made at $C = \overline{C}$ with (21) slack, then the shareholders' *effective* risk-aversion vanishes at $\overline{C}$.

Thus, whenever (21) holds with equality and $v''(\overline{C}) < 0$, the shareholders' limited commitment combined with moral hazard κ constrain the firm in optimally managing liquidity risks. Note that constraint (21) is always slack if $\frac{\theta}{1-\kappa} < 1$, which is the case when a liquidation would not violate promise keeping, as $(1-\theta)M < Y \iff 1 < \frac{\theta}{1-\kappa}$.[15] For $\frac{\theta}{1-\kappa} > 1$, we simply check condition (21) at the *candidate* payout boundary $\overline{C}^*$ defined by (22).

Proposition 2. *Let $\mathcal{C}$ solve (5). Then, the following holds true:*

i) The shareholders' value function $V(\cdot)$ satisfies $V(\cdot) = v(C)$, where the function $v(\cdot)$ is twice continuously differentiable, i.e., $v \in C^2$.

ii) The principal's payoff is given by a function v, that solves the HJB-equation (17) subject to $v(0) - L = v'(\overline{C}) - 1 = 0$ and either $v''(\overline{C}) = 0$ or $v(\overline{C}) = \theta\overline{C}/(1-\kappa) + L$.

iii) The value function v is strictly concave $[0, \overline{C})$ with $v'''(C) > 0$.

4 Analysis

Unless specified otherwise, we assume that parameters are such that the payout boundary is optimally determined by the super-contact condition, i.e., $v''(\overline{C}) = 0$.[16]

4.1 Performance-Pay & Hedging Through Labour Contracts

In this section, we analyze the pay-performance sensitivity β. For clarity of exposition, let us for the time being assume that $\lambda = \theta = 0$, so that $\beta = \beta^*$. The assumption $\lambda = 0$ is equivalent to the absence of the agency problem in terms of stealing out of cash-flow, but does not preclude stealing from cash-stock, i.e., $\kappa > 0$.

Absent liquidity concerns, it is optimal for the principal not to expose the risk-averse manager to any cash-flow shocks, i.e., to set $\beta^* = \lambda = 0$. However, in the *presence* of liquidity concerns,

[15] This is because $v(\bar{C}) - L = v(\bar{C}) - v(0) > \bar{C}$, as $v'(C) \geq 1$ with the inequality being strict for some C. Hence, the super contact condition holds if $\theta\bar{C} \geq \frac{\bar{C}\theta}{1-\kappa}$.

[16] As mentioned in the preceding footnote, a sufficient condition for this is $\theta < 1 - \kappa$.

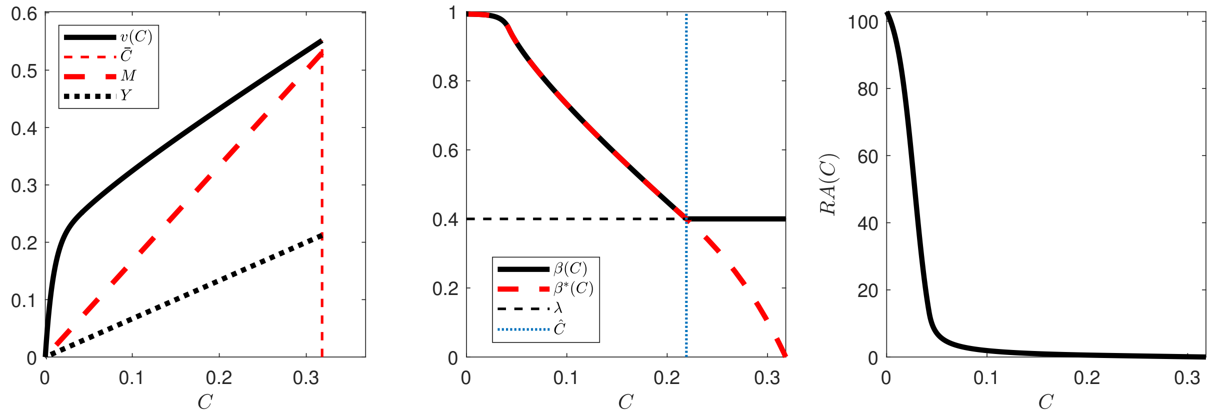

Figure 2: **Benchmark value function**: The parameters are $\mu = 0.25, r = 0.1, \delta = 0.25$, $\lambda = \kappa = 0.4$, $\sigma = 0.75$, $\theta = 0$ and $\rho = 7$.

shareholders become increasingly risk-averse as cash-reserves dwindle and would optimally like to hedge liquidity risk through *labour contracts* by setting incentive pay $\beta^* > 0$.

Incentive pay transfers risk to the agent, in that the volatility of the liquidity reserves, $dC/dX = \sigma(1-\beta)$, decreases in β for $\beta < 1$. Consider the benefit of increasing β:

$$\frac{\partial v(C)}{\partial \beta} \propto \underbrace{-v'(C)\rho r \beta \sigma^2}_{\text{Risk-Compensation;}<0} + \underbrace{(1-\beta)\sigma^2(-v''(C))}_{\text{Reduction in Cash-Flow volatility;}>0} .$$

Increasing β makes C less volatile and reduces the likelihood that the firm runs out of cash but also requires a risk-compensation to the agent, as her wage has become more volatile. When the firm has low cash holdings, a reduction in volatility is particularly beneficial, since $-v''(C)$ is large. On the other hand, the marginal value of cash of the firm $v'(C)$ is pronounced under distress, so that the drift of promised wages required as risk-compensation is also very costly.

Intuitively, the optimal β^* implements a risk-sharing solution that balances the agent's constant absolute risk-aversion ρ against the shareholders' state-dependent absolute risk-aversion $-v''(C)/v'(C)$. The firm hedges more strongly through labour contracts for low net-cash positions, i.e., $\beta^*(C) > 0$ for $C > 0$, whereas it absorbs all risk at the payout boundary, $\beta^*(\overline{C}) = 0$. That is, compensation becomes more equity like when the firm undergoes financial distress and has little cash. In practice, firms with little cash often are start-ups and young firms, where it is indeed well documented that their employees are often rewarded with stock.

When $\lambda > 0$, the firm's risk-sharing is constrained by $\beta \geq \lambda$. Thus, risk-sharing is constrained for high levels of C in that due to IC constraint the principal can never fully insure the agent, even at the payout boundary as $\beta^*(0) = \lambda$.

Furthermore, the incidence of negative wages $dw < 0$ rises for low C, in that $0 = d\varphi = d(Y/M)$ implies $dw = \mu_w dt + \frac{\beta(C)-\kappa}{1-\kappa} dZ$. In other words, the model predicts an increased propensity of managers to pledge private assets in response to *negative* cash-flow shocks for low liquidity firms, something that is common in both start-ups and firms in financial distress. In the special case of $\lambda = \kappa$, we have $\frac{\beta(C)-\kappa}{1-\kappa} \geq 0$. Therefore, negative wages in response to cash-flow shocks occur exactly when the risk-sharing considerations outweigh the agency issues.

We summarize our findings in the following corollary.

Corollary 1. *Let $\mathcal{C}$ solve (5). Then, the following holds true:*

i) There exists $C' \in [0, \overline{C})$, so that the pay-performance sensitivity β^* (weakly) decreases in on $[C', \overline{C}]$. In particular, $\frac{\partial \beta^*(C)}{\partial C} < 0$ on $[C', \overline{C}]$. If σ is sufficiently low, then $C' = 0$

ii) There exists a unique value $\hat{C} \in [0, \overline{C}]$, such that $\beta(C) > \lambda$ for $C < \hat{C}$. If λ is sufficiently small, it follows that $\hat{C} > 0$.

iii) The loading of wages on the cash-flow shocks is given by $\frac{\beta(C)-\kappa}{1-\kappa}$ and thus negative wages are more prevalent for low-cash firms.

4.2 Risk-sharing vs retained earnings as liquidity management tools

In our setting, the firm has two distinct but connected tools to manage liquidity risks:

- The firm can hedge liquidity risk through labour contracts and provide particularly high-powered incentives β during financial distress when C is close to zero.
- The firm can increase retained earnings accumulation, as proxied by the dividend boundary $\overline{C}$. All else equal, a higher payout boundary $\overline{C}$ implies higher average net-cash holdings.

Let us first establish the following analytic results regarding comparative statics:

Corollary 2 (Hedging through high powered incentives)**.** *For a firm under distress, i.e., $C \simeq 0$, $\beta(C)$, the analytic comparative statics are summarized in the first row of Table 1.*

Corollary 3 (Hedging through cash reserves)**.** *For the target cash-holdings $\bar{C}$, the analytic comparative statics are summarized in the second row of Table 1.*

	$\beta\,(C \approx 0)$	$\overline{C}$
κ	$-$	$-$ (κ sufficiently large)
σ	$-$	$+$ (ρ, λ sufficiently low), $-$ otherwise
μ	$+$	
ρ	$-$ (low ρ), 0 (otherwise)	$-$ (low ρ), $+$ (high ρ)
λ	$-$ (low λ), $+$ (otherwise)	$-$ (low & high λ)
δ	$-$	$-$ (κ sufficiently large)
θ	$-$ (high θ, κ), 0 (otherwise)	$-$ (high θ, κ), 0 (otherwise)

Table 1: Comparative statics.

Next, we will show numerically that these two liquidity management tools are substitutes by analyzing the following experiments: consider constraining the principal to a sub-optimal strategy in one of the two liquidity management tools – (i) an *exogenously* too high $\beta(C)$, or (ii) an *exogenously* too low $\overline{C}$. From our previous discussions, a situation in which the IC constraint (19) is binding is essentially experiment (i) and can thus be proxied for by comparative statics w.r.t. λ, whereas a situation in which the commitment constraint (21) is binding is essentially experiment (ii) and can thus be proxied for by comparative statics w.r.t. θ. In our discussion below, "avg β" refers to the equal-weighted integral $\int_0^{\overline{C}} \beta(C)dC/\overline{C}$.

Changing $\lambda = \kappa$. Let us consider varying the degree of agency friction as measured by the stealing efficiency $\lambda = \kappa$. Column 1 of Figure 3 shows the behaviour of $\overline{C}$ and avg β (solid black lines) when varying $\lambda = \kappa$. The avg β increases mechanically as we are raising the floor on $\beta(C)$ (dashed red line) via the IC constraint. In response to this increased risk-sharing through labor contracts, the need for retained earnings decreases and $\overline{C}$ optimally shrinks. Moreover, more severe moral hazard reduces firm value and thereby also overall hedging demand. Not shown here is that numerically there is almost no movement in $\beta(0)$.

Changing θ. Let us consider varying the degree of commitment by the manager as measured by the bargaining weight θ. As long as (21) is slack changes in θ have no impact on any of the principal's choices. However, once θ is high enough and (21) starts binding the firm has to use an inefficiently low payout boundary $\overline{C}$. Column 2 in Figure 3 illustrates. Constraint (21) starts binding at $\theta \approx .85$, and any further increase in θ reduces the payout-boundary $\overline{C}$. To counteract this deterioration in liquidity management via retained earnings, the principal increases hedging through labor contracts by increasing the pay-performance sensitivity of wages, as indicated by an

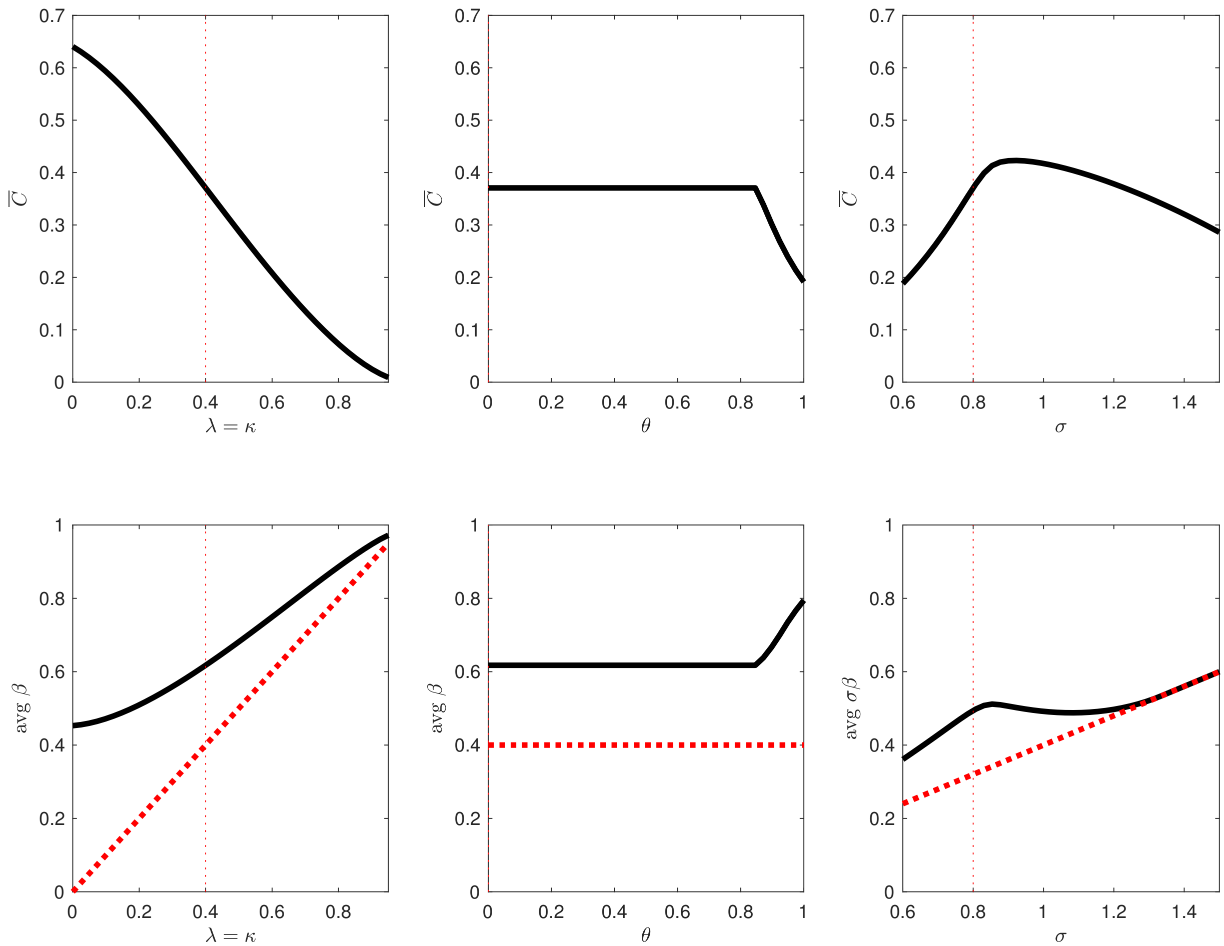

Figure 3: **Comparative statics** w.r.t. $\lambda = \kappa$ (Column 1), w.r.t. θ (Column 2), w.r.t. σ (Column 3), top row $\overline{C}$, bottom row σ-scaled avg β. The solid black lines depict the object described on the y-axis, the dashed red line depicts the IC constraint (19), the thin vertical dashed red line depicts the parameter value in our benchmark.

increase in avg β.

Changing σ. Let us discuss changing the dynamics of the cash-flow generating process. Here, the effects are more complex in that some non-monotonicity appears. First, consider an increase in σ. A higher σ in a pure risk-sharing model, that is with $\lambda = 0$, will lead to a higher payout boundary $\overline{C}$ as default has now become more likely, holding everything else constant. Non-monotonicity can only arise when the commitment constraint (21) starts binding and then follows closely the explanations in the discussion regarding θ. Column 1 in Figure 8 shows the situation in which $\lambda > 0$. We see that $\overline{C}$ is non-monotone even in the absence of (21) binding. The intuition is as follows: higher σ raises the risk of liquidation and requires more intense risk-management, so that $\overline{C}$ and avg β increase. However, due to agency conflicts, the agent must be provided costly incentives $\beta \geq \lambda$, even if this is not optimal from a pure risk-management perspective. Consequently, severe agency conflicts drain the firm value and reduce the overall hedging demand. The latter effect dominates, when σ and λ are sufficiently large and the agent requires a high risk-premium in response to performance-pay.

Changing ρ, δ and μ. The comparative statics of ρ, δ and μ are relegated to appendix E. Since δ essentially captures carry-cost of cash, $\overline{C}$ not surprisingly decreases in δ. Moreover, when the agent is more risk-averse, incentive-pay and therefore hedging through labour contracts becomes more costly, so that the firm hedges more through retained earnings instead, in that $\overline{C}$ increases in ρ. On the other hand, moral hazard has more bite for larger ρ, which in turn implies that overall firm value decreases in ρ. As a result, liquidation gets less inefficient, which calls for less hedging of liquidity risks. This leads to non-monotonic comparative statics of $\overline{C}$ wrt. ρ.

4.3 Stock Return Volatility and Agency Conflicts

In this section, we discuss how firm agency conflicts impact the firm's stock returns:

$$dR_t = \frac{dDiv_t + dv(C_{t^-})}{v(C_{t^-})} = r + \delta + \frac{dDiv_t}{v(C_{t^-})} + \Sigma_t dZ_t. \tag{23}$$

Of particular interest is the stock-return volatility $\Sigma_t = \Sigma(C_t)$ where

$$\Sigma(C) = \sigma(1 - \beta(C)) \times \frac{v'(C)}{v(C)}. \tag{24}$$

Recall our assumption that the firm is 100% equity financed and that we do not take a stance on the implementation of the manager's contract. In case the contract is implemented via stock, vesting stock or stock options, dR_t is best interpreted as the return on *outside* equity, owned by shareholders, rather than *inside* equity, owned by management.

First, contrary to the existing literature on dynamic cash-management (compare e.g. Décamps et al. (2011)) the firm's stock return volatility does not necessarily decrease in the firm's level of financial slack.[17] In fact, we find that firms with relatively low levels of cash can have less volatile stock returns than otherwise comparable firms with high cash-levels. The reason is that in our model firms hedge liquidity risk intensely through labor contracts under financial distress. Under these circumstances, the agent's compensation package is highly contingent on cash-flow realizations and firm performance, so that a substantial amount of risk is absorbed through labor contracts. This in turn lowers the stock return volatility Σ_t of *outside* equity owned by shareholders. Especially when cash-flow uncertainty σ is low, the agent's compensation scheme is exposed to a considerable amount of cash-flow risk, so that stock-return volatility may follow a hump-shaped pattern in C. As a consequence of intense hedging through labour contracts, stock-return volatility is then even lowest under financial distress.

Second, we find that the nature of agency conflicts determines its impact on the firm's stock return volatility. Severe moral hazard λ over cash-*flows* requires the manager to be sufficiently exposed to cash-flow realizations dX by means of high-powered incentives β, thereby leading to a low stock-return volatility. In contrast, severe moral hazard κ over cash-*holdings* or high δ imply large carry cost of cash. This leads to little hedging of liquidity risks and thereby a high stock-return volatility.

Corollary 4. *Stock return volatility $\Sigma(C)$ decreases in a neighbourhood of $\overline{C}$, and also decreases for low levels of C when L is sufficiently low. Further, we have the following comparative statics:*

i) More severe moral hazard λ reduces the stock return volatility:

- *For any C, $\Sigma(C)$ decreases in moral hazard, provided λ is sufficiently large. That is, for all $C \geq 0$ there exists $\bar{\lambda} \in (0,1)$, such that $\frac{\partial \Sigma(C)}{\partial \lambda} < 0$ for $\lambda \geq \bar{\lambda}$.*
- *For ρ or λ sufficiently small, $\Sigma(C)$ decreases in λ for C close to $\overline{C}$.*

[17] In dynamic liquidity management models without labor contracts, stock return volatility is given by $\frac{v'(C)}{v(C)}\sigma$ where C is the firm's cash stock. Since the value function is regardless of labor contracts strictly increasing and concave, stock return volatility *always* decreases in financial slack.

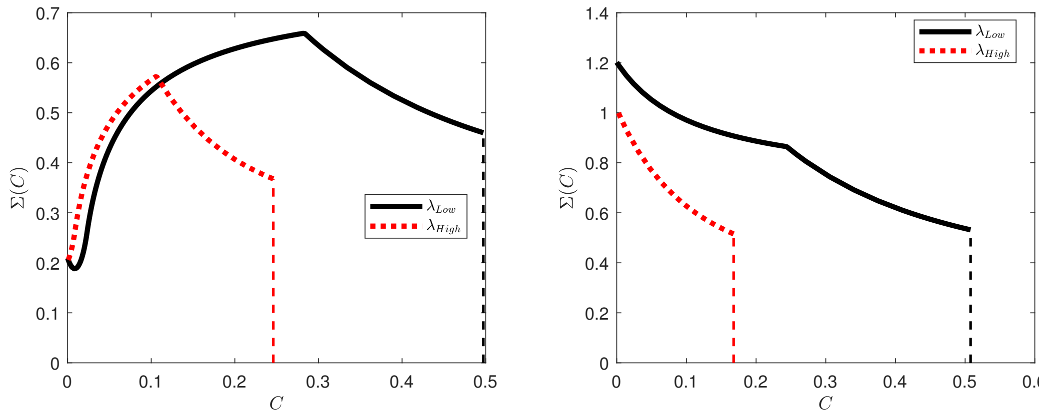

Figure 4: **Stock Return** is non-monotonic in C and increases in λ. The left panel depicts the case of low volatility, $\sigma = 0.8$, and the right panel the case of high volatility, $\sigma = 0.9$. The Parameters are $\mu = 0.25, r = 0.1, \delta = \kappa = 0.25$, $\theta = 0$, $L = 0.25$, $\lambda_{Low} = 0.5 < 0.75 = \lambda_{High}$ and $\rho = 7$.

ii) More severe moral hazard κ increases the stock return volatility. For C close to $\overline{C}$, $\Sigma(C)$ increases in κ.

5 The Model with Refinancing

In this section, we introduce the possibility of refinancing. Similar to Hugonnier et al. (2014), we assume that there are search frictions in capital markets, in that finding new outside investors requires some time and search effort. In particular, conditional on seeking refinancing, the firm finds investors willing to contribute funds with probability πdt during a short-period of time $[t, t + dt]$, so that a financing opportunity arrives according to some jump process $d\Pi$ with intensity $\pi \geq 0$.

Upon finding investors, we assume without loss of generality that there are no further cost to refinancing – the firm can issue equity at a fair price to raise cash and therefore appropriates all generated surplus.[18] In particular, when the firm raises an amount Δ of cash from outside investors, these outside investors obtain equity worth exactly Δ. For simplicity, looking for investors is costless and not subject to moral hazard, and for technical reasons we suppose that $d\Pi_t = 0$ with probability one at all times t, where the firm chooses $\Delta_t = 0$.[19]

Since refinancing raises the amount of cash the manager can steal from, the optimal contract

[18]If outside investors and existing shareholder were to split the surplus according to the Nash-Bargaining protocol with respective weights $\eta, 1 - \eta$, then the problem were isomorphic to one where the arrival rate is altered from π to $\eta\pi$, so that the choice $\eta = 1$ is indeed wlog.

[19]This means that either shareholders look for new investors or the manager does so, in which case her search activity is observable and contractible to shareholders. Furthermore, it is straightforward to incorporate monetary search cost but as endogenous cost due to agency arise, this modification is unlikely to alter our findings.

must align her incentives during the refinancing event. This alignment of incentives could in principle be reached via three mechanisms: (1) rewarding the manager with a (lumpy) increase in future promised payments Γ (sometimes referred to as "payment for luck"), (2) rewarding the manager with a (lumpy) wage payment, and (3) requiring the agent to contribute a prescribed amount of funds. Recall that Assumption 1 restricts cumulative wages to $\lim_{\varepsilon \to 0} w_{t+\varepsilon} - w_t \geq 0$, which leads to two outcomes: it rules out (3) and make φ a state-variable in the refinancing event as it cannot be adjusted freely anymore.[20] For the following discussion, let us briefly ignore (2), the lumpy wage payments.

Let us now consider a firm at time t^-, i.e., just before time t, with cash-holdings M_{t^-} and $C_{t^-} = M_{t^-} - Y_{t^-}$. Assume for the moment that the firm is not refinancing all the way to the payout boundary so that $dDiv = dw = 0$. When a refinancing opportunity arises, i.e., $d\Pi_t = 1$, the agent can potentially abscond with the total cash-balance just after outside investors put in amount Δ_t. From doing so, she receives $\kappa(M_{t^-} + \Delta_t)$ but loses her deferred compensation Y_{t^-} and the lump-sum reward Γ_t, so that stealing is not optimal if

$$\kappa(M_{t^-} + \Delta_t) \leq Y_{t^-} + \Gamma_t \Longleftrightarrow \varphi_{t^-} := \frac{Y_{t^-}}{M_{t^-}} \geq \frac{\kappa(C_{t^-} + \Delta_t) - \Gamma_t}{C_{t^-} + \kappa\Delta_t - \Gamma_t} \tag{25}$$

or equivalently

$$\Gamma_t \geq \frac{\kappa\Delta(1 - \varphi_{t^-}) - (\varphi_{t^-} - \kappa)C_{t^-}}{1 - \varphi_{t^-}}. \tag{26}$$

Hence, in order to align incentives during a financing round, the principal must either give the agent a high reward Γ_t or must have chosen higher deferred compensation $Y_{t^-} > \kappa M_{t^-}$ beforehand, resulting in $\varphi_{t^-} > \kappa$, both of which are costly. Since the incentive constraint (25) tightens when more funds Δ_t are raised, the firm might decide to raise less funds due to agency conflicts. At the optimum, inequalities (25) and (26) hold as equalities, which essentially means that the principal – ceteris paribus – designs the contract to minimize carry cost of cash and flotation cost of refinancing.

Cash-holdings and the firm's financial slack in a refinancing event change according to

$$\frac{dM_t}{d\Pi_t} = \Delta_t \text{ and } \frac{dC_t}{d\Pi_t} = \frac{dM_t}{d\Pi_t} - \frac{dY_t}{d\Pi_t} = \Delta_t - \Gamma_t.$$

Because the manager is paid for luck $\Gamma \geq 0$ upon refinancing, she requires a lower flow wage and

[20]Without this Assumption 1 there would be a complete separation between φ, the variable controlling cash-holding in the firm, and (Δ, Γ), the amount of cash raised and the payment for luck required. This difference does not matter as for most of our analysis $\varphi \geq \kappa$ holds with equality.

W_t features a lower required drift by $\frac{\pi(1-e^{-\rho r\Gamma_t})}{\rho r} > 0$. Essentially, $\Gamma > 0$ shifts part of the manager's compensation from distress states towards states, in which the firm is flush with liquidity. While this is beneficial from a risk-management point of view, it comes at the cost of exposing the agent to jump risk $d\Pi$.

The dynamics of C then follow

$$dC_t = \mu_{Ct}dt + (1-\beta_t)dZ_t + (\Delta_t - \Gamma_t)d\Pi_t - dDiv_t \tag{27}$$

with

$$\mu_{Ct} := rC_{t^-} + \mu - \frac{\rho r}{2}(\beta_t\sigma)^2 - \delta A\left(\frac{\varphi_t C_{t^-}}{1-\varphi_t}\right) + \frac{\pi(1-e^{-\rho r\Gamma_t})}{\rho r}. \tag{28}$$

We again consider the relaxed problem (allowing φ to be freely chosen on $[\kappa, 1]$ *outside* a refinancing event) via the following HJB equation:

$$(r+\delta)v(C) = \max_{\beta\geq\lambda,\varphi,\Gamma,\Delta}\left\{v'(C)\mu_C + \pi\big[v(C+\Delta-\Gamma) - v(C) - \Delta\big] + \frac{v''(C)(1-\beta)^2}{2}\right\} \tag{29}$$

subject to

$$\varphi \geq \max\left\{\kappa, \frac{\kappa(C_{t^-}+\Delta_t)-\Gamma_t}{C_{t^-}+\kappa\Delta_t-\Gamma_t}\right\}.$$

Define

$$C^*(C) := C + \Delta(C) - \Gamma(C), \tag{30}$$

A firm's *refinancing policy* is then given by two of $C^*(C), \Delta(C), \Gamma(C)$. Next, we have to consider two scenarios: (1) shareholders can ex-ante commit to a refinancing policy or (2) shareholders cannot commit ex-ante, but instead maximize their refinancing policy *conditional* on a refinancing opportunity arising. We will discuss these scenarios in turn. Importantly, in the discussions we maintain the counter-factual assumption that the same $\overline{C}$ applies in all considered scenarios for ease of comparison. Of course, once fully solved, different payout thresholds apply in different scenarios.

Lastly, it is during the refinancing event that our restriction on the wage process, Assumption 1, possibly has bite: A slack $\varphi > \kappa$ helps the firm raise more Δ for the same amount of pay-for-luck Γ by relaxing constraint (26), as the firm cannot freely adjust φ during the refinancing event. For expositional clarity, however, we assume parameters that result in $\varphi = \kappa$ for all $C \in [0, \overline{C}]$ in our discussion below.

5.1 No ex-ante commitment and constant proportional flotation costs

Suppose shareholders cannot ex-ante commit to any refinancing policy. This means that upon finding outside investors, i.e., $d\Pi_t = 1$, the firm raises the *ex-post* optimal amount Δ rather than the *ex-ante* one. More specifically, inspecting the HJB, it is as if the shareholders ignore the impact that the optimal Γ has on the drift of C, i.e., they ignore $\frac{\partial \mu_{Ct}}{\partial \Gamma}$, and maximize the *static* problem

$$\max_{\Delta \geq 0, \Gamma} \left\{ v(C + \Delta - \Gamma) - v(C) - \Delta \right\} \text{ s.t. (26).}$$

Inspecting the FOC, we see that this results in an implied *constant* proportional flotation cost

$$v'(\underbrace{C + \Delta - \Gamma}_{=:C^*_{LC}}) = 1 + \underbrace{\frac{\kappa}{1-\kappa}}_{\text{Flotation Cost}}. \tag{31}$$

Further, the firm refinances to the *same* target cash-level $C^*_{LC} < \overline{C}$ regardless of current C, and there is no lumpy wage payment. Consider $\kappa = 0$. The FOC implies $v'(C^*_{FB}) = 1$, which in turn implies $C^*_{FB} = \overline{C} > C^*_{LC}$. Absent agency conflicts, the firm refinances to the payout boundary.

5.2 Ex-ante commitment and state-dependent flotation costs

In the ex-ante commitment case, the principal optimally takes into account that any choice of (Δ, Γ) via Γ affects increases the drift μ_{Ct}. Let $\hat{C}^*(C)$ solve the resulting FOC:

$$v'(\hat{C}^*(C)) = 1 + \underbrace{\frac{\kappa}{1-\kappa}\left[1 - v'(C) e^{-\rho r \frac{\kappa}{1-\kappa}[\hat{C}^*(C) - C]}\right]}_{\text{State-Dependent Flotation Cost}}. \tag{32}$$

The shareholders essentially commit to act as if they are facing an endogenously *lower* state-dependent flotation cost than in the static optimization problem above. Note that (marginal) flotation cost ceteris paribus decrease in $\hat{C}^*(C)$. If $\hat{C}^*(C)$ is strictly lower than $\overline{C}$, then it is the optimal refinancing level, i.e., $C^*(C) = \hat{C}^*(C)$. In this case, flotation cost are strictly positive but less than in the ex-ante commitment case, and the marginal value of cash after refinancing equals marginal cost of raising funds.

If however, $\hat{C}^*(C) > \overline{C}$, we have *negative* flotation cost, which occurs exactly when

$$\ln(v'(C)) \times \frac{(1-\kappa)}{\rho r \kappa} + C \geq \overline{C}, \tag{33}$$

This is more likely to for low C firms, as then $v'(C)$ is high. Consider refinancing all the way to $\hat{C}^*(C) > \overline{C}$. This would trigger *immediate* dividend and wage payouts to reset to $\overline{C}$. The key observation now is that such dividend payouts would be a wash[21] but the required jump in managerial compensation, $\Gamma + dw_{refi}$, is not. Consequently, define

$$C^*(C) := \min\{\hat{C}^*(C), \overline{C}\}. \tag{34}$$

The jump in managerial compensation in a refinancing event, $\Gamma + dw_{refi}$, is then given by

$$\Gamma = \frac{\kappa}{1-\kappa}[C^*(C) - C] \geq 0 \text{ as well as } dw_{refi} = 1_{\{\hat{C}^*(C)>\overline{C}\}} \left[\frac{\ln(v'(C))}{\rho r} - \frac{\kappa}{1-\kappa}(\overline{C} - C)\right] \geq 0, \tag{35}$$

there is no dividend payments, and the firm raises an amount of cash of

$$\Delta(C) = C^*(C) - C + \Gamma + dw_{refi} = \begin{cases} \frac{1}{1-\kappa}[\hat{C}^*(C) - C] & \hat{C}^*(C) \leq \overline{C} \\ \frac{\kappa}{1-\kappa}[\hat{C}^*(C) - \overline{C}] + \frac{1}{1-\kappa}[\overline{C} - C] & \hat{C}^*(C) > \overline{C} \end{cases}. \tag{36}$$

We note that pay-for-luck is *excessive*, as it is more — by $\frac{\kappa}{1-\kappa}[\hat{C}^*(C) - \overline{C}]$ — than the amount of cash needed to reset to $\overline{C}$ while simultaneously preserving incentive compatibility.

Figure 5 demonstrates that the refinancing threshold $C^*(C)$ can be non-monotonic in C. While the target refinancing level follows a U-shaped pattern, the amount raised within a financing round, $\Delta(C)$, unambiguously decreases in C.

Corollary 5. *Under full ex-ante commitment to a refinancing strategy:*

i) The amount raised Δ and Γ decrease in C

ii) The target level $C^*(C)$ increases in a neighbourhood of $\overline{C}$

iii) The target level $C^*(C)$ decreases in a neighbourhood of zero, provided κ or ρ is sufficiently small.

Setting $\kappa = 0$ implies the first-best $C^*_{FB} = \overline{C}$. In the ex-ante commitment scenario, $v'(C) > 0$ and holding the payout boundary constant, the principal commits to more aggressive refinancing than implied by the static problem, i.e., $C^*_{FB} \geq C^*(C) \geq C^*_{LC}$. Committing to *over*-refinancing and even *excessive* pay-per-luck, the firm increases the drift of C and thus relaxes the liquidity

[21] Any dollar raised to be used for an *immediate* dividend payment is paid for by shareholders themselves. Thus, a small *exogenous* refinancing cost would eliminate any part of refinancing used for such immediate dividend payouts.

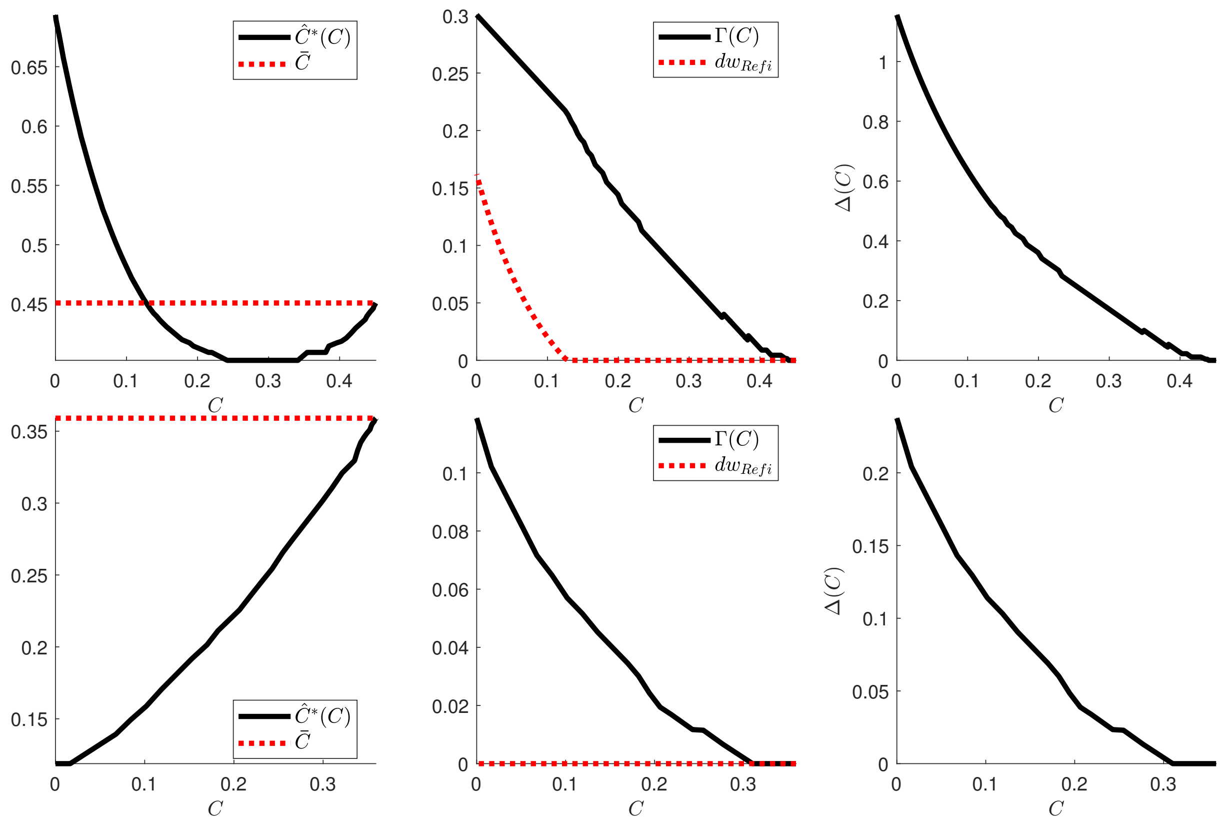

Figure 5: **Optimal Refinancing** under full ex-ante commitment w.r.t. the refinancing strategy. Parameters are $\mu = 0.25, r = 0.1, \kappa = \lambda \in \{0.4, 0.5\}$, $\theta = 0$, $L = 0$, $\sigma = 0.8$, $\delta = 0.25$, $\pi = 0.2$ and $\rho = 7$.. The upper three panels use $\lambda = \kappa = 0.4$, the lower three panels $\lambda = \kappa = 0.5$.

problem at the cost of larger than statically optimal payments to agent in the event of refinancing. However, as the marginal utility of cash to the shareholders is higher pre- than post-refinancing due to $v''(C) \leq 0$, this is a beneficial trade-off.

5.3 Capital Market Access and Hedging

How does the possibility to raise funds in capital markets impact the firm's risk-management? Intuitively, one could argue that better refinancing opportunities render altogether less hedging needed, as demonstrated in e.g. Hugonnier et al. (2014). However, our model yields a different prediction. Under less frictional capital markets, finding outside investors becomes easier and liquidation less likely, so that there is less need to hold large liquidity reserves, in that $\overline{C}$ decreases in π. In addition, the access to outside funds boosts the firm's going concern value and liquidation becomes more inefficient. Thus, *conditional* in being in a low C state, shareholders have more incentives to avert termination when π is high, in which case it becomes optimal to hedge more intensely via labour contracts. Furthermore, surviving the next instant $[t, t + dt]$ entails the additional benefit of possibly having a refinancing opportunity, which happens with probability πdt, further increasing the hedging demand. Inspecting the first-order conditions for both the ex-ante commitment and no commitment case, we see that π only indirectly affects the choice of $C^*(C)$ via $v(\cdot)$, but does not directly enter either (31) or (32).

Therefore, firms with better access to capital markets tend to hedge *less* through internal cash but *more* through labor contracts. This holds true regardless of the commitment structure. When shareholders cannot commit to a refinancing policy, they also raise less cash during a single financing round, when financing opportunities arrive more frequently, i.e., C^* decreases in π. We summarize these findings in the following corollary.

Corollary 6. *For a firm under distress, i.e., $C \simeq 0$, $\beta(C)$ increases in π. Target cash-holdings $\bar{C}$ decrease in π. In the limited commitment case, the refinancing target C^* decreases in π.*

6 Conclusion

We present a model of liquidity management and financing decisions under moral hazard in which a firm accumulates cash to forestall liquidity default. When the cash balance is high, a tension arises between accumulating more cash to reduce the probability of default and providing incentives for the manager. When the cash balance is low, the firm hedges against liquidity default by transferring cash flow risk to the manager via high powered incentives. This risk transfer occurs even though the manager is risk averse and the firm's owners are risk neutral because default is costly. Firms with more volatile cash flows transfer less risk to the manager and hold more cash. Agency conflicts lead to endogenous flotation costs related to the severity of the moral hazard problem, even in a market with no physical cost of raising financing. These flotation costs are state-dependent, lead to raising more than a static optimization would imply, and sometimes even lead to large cash-payouts to the agent in case of successful refinancing. Finally, because the manager's incentive-pay absorbs part of the liquidity risk, the firm's stock return volatility can be non-monotonic in the level of cash and decreases in the severity of moral hazard.

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Appendix

A Preliminaries

A.1 Regularity Conditions

Throughout the paper and for all problems, we impose finite utility for any consumption process c

$$\mathbb{E}\left[\int_0^\infty e^{-rt}|u(c_t)|dt\right] < \infty$$

and square integrability conditions of dividend payouts Div and payments w:

$$\mathbb{E}\left[\int_0^\infty e^{-rt}dDiv_t\right]^2 < \infty \text{ and } \mathbb{E}\left[\int_0^\infty e^{-rt}dw_t\right]^2 < \infty. \tag{A.1}$$

Finite utility implies that

$$\lim_{t\to\infty} e^{-rt}U_t(\cdot) \equiv \lim_{t\to\infty} e^{-rt}\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}u(c_s)ds\right] = 0, \tag{A.2}$$

where $U_t(\cdot)$ represents the agent's continuation value under any, admissible strategy, suppressed for convenience. Condition (A.2) is also known as the *transversality condition* for the co-state, when solving the contracting problem by means of Pontryagin's maximum principle (compare e.g. Williams (2009), Williams (2011)).

Next, note that

$$\hat{S}_t = \int_0^t e^{r(t-s)}dw_s - \int_0^t e^{r(t-s)}\hat{c}_s ds + \hat{S}_0 e^{rt}$$

for the consumption process $\hat{c}$ specified by contract $\mathcal{C}$, while c is the agent's actual consumption. Savings $\hat{S}$ corresponds to consumption $\hat{c}$ and savings S to consumption c.

We impose the no-Ponzi condition for all feasible consumption processes $c, \hat{c}$:

$$\mathbb{P}(\lim_{t\to\infty} e^{-rt}S_t \geq 0) = \mathbb{P}(\lim_{t\to\infty} e^{-rt}\hat{S}_t \geq 0) = 1.$$

Further, $c, \hat{c}$ must satisfy the transversality condition:

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}u'(c_t)S_t = 0 = \lim_{t\to\infty} e^{-rt}\mathbb{E}u'(c_t)S_t = 0$$

Due to finite utility it follows that marginal utility – which is proportional to flow utility – must also be finite $\mathbb{P}$-almost surely, so that one can disregard $u'(c_t)$ in the transversality condition, which leads to

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}S_t = \lim_{t\to\infty} e^{-rt}\mathbb{E}\hat{S}_t = 0.$$

Combined with the no-Ponzi condition, it follows after invoking Fatou's Lemma in fact that

$$\mathbb{P}(\lim_{t\to\infty} e^{-rt}S_t = 0) = \mathbb{P}(\lim_{t\to\infty} e^{-rt}\hat{S}_t = 0) = 1,$$

which we refer to as the *transversality condition*, even though it emerges as a combination of transversality and No-Ponzi condition. By the triangle inequality:

$$\lim_{t\to\infty} e^{-rt}|S_t - \hat{S}_t| = 0 \ \mathbb{P} - a.s. \Longrightarrow \lim_{t\to\infty} e^{-rt}|\hat{c}_t - c_t| = 0 \ \mathbb{P} - a.s.$$

For technical reasons, we postulate that the processes β, α are almost surely bounded, so that

$|\beta_t|, |\alpha_t| < M$ almost surely, i.e. $\mathbb{P}\big(|\psi_t| < M\big) = 1$ for $\psi \in \{\alpha, \beta\}$, for any t. The equivalence of the measures $\mathbb{P}, \mathbb{P}^b$ (to be discussed in the next paragraph) ensures that the sensitivities are almost surely bounded under each probability measure used throughout the paper. We assume $M \in \mathbb{R}_+$ to be sufficiently large, so that this imposed constraint actually never binds in optimum. Furthermore, we impose that β and α are of bounded variation.

Let us further assume throughout the appendix that the agent cannot own the firm. Otherwise, it would be

$$v_\tau = \max\left\{L, \frac{\mu - \rho r \sigma^2/2}{r+\delta}\right\},$$

so that whenever $v_\tau > L$, the shareholders would sell the firm to the agent, who would run it forever. The above assumption is always satisfied for L, ρ or σ sufficiently large.

A.2 Change of Measure

To start with, fix a probability measure $\mathbb{P}$, such that $dX_t = \mu dt + \sigma dZ_t$ with a $\mathbb{F}$-progressive standard Brownian Motion Z under the measure $\mathbb{P}$. Take a progressive process b, that is absolutely continuous and one can write $db_t = b_t^0 dt$ for some process b^0. Define the process χ via $\chi_t = \frac{db_t}{\sigma dt}$ for all $t \geq 0$, almost surely. Further, let

$$\Gamma_t = \Gamma_t(b) = \exp\Big(\int_0^t \chi_u dZ_u - \frac{1}{2}\int_0^t \chi_u^2 du\Big).$$

Assuming that the so-called Novikov condition is satisfied, i.e.,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^\tau \chi_t^2 dt\right)\right] < \infty,$$

it follows that Γ follows a martingale. Given our restriction of bounded sensitivities, the Novikov-Condition is evidently met. Due to $\mathbb{E}[\Gamma_0] = 1$, it is evident that Γ is a progressive density process and defines a probability measure $\mathbb{P}^b$ via the Radon-Nikodym derivative

$$\left(\frac{d\mathbb{P}^b}{d\mathbb{P}}\right)\Bigg|_{\mathcal{F}_t} = \Gamma_t = \Gamma_t(b).$$

Under the probability measure $\mathbb{P}^b$, the process Z^b with

$$Z_t^b = Z_t - \int_0^t \chi_u du = \frac{X_t - \mu t - \int_0^t b_u^0 du}{\sigma}$$

follows a standard Brownian Motion up to the stopping time τ. All measures $\{\mathbb{P}, \mathbb{P}^b : b\}$ are equivalent for suitable absolutely continuous processes b, that satisfy the above stated conditions, such that the measures share the same null sets.

Girsanov's theorem is only applicable if b is absolutely continuous $\mathbb{P}$-almost surely, in which case Z^b follows a Brownian Motion under $\mathbb{P}^b$.

B The Agent's Problem: Proof of Proposition 3

We split up the proof in two parts. First, we establish the representation of U by means of a stochastic differential equation, given a contract $\mathcal{C}$. From there, we proceed to show the claim regarding incentive compatibility.

B.1 Martingale Representation: Proof of Proposition 1 i)

Proof. Let in the following $\mathcal{C} = (\hat{c}, w, \hat{b})$ represent the manager's contract with $\mathcal{C} \in \mathbf{C}$. We denote the manager's continuation value by

$$U_t = U_t(\mathcal{C}) = \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(\hat{c}_s)) ds \right],$$

where $\hat{c}$ is prescribed consumption, which might differ from actual consumption c. Define

$$A_t \equiv \mathbb{E}_t \left[\int_0^\infty e^{-rt} u(\hat{c}_s) ds \right] = \int_0^t e^{-rs} u(\hat{c}_s) ds + e^{-rt} U_t(\mathcal{C}) \tag{B.1}$$

By construction, $\{A_t : 0 \leq t \leq \infty\}$ is a square integrable martingale, progressive with respect to $\mathbb{F}$ under $\mathbb{P}$. By the martingale representation theorem, there exist now $\mathbb{F}$-predictable processes $\alpha, \beta, \hat{\Gamma}$ such that

$$\begin{aligned} e^{rt} dA_t = &(-\rho r U_{t^-}) \beta_t \big(dX_t - \mu dt \big) - (-\rho r U_{t^-}) \alpha_t \big(dN_t - \delta dt \big) \\ &+ (-\rho r U_{t^-}) \Gamma_t (d\Pi_t - \pi dt) \mathbf{1}_{\{C_{t^-} < C^*\}}. \end{aligned}$$

and therefore

$$\begin{aligned} dU_t = &r U_{t^-} dt - u(\hat{c}_t) dt + (-\rho r U_{t^-}) \beta_t \big(dX_t - \mu dt \big) - (-\rho r U_{t^-}) \alpha_t \big(dN_t - \delta dt \big) \\ &+ (-\rho r U_{t^-}) \Gamma (d\Pi_t - \pi dt) \mathbf{1}_{\{C_{t^-} < C^*\}}. \end{aligned}$$

□

B.2 Incentive Compatibility: Proof of Proposition 1 ii) and iii)

We consider for brevity the case $\pi = 0$. It is straightforward to adapt the proof for $\pi > 0$.

Proof. We prove first the following auxiliary Lemma

Lemma 1. *Fix a $\mathbb{F}$-predictable process $\hat{c}$ and let $\mathcal{S} \in \mathbb{R}$. Consider the problem*

$$\begin{aligned} &U_t = \max_{\{c_s\}_{s \geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s) ds \right] \\ &\text{subject to } d\Delta_s = r\Delta_s ds + d\hat{c}_s ds - c_s ds, \Delta_t = 0 \text{ and } \lim_{s \to \infty} e^{-r(s-t)} |\Delta_t - \Delta_s| = 0 \text{ a.s.} \end{aligned}$$

Next consider the problem

$$\begin{aligned} &U_t' = \max_{\{\tilde{c}_s\}_{s \geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(\tilde{c}_s) ds \right] \\ &\text{subject to } d\Delta_s = r\Delta_s ds + d\hat{c}_s ds - \tilde{c}_s ds, \Delta_t = \mathcal{S} \text{ and } \lim_{s \to \infty} e^{-r(s-t)} |\Delta_t - \Delta_s| = 0 \text{ a.s.} \end{aligned}$$

Then, $c_t + r\mathcal{S} = \tilde{c}_t$ and $U_t' = e^{-\rho r \mathcal{S}} U_t$.

Proof. Suppose that there exists a process $c' \neq \tilde{c}$, which satisfies the transversality condition, such that

$$U_t'(c') > U_t'(\tilde{c}) = e^{-\theta r \mathcal{S}} U_t.$$

Define the process c'' via $c''_t = c'_t - r\mathcal{S}$. Then c'' satisfies the transversality condition and

$$\mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c''_s)ds\right] = e^{\rho r\mathcal{S}}U'_t(\{c'\}) > U_t,$$

a contradiction. □

Next, we provide necessary and sufficient conditions for $\mathcal{C}$ to be incentive-compatible, in that $\hat{S}_t = S_t$ and $\hat{b}_t = b_t = 0$ for all $t \geq 0$ holds almost surely.

For this sake write, $db_t = (b^0_t - b^2_t)dt + db^1_t$, where b^0 and b^2 are absolutely continuous and almost surely positive, i.e, write $db_t = \hat{b}_t dt + db^1_t$, where $b^0_t = \max\{0, \hat{b}_t\}$ and $b^2_t = -\min\{0, \hat{b}_t\}$. Here, b^0 corresponds to cash-flow diverted, while b^2 is the amount by which cash-flow is boosted by means of the agent's savings account. Define $\Delta_t \equiv S_t - \hat{S}_t$ the deviation state with $\Delta_0 = 0$ and note that

$$d\Delta_t = r\Delta_t dt + \hat{c}_t dt - c_t dt + \lambda b^0_t dt + \kappa db^1_t - b^2_t dt,$$

where $\hat{c}$ is the prescribed consumption and is such that $S_t = \hat{S}_t$, i.e. $\Delta_t = 0$ for all t. Note that $dZ^b_t \equiv (dX_t - \mu dt + b^0_t dt)/\sigma$ is the increment of a standard Brownian Motion under the measure $\mathbb{P}^b$. We rewrite for $t < \tau$:

$$dU_t = rU_{t^-}dt - u(\hat{c}^M_t)dt + (-\rho r U_{t^-})\beta_t\big(dZ^b_t + b^0_t dt\big) - (-\rho r U_{t^-})\alpha_t\big(dN_t - \delta dt\big).$$

Let $\hat{U}$ the agent's actual continuation value, so that

$$\hat{U}_t(c) = \hat{U}_t \equiv \mathbb{E}^b_t\left[\int_t^\infty e^{-r(s-t)}u(c_s)ds\right],$$

where the expectation $\mathbb{E}^b_t$ is taken under the measure $\mathbb{P}^b$, induced by the choice of b.

Define the agent's certainty equivalent $W_t = \frac{-\ln(-\rho r U_t)}{\rho r}$ and $Y_t \equiv W_t - S_t$.

First, let us consider the agent deviates at time t^- through specifying $M_{t^-} \geq db^1_t > 0$, so that $db_t \notin o(dt)$. The principal can detect this deviation and accordingly punish the agent through reducing her certainty equivalent by the same amount. The agent can either leave the firm and avoid the punishment or take the punishment and stay, in which case the deviation does not yield any profit for her. In case she leaves the firm, her savings equal $S_t = S_{t^-} + \kappa db^1_t$, yielding continuation value by Lemma 1:

$$\int_t^\infty e^{-r(s-t)}u(c_s)ds = \frac{u[r(S_{t^-} + \kappa db^1_t)]}{r},$$

as the agent perfectly smoothes consumption after contract termination and consumes at each time flow interest of savings. The continuation value is maximized for $db^1_t = M_{t^-}$. The deviation is not profitable, if and only if

$$\frac{u[(r(S_{t^-} + \kappa db^1_t)]}{r} \leq U_{t^-} \Longleftrightarrow Y_{t^-} \geq \kappa M_{t^-}.$$

Hence, a necessary condition for the contract $\mathcal{C}$ to be incentive compatible is that $Y_{t^-} \geq \kappa M_{t^-}$ with probability one for all times $t \geq 0$.

Second, let us turn to strategies where $db^1_t = 0$ for all $t \geq 0$. Let $t > 0$ and suppose the manager follows the recommended policy from time t onwards, in that $b^0_s = 0$ and $c_s = \hat{c}_s + r\Delta_t$ for all $s \geq t$ by Lemma 1. The payoff from following this strategy is represented by the auxiliary gain process

$$G^M_t \equiv G^M_t(c, b) = \int_0^t e^{-rs}u(c_s)ds + e^{-\rho r\Delta_t}e^{-rt}U_t$$

and by means of Lemma 1, it suffices to consider deviations of this type, which yield weakly higher payoff than deviations of any other type. In addition, $\hat{U}_s = e^{-\rho r \Delta_t} U_s$ for $s \geq t$.

Next, note that the transversality condition and finite utility imply that $e^{-\rho r \Delta} U_t < \infty$ for all $t \geq 0$, so that $\lim_{t\to\infty} \mathbb{E}^b e^{-\rho r \Delta_t} e^{-rt} U_t = 0$ for any possible strategy of the manager. which implies that the manager's actual payoff equals

$$\hat{U}_{0^-} = \max_{c,b^0} \mathbb{E}^b \int_0^\infty e^{-rs} u(c_s) ds = \max_{c,b^0} \mathbb{E}^b G_\infty^M = \max_{c,b^0} \mathbb{E}^b \lim_{t\to\infty} G_t^M.$$

By Itô's Lemma:

$$\begin{aligned}
& e^{\rho r \Delta_t} e^{rt} dG_t^M \\
&= \left(u(c_t) e^{\rho r \Delta_t} - u(\hat{c}_t) - \rho r U_{t^-} (r\Delta_t + \hat{c}_t - c_t + \lambda b_t^0 - b_t^2) - (-\rho r U_{t^-}) \beta_t b_t^0 \right) dt \\
&+ (-\rho r U_{t^-}) \beta_t dZ_t^b - (-\theta r U_{t^-}) \alpha_t (dN_t - \delta dt) \\
&\equiv \mu_{tG}^M(\cdot) dt + (-\rho r U_{t^-}) \beta_t dZ_t^b - (-\rho r U_{t^-}) \alpha_t (dN_t - \delta dt)
\end{aligned}$$

Observe that, because α, β are bounded and finite utility is imposed, we have

$$\mathbb{E}^b \left(\int_0^t e^{-rs} \beta_s (-\rho r U_{s^-}) dZ_s^b \right) = \mathbb{E}^b \left(\int_0^t e^{-rs} \alpha_s (-\rho r U_{s^-}) (dN_s - \delta ds) \right) = 0,$$

for any absolutely continuous b. It is then evident that by choosing $b_t^0 = 0$, $c_t = \hat{c}_t$, the manager can ensure that $\Delta_t = \mu_{tG}^M(\cdot) = 0$ for all $t \geq 0$, in which case $\{G^M(\hat{c}, 0)\}$ follows a martingale under $\mathbb{P}$ with last element $G_\infty^M(\cdot)$, such that $\mathbb{E}|G_\infty^M(\hat{c}, 0)| < \infty$ due to the regularity conditions we impose. Hence, by optional sampling

$$\hat{U}_{0^-} = \max_{c,b^0} \mathbb{E}^b G_\infty^M(c, b^0) \geq \mathbb{E} G_\infty^M(\hat{c}, 0) = \lim_{t\to\infty} \mathbb{E} G_t^M(\hat{c}, 0) = U_{0^-}.$$

Next, observe that the highest value that $\mu_{tG}^M(\cdot)$ can obtain given Δ_t is given by the maximization over c_t and b_t^0, where the solution satisfies the following FOC:

$$u'(c_t) e^{r\rho\Delta_t} = -\rho r U_{t^-},$$

which implies

$$u(c_t + r\Delta_t) = rU_{t^-},$$

and $b_t^0 = b_t^2 = 0$ if and only if:

$$\lambda \Delta \rho r u(c_t) e^{\rho r \Delta_t} - \lambda \rho r U_{t^-} + (\rho r U_{t^-}) \beta_t \leq 0 \text{ and } \Delta \rho r u(c_t) e^{\rho r \Delta_t} + \rho r U_{t^-} + (\rho r U_{t^-}) \beta_t \leq 0.$$

If $\mathcal{C}$ is such that $rU_{t^-} e^{-\rho r \Delta_t} = u(\hat{c}_t)$ and $1 \geq \beta_t \geq \lambda$ hold for all $t \geq 0$, it follows $c_t = \hat{c}_t$ and $b_t^0 = b_t^1 = 0$ for all $t \geq 0$, in which case $\Delta_t = \mu_{tG}^M(\cdot) = 0$. Indeed, because the deviation gains are concave in the state Δ, the first order conditions are sufficient.

Hence, any other strategy tuple (c, b^0) makes the process $G^M(c, b^0)$ a supermartingale under the measure $\mathbb{P}^b$, i.e.

$$U_{0^-} = G_0^M(\hat{c}, 0) \geq \mathbb{E}^b G_t^M(c, b^0)$$

Because our regularity conditions ensure that $G^M(c, b^0)$ is bounded from below, we can thus take limits on both sides and apply optional sampling to obtain

$$U_{0^-} \geq \lim_{t\to\infty} \mathbb{E}^b G_t^M(c,b^0) = \mathbb{E}^b \lim_{t\to\infty} G_t^M(c,b^0) = \mathbb{E}^b G_\infty^M(c,b^0)$$

and in particular

$$U_{0^-} \geq \max_{c,b^0} \mathbb{E}^b G_\infty^M(c,b^0) = \hat{U}_{0^-}.$$

While we focused on strategies $(c,0,b^0)$ and $(\hat{c},b^1,0)$ separately, it follows immediately – as there is no persistent deviation state and $db_t^1 > 0 \Rightarrow t = \tau$ – that

$$U_{0^-} \geq \max_{c,b^0} \mathbb{E}^b G_\infty^M(c,0,b^0) \text{ and } U_{0^-} \geq \max_{b^1} \mathbb{E}^b G_\infty^M(\hat{c},b^1,0) \Longrightarrow U_{0^-} \geq \max_{c,b^1,b^0} \mathbb{E}^b G_\infty^M(c,b^1,b^0).$$

This is because the maximal utility the agent can obtain at time t equals $e^{-\rho r\Delta_t}U_{t^-}$ under any consumption c, while the deviation utility is given by

$$e^{-\rho r\Delta_t}\frac{u[(r(S_{t^-}+\kappa db_t^1)]}{r},$$

which is smaller than $e^{-\rho r\Delta_t}U_{t^-}$ if and only $Y_{t^-} \geq \kappa M_{t^-}$.

Therefore, $U_{0^-} = \hat{U}_{0^-}$ and $(c_t,b_t) = (\hat{c}_t,0)$ for all $t \geq 0$ is the optimal strategy for the agent if and only if $1 \geq \beta_t \geq \lambda, rU_{t^-} = u(\hat{c}_t), Y_{t^-} \geq \kappa M_{t^-}$ are satisfied for all $t \geq 0$ with probability one. In this case, the contract $\mathcal{C}$ is incentive compatible. □

C The Principal's Problem: Proof of Proposition 2

C.1 Reduction of the State Space: Proof of Proposition 5 i)

Per se, the state space is three dimensional and we have to keep track of three state M, W, S.

Due to the absence of wealth effects, it is clear that the value function takes the form $V(M,Y) = V(M,W-S) = \hat{V}(M,W,S)$. This relationship is straightforward to verify and we omit this here, to conserve space. We go on now to demonstrate that the state space $\mathcal{M}$ within an optimal contract must be one-dimensional.

Let us for simplicity assume that $\pi = 0$. We start with the following auxiliary Lemma, which analyzes the value function $V(M,Y)$.

Lemma 2. *Let $V(M,Y) = V$ the principal's value function and define $\tau = \inf\{t \geq 0 : M_{t^-} = 0\}$. Then, under the optimal contract $\mathcal{C}$ the space of states $\mathcal{M} \subset \mathbb{R}^2$, which are reached with positive probability before time τ, must be one-dimensional, i.e., a one dimensional manifold. In particular, there exists a mapping φ so that $Y = \varphi(M)M$ for $M > 0$.*

Proof. Assume to the contrary the state space $\mathcal{M}$ is two-dimensional. In order to maintain incentive compatibility, it must be that $Y \geq M$ for all $(M,Y) \in \mathcal{M}$. For any interior point $(M,Y) \in \mathcal{M}$ with $M > 0, Y \geq \kappa M$, the firm's value function is given by:

$$rV = \max_{dDiv\geq 0, dw\leq M,\beta} \Big\{ dDiv + V_M\Big(\mu + rM - dw - dDiv\Big) + \frac{V_{MM}\sigma^2}{2} \\ + V_Y\Big(rY + \frac{\rho r}{2}(\beta\sigma)^2 dt + \delta A(Y) - dw\Big) + \frac{V_{YY}(\beta\sigma)^2}{2} + V_{MY}\beta\sigma^2\Big\}$$

Since $Y \geq \kappa M > 0$, the IC-constraint does not bind, it must be that $V_M + V_Y = 0$. Otherwise, the principal would optimally move from state (M,Y) to state $(M-\varepsilon, Y-\varepsilon)$ through setting some non-infinitesimal adjustment $dw = -\varepsilon$ for some ε. The adjustment dw is feasible, as long as the constraint $Y \geq \kappa M$ and $M \geq 0$ does not bind.

The relation $V_M + V_Y = 0$ must hold for any interior point over the whole state space, when the state space $\mathcal{M}$ is a two-dimensional subspace of $\mathbb{R}^2$. Differentiating on this space the identity $V_M + V_Y = 0$ yields:

$$V_{MM} + V_{MY} = V_{YY} + V_{MY} \Longrightarrow V_{MM} = V_{YY} = -V_{MY}.$$

In order to satisfy the above relation, the function V must be such that $V(M,Y) = v(M-Y)$ for some function $v \in C^2$ and therefore for $C \equiv M - Y$:

$$(r+\delta)v(C) = \max_{\beta, dDiv \geq 0} \left\{ dDiv + v'(C)\left(rC - \frac{\rho r}{2}(\beta\sigma)^2 dt - \delta A(Y) + \mu \right) + \frac{\sigma^2(1-\beta)^2}{2} v''(C) \right\}.$$

As dividend payouts $dDiv > 0$ are always possible but not necessarily optimal, the marginal value of cash must satisfy $V_M = v' \geq 1$. However, for $\delta > 0$ and due to $A' > 0$ it follows that

$$V(M-\varepsilon, Y-\varepsilon) - V(M,Y) = v'(C)\delta(A(Y) - A(Y-\varepsilon)) > 0$$

for any $\varepsilon > 0$ with $M - \varepsilon > 0$, which contradicts $V_M + V_Y = 0$. Hence, within the optimal contract (M,Y) cannot be an interior point of $\mathcal{M}$ and in particular $\mathcal{M}$ cannot have interior points, so that this set must be one-dimensional. □

The previous Lemma shows that the state space is one-dimensional before time τ and therefore can be parameterized by

$$\mathcal{M} = \{(M, \varphi(M)M) : M \geq 0\}$$

for some function φ, determined by the optimal contract. We show in the following Lemma that $Y_{t^-} \leq M_{t^-}$ or equivalently $\varphi(M_{t^-}) \leq 1$ must hold for all $t \geq 0$.

Lemma 3. *Let $\mathcal{C}$ a contract and Div a dividend process. Further, define $t_F = \inf\{t \geq 0 : M_{t^-} < Y_{t^-}\}$. The contract is feasible, only if $\mathbb{P}(t_F = \infty) = 1$ and in particular $\mathbb{P}(\tau > t_F) = 1$. Hence, the event $\{Y_t > M_t\}$ must have zero probability.*

Proof. Fix the dividend process Div. Take a contract $\mathcal{C} \in \mathbf{C}$ and assume to the contrary that there exists a time $t_F < \tau$ with $\mathbb{P}(t_F < \infty) > 0$ and $Y_{t_F^-} > M_{t_F^-}$.

If the principal terminates the firm at time t_F, i.e., $\tau = t_F$, and sets optimally $dw_t = 0$ for $t \geq t^F$, the manager receives due to Nash-Bargaining amount $(1-\theta)M_{t_F^-} < Y_{t_F^-}$ and promise keeping is violated as $Y_\tau > 0$, evidently contradicting $\mathcal{C} \in \mathbf{C}$.

If the principal does not terminate, set $\tau_F = \inf\{t \geq t_F : M_t = Y_t\}$ and note that a contract $\mathcal{C} \in \mathbf{C}$ must satisfy $\mathbb{P}(\tau_F \leq \tau) = 1$, i.e., promise-keeping and in particular $Y_\tau = M_\tau = 0$. We consider now different cases.

i) First, let us assume that $M_{t_F^-} = 0 < Y_{t_F^-}$ and the principal would not like to specify $dw_{t_F^-} = -\varepsilon$, in order to continue at state $(\varepsilon, Y_{t_F^-} + \varepsilon)$ for some non-infinitesimal $\varepsilon > 0$ with $\varepsilon \notin o_p(dt)$. The other case will be – among others – analyzed in part ii) of the proof. Note that $t_F < \tau$, which requires $\beta_{t_F} = 1$, as a termination policy $\tau > t_F$ implies the agent must cover potential operating losses. Next, define $\tau_0 = \inf\{t \geq t_F : M_t > 0\}$. For all $t < \tau_0 \wedge \tau$, it must be $\beta_t = 1$ and as the agent covers operating losses:

$$\frac{dY_t}{dX_t} = \frac{dW_t}{dX_t} - \frac{dS_t}{dX_t} = 0,$$

so that Y_t has zero volatility for $t < \tau_0 \wedge \tau$. Furthermore, under contract $\mathcal{C}$, the agent consumes rW_t while earning interest $rS_t < rW_t$, so that the agent must borrow amount $-r(S_t - W_t) = -rY_t >$ and therefore $\mathbb{E}dS_t < 0$, while $\mathbb{E}dW_t > 0$ owing to the risk-premia earned. Hence, $\mathbb{E}dY_t \geq rY_t dt > 0$. Since Y_t grows at least at rate r, also the growth rate of the agent's borrowings is bounded from below by r, so that savings S_t shrink at least at rate r for $t_F \leq t \leq \tau_0$. In particular, $S_t = S_{t^F} - \int_{t_F}^{t} e^{r(t-s)} rY_s ds$. However, with positive probability there is a sample path of shocks $\{Z\}_{t \geq t_F}$, in which case $\tau_0 = \infty$. Then, either $\tau_0 > \tau$ promise keeping is violated with $Y_\tau > 0$ or

$$\lim_{t\to\infty} e^{-rt} S_t \leq \lim_{t\to\infty} e^{-rt}\left(-\int_{t_F}^{t} e^{r(t-s)} rY_s ds\right) = \lim_{t\to\infty}\left(-\int_{t_F}^{t} e^{-rs} rY_s ds\right) < 0$$

with positive probability, so that the no-Ponzi condition (6) is violated. Hence, $\mathcal{C} \notin \mathbf{C}$, a contradiction.

ii) Let us now consider $M_{t_F^-} > 0$. Define now $t_0 = \inf\{t \geq t_F : Y_{t^-} > M_{t^-} = 0\}$. Since $C_{t_F} < 0$ and $vol(dC_t) = \sigma(1-\beta_t)$, there must exist a random time $\tau_1 < t_0$ a.s. and $\mathbb{P}(\tau_1 < \tau) > 0$ such that $\beta_{\tau_1} > 1$, in order to ensure that $\mathbb{P}(t_0 > \tau) = 1$. However, when $\beta_{\tau_1} > 1$ the agent would like to boost cash-flow and incentive compatibility is violated. Since τ_1 is reached with positive probability (before time τ), it follows that $\mathcal{C} \notin \mathbf{C}$.

Hence, continuing from time t_F, it must be that t_0 is reached with positive probability. By step i), we get either a violation of the no-Ponzi condition, in which case $\mathcal{C} \notin \mathbb{C}$, or the principal asks the agent to put in money into the firm through setting $dw_{t_0} = -\varepsilon_0 < 0$, in which case the game continues at state $(\varepsilon_0, Y_{t_0} + \varepsilon_0)$. The principal has then cash-reserves compensate the agent for her lack of interest earned rY_{t^-}, so that we may consider that the principal does so. Moreover, we may now without loss of generality assume, that at each time the firm runs out of cash, the principal asks the manager to put in some strictly positive amount of cash.

However, then there exists a sequence of random times $(t_n)_{n\geq 1}$ and discrete amount $(\varepsilon_n)_{n\geq 0}$, defined via

$$t_n = \inf\{t \geq t_{n-1} : 0 = M_{t^-} < Y_{t^-}\} \text{ and } \varepsilon_n = -dw_{t_n} > 0.$$

All t_n are reached with positive probability before time $\tau_F \leq \tau$, so that $\mathbb{P}_{t_F}(t_n < \tau_F) > 0$ for all $n \geq 0$. With positive probability for any chosen sequence $(\varepsilon_n)_{n\geq 0}$, we get

$$\mathcal{O}_t \equiv \int_{t_0}^{\tau\wedge t} e^{r(t-s)} \sum_{t_i \leq s} \varepsilon_{t_i} ds \notin o(e^{rt})$$

or equivalently $\mathcal{O}_t \notin o_p(e^{rt})$, in that the manager puts cash into the firm on a rate higher than r with positive probability. As a consequence

$$\lim_{t\to\infty} e^{-rt} S_t \leq \lim_{t\to\infty} e^{-rt}(-\mathcal{O}_t) < 0$$

with positive probability and the no-Ponzi condition is violated.

□

By the previous Lemma, $\{Y_t > M_t\}$ must be a zero probability event. Hence, the firm must be terminated at time $\tau = \inf\{t \geq 0 : M_{t^-} = Y_{t^-}\} = \inf\{t \geq 0 : C_{t^-} = 0\}$ or the principal eliminates volatility through setting $vol(dC_\tau) = 0 \Leftrightarrow \beta_\tau = 1$, in order to prohibit that a state with $C_t < 0$ is reached with positive probability.

We show now in the following Lemma that the principal never would like to refinance by the agent when it runs out of cash, in that it does not ask the agent to put in any non-infinitesimal amount $-dw_{\tau_0}$ at any time τ_0 with $M_{\tau_0^-} = 0$ and in fact the equivalence $M_{t^-} = 0 \Leftrightarrow C_{t^-} = 0$.

Lemma 4. *Let $\mathcal{C}$ the optimal contract. Then, at any time $t_F = \inf\{t \geq 0 : C_{t^-} = 0\}$ it follows that dw_t is infinitesimal, that is, $dw_t \in o_p(dt)$, and the principal does not raise any strictly positive amount of debt from the agent. Moreover, $M_{t^-} = 0 \Leftrightarrow C_{t^-} = 0$.*

Proof. We prove now that once $C = M - Y = 0$ with $M = Y = 0$, the principal cannot profitably switch to a state (M, M) with $M > 0$. Let us assume the principal sets $\tau > t_F$ with $M_{t_F} = Y_{t_F} = 0$ and in particular $dw_{t_F} = \Delta^w \notin o_p(dt)$ and let payoff under this strategy be $v(\Delta^w, \Delta^w)$ with dividend payouts Div.

Let $\tau_F > t_F$ a stopping time, as follows. The principal can improve upon setting $dw_{t_F} = -\Delta_w + \varepsilon > 0$ and setting $dw_{\tau_F} = -\varepsilon < 0$, where $\tau_F = \inf\{t \geq t_F : M_{t^-} = \Delta^w - \delta\}$ for some arbitrary $\delta > 0$. Then, $\mathbb{P}(\tau_F > t_F) = 1$ and $\mathbb{P}(\tau_F - t_F > \delta') > 0$ for some arbitrary $\delta' > 0$. Setting payouts under the new strategy for $t_F \leq t \leq \tau_F$ according to $d\hat{D}iv_t = \delta(A(Y_t) - A(Y_t - \varepsilon))dt + dDiv_t$. All other features of the previous strategy will be mimicked. Then, the payoff under the modified strategy equals

$$v(\Delta^w, \Delta^w) + \mathbb{E}_{t_F}\left(\int_{t_F}^{\tau\wedge\tau_F} \delta(A(Y_t) - A(Y_t - \varepsilon))dt\right) > v(\Delta^w, \Delta^w).$$

As this holds for any $\varepsilon < \Delta^w$, it follows that the best the principal can do is to just raise the amount needed, that is set $\beta_t = 1$ and $dw_t \in o_p(dt)$ for $t_F \leq t \leq \tau_0$ with $\tau_0 = \inf\{t \geq 0 : C_{t^-} > 0\}$, in case $\tau > t_F$ is indeed optimal.

The second claim of the Lemma is immediate by the previous arguments. This is because being at state (M, M), the principal prefers to set payouts $dw = M$ and switch to state $(0, 0)$. Because $Y > M$ is not feasible, this implies the equivalence $M_{t^-} = 0 \Leftrightarrow C_{t^-} = 0$ for all $t \geq 0$. □

As a consequence, we obtain $M = 0 \Leftrightarrow C = 0$, so that C indeed summarizes the whole contract relevant history and serves as the only relevant state-variable. Hence, firm value – i.e., the principal's payoff – can be written as a function $v = v(C)$ of the state C only. The state space by means of C is contained in $\mathbb{R}_+$, i.e., C exceeds zero.

Either the firm defaults if and only if it runs out of cash and therefore $\tau = \inf\{t \geq 0 : C_{t^-} = M_{t^-} = 0\}$. Or there is an absorbing state, so that $\beta_t = 1$ whenever $C_{t^-} = c \geq 0$ for some constant c. As we verify, in the next section, there will not be an absorbing state, in that $\beta_t < 1$ for all $t \geq 0$ with probability one.

C.2 Verification: Proof of Proposition 5 ii)

Proof. A formal existence proof of the solution is beyond the scope of the paper and therefore omitted. Therefore, we assume $v(\cdot)$ is twice continuously differentiable and solves uniquely (29).

We verify that $v(C_{t^-})$ indeed represents shareholders' profit in optimum.

Let $\mathcal{C} \in \mathbf{C}$ the optimal contract and Div the optimal payout policy, solving the principal's problem and consider any other contract $\hat{\mathcal{C}} \in \mathbf{C}$ and any other payout policy $\hat{D}iv$.

For convenience, let the contract contain the optimal refinancing sum Δ. We denote the n'th refinancing time by τ^n. Ex-post optimality owing to the shareholders' limited commitment pins down at time τ^n for each $n \geq 1$:

$$\max_{\Delta,\Gamma}\Big(v(C + \Delta - \Gamma) - v(C) - \Delta\Big) \text{ s.t. } (26),$$

given the solution v.

We show now that the value function $v(\cdot)$ solving (17) represents the principal's optimal profit, in that the contracts $\mathcal{C}$, the payout policy Div and the refinancing quantity Δ outlined in the Proposition are indeed optimal.

Let us for brevity write:

$$dC_t = \mu_{Ct}dt + \sigma_{tC}dZ_t + (\Delta_t - \Gamma_t)d\Pi_t - dDiv_t$$

with

$$\mu_{Ct} \equiv rC_{t-} + \mu - \frac{\rho r}{2}(\beta_t \sigma)^2 - \delta A\left(\frac{\varphi_t C_{t-}}{1-\varphi_t}\right) + \pi \frac{1-e^{-\rho r \Gamma_t}}{\rho r} \text{ and } \sigma_{tC} = (1-\beta_t)\sigma,$$

where we suppress the dependence of drift and volatility on controls and model parameters. Introduce the linear functional $\mathcal{L}$, operating on functions dependent on $C \geq 0$ with $\mathcal{L}f(C) = f'(C)\mu_C + \frac{\sigma_C^2 f''(C)}{2}$. Define for $t < \tau$ the auxiliary gain process upon following an arbitrary strategy $(\hat{\mathcal{C}}, \hat{D}iv)$ up to time t and then switching to $(\mathcal{C}, Div)$

$$G_t^P = G_t^P(\hat{\mathcal{C}}, \hat{D}iv) = \int_0^t e^{-rs} d\hat{D}iv_s + e^{-rt} v(C_{t-}).$$

By Itô's Lemma:

$$\begin{aligned}
e^{rt} dG_t^P = &\Big\{ -(r+\delta+\pi)v(C_{t-}) + \mathcal{L}v(C_{t-}) + \pi\big[v(C_{t-} + \Delta_t - \Gamma_t) - \Delta_t\big] \Big\} dt \\
&+ \big(1 - v'(C_{t-})\big) d\hat{D}iv + \sigma_{tC} v'(C_{t-}) dZ_t - v(C_{t-})(dN_t - \delta dt) \\
&\equiv \mu_t^G(\hat{\mathcal{C}}, \hat{D}iv)dt + \big(1 - v'(C_{t-})\big) d\hat{D}iv + \sigma_{tC} v'(C_{t-}) dZ_t - v(C_{t-}(dN_t - \delta dt).
\end{aligned}$$

By the HJB equation (29), the drift term in curly brackets is zero under the optimal controls under contract $\mathcal{C}$ and optimal dividend payout Div, while each other strategy/contract will make this term (weakly) negative, i.e $\mu_t^G(\hat{\mathcal{C}}, \hat{D}iv) \leq 0$. Because the process $\hat{D}iv$ is almost surely increasing and the fact that $v'(C_{t-}) \geq 1$, the term $\big(1 - v'(C_{t-})\big)$ is (weakly) negative under any dividend payout policy $\hat{D}iv$ and zero under the payout policy Div.

Next, our regularity conditions ensure that α, β are bounded and so is σ_C. Further, v' and v must be bounded over $(0, \infty)$. Evidently, $v < \mu/r$. If now v' were not bounded, then v could not be bounded either. Hence, there exists $\infty > K > 0$ with $v, v' < K$. Hence:

$$\mathbb{E}\bigg(\int_0^t e^{-rs} \sigma_{tC} v'(C_{t-}) dZ_s\bigg) = \mathbb{E}\bigg(\int_0^t e^{-rs} v(C_{t-})(dN_s - \delta ds)\bigg) = 0$$

for all $t < \tau$. Therefore, $G^P(\hat{\mathcal{C}}, \hat{D}iv)$ follows a supermartingale, while $G^P(\mathcal{C}, Div)$ follows a martingale under the measure $\mathbb{P}$ and so do the stopped processes $\{G^P(\hat{\mathcal{C}}, \hat{D}iv)_{t\wedge\tau}\}$ and $\{G^P(\mathcal{C}, Div)_{t\wedge\tau}\}$. Hence, the payoff under strategy $(\hat{\mathcal{C}}, \hat{D}iv)$ satisfies

$$\hat{v}(C_{0-}) \equiv G_{0-}^P(\hat{\mathcal{C}}, \hat{D}iv) \geq \mathbb{E}G_{t\wedge\tau^I}^P(\hat{\mathcal{C}}, \hat{D}iv)$$

Then it follows for any t:

$$\begin{aligned}
\hat{v}(C_{0-}) &= \mathbb{E}\bigg(\int_0^\tau e^{-rs} d\hat{D}iv_s + e^{-r\tau} L\bigg) = \mathbb{E}G_\tau^P(\hat{\mathcal{C}}, \hat{D}iv) + e^{-r\tau} L \\
&= \mathbb{E}\bigg(G_{t\wedge\tau^I}^P(\hat{\mathcal{C}}, \hat{D}iv) + \mathbf{1}_{t\leq\tau}\Big[\int_t^\tau e^{-rs} d\hat{D}iv_s + e^{-r\tau} L - e^{-rt} v(C_{t-})\Big]\bigg) \\
&= \mathbb{E}G_{t\wedge\tau}^P(\hat{\mathcal{C}}, \hat{D}iv) + e^{-rt}\mathbb{E}_t \mathbf{1}_{t\leq\tau}\bigg(\int_t^\tau e^{-r(s-t)} d\hat{D}iv_s + e^{-r(\tau-t)} L - v(C_{t-})\bigg) \\
&\leq v(C_{0-}) + e^{-rt}\big(v^{FB} - L\big),
\end{aligned}$$

where we used the supermartingale property and the fact that

$$\mathbb{E}_t\bigg(\int_t^\tau e^{-r(s-t)} d\hat{D}iv_s + e^{-r(\tau-t)} L\bigg) \leq v^{FB} \equiv \frac{\mu}{r}$$

and $v(C_{t^-}) \geq L$.

From the above arguments, we readily obtain $\hat{v}(C_{0^-}) \leq v(C_{0^-})$ for any contract $\hat{\mathcal{C}}$ and any payout policy $\hat{Div}$. On the other hand, under $(\mathcal{C}, Div)$ the principal's payoff $\hat{v}(C_{0^-})$ achieves $v(C_{0^-})$, as the above weak inequality holds in equality when $t \to \infty$. This concludes the proof. $\square$

C.3 Concavity of value function: Proof of Proposition 2 iii)

Proof. Wlog, we prove the claim only under limited commitment w.r.t. a refinancing strategy. The proof for full commitment works analogously. Note that in optimum $C + \Delta - \Gamma = C^*$ for a constant C^*. Differentiating the above identity yields

$$0 = 1 + \frac{\partial \Delta}{\partial C} - \frac{\partial \Gamma}{\partial C} = 1 + \frac{\partial \Delta}{\partial C} - \frac{\partial \Gamma}{\partial \Delta}\frac{\partial \Delta}{\partial C} \Longrightarrow \frac{\partial \Delta}{\partial C} = -\frac{1}{1-\kappa},$$

because by (26) – which is tight in optimum – it follows that $\frac{\partial \Gamma}{\partial \Delta} = \kappa$. By the envelope theorem:

$$v'''(C) = \frac{2}{(1-\beta)^2\sigma^2} \times \left\{ \left[\delta + \pi + \frac{\delta\varphi \mathbf{1}_{\{\varphi=\kappa\}}}{(1-\varphi)} A'\left(\frac{\varphi C}{1-\varphi}\right) + \frac{\pi\kappa e^{-\rho r \Gamma}}{1-\kappa}\mathbf{1}_{\{\varphi=\kappa\}}\mathbf{1}_{\{\Delta>0\}} \right] v'(C) - v''(C)\mu_C - \frac{\pi}{1-\kappa}\mathbf{1}_{\{\Delta>0\}} \right\}$$

Let us evaluate $v'''(\cdot)$ at the boundary, in which case $\Delta = 0$ due to $\kappa > 0$ and therefore $\varphi = \kappa$.

First, assume that $v''(\bar{C}) = 0$ and the super-contact condition holds. Due to $A' \geq 1$, $v''(\bar{C}) = v'(\bar{C}) - 1 = 0$ and $\beta = \lambda$, it is immediate that $v'''(\bar{C}) > 0$. Hence, by continuity, there exists $\varepsilon > 0$, so that $v'' < 0$ on an interval $(\bar{C} - \varepsilon, \bar{C})$. Second, assume $v''(\bar{C}) \neq 0$. If $v''(\bar{C}) > 0$, there exists a point $C' < \bar{C}$ with $v'(C') < 1$, a contradiction to $\bar{C}$ being the payout boundary. Hence, also in this case $v'' < 0$ on an interval $(\bar{C} - \varepsilon, \bar{C})$.

Let us assume that v is not strictly concave on $[0, \bar{C})$ and define $C' \equiv \sup\{C \in [0, \bar{C}] : v''(C) > 0\}$. By assumption, the set over which we take the supremum is non-empty, so that $C' < \infty$. As $v'' < 0$ in a left-neighbourhood of $\bar{C}$, we also have that $C' < \bar{C}$. Due to continuity, $v''(C') = 0$. As $\Delta > 0$ implies $v'(C) \geq 1/(1-\kappa)$, it follows that $v'''(C') > 0$, so that there exists $C'' > C'$ with $v''(C'') > 0$, a contradiction to the definition of C'. Hence, $v'' < 0$ on $[0, \bar{C})$. In addition, strict concavity of v implies $v''' > 0$ on $[0, \bar{C}]$, thereby concluding the proof. $\square$

D Additional Analytic Results

D.1 Proof of Corollary 1

Proof. Differentiating the expression for $\hat{\beta}$ w.r.t. C yields

$$\partial_C \beta^*(C) = \frac{\partial \beta^*(C)}{\partial C} \propto -v'(C)v'''(C) + v''(C)v''(C),$$

so that there exists $C' := \inf\{C \geq 0 : \partial_C \beta^*(C) < 0\} < \bar{C}$ with $\partial_C \beta^*(C) < 0$ on $[C', \bar{C})$ and β^* strictly decreases in an open left neighbourhood of $\bar{C}$. Further, it is immediate to verify that

$$\partial_C \left(\frac{-v''(C)}{v'(C)} \right) \propto (\beta^*)'(C).$$

As $\sigma \to 0$, clearly $\bar{C} \to 0$. By the super-contact condition, $v''(C) = o(\bar{C})$, while $v'(C)v'''(C) \neq o(\bar{C})$. Hence, $C' \uparrow 0$ as $\sigma \to 0$, which proves that for σ sufficiently low β^* decreases on $[0, \bar{C})$.

Let $\hat{C} = \inf\{C \in [0, \bar{C} : \beta^*(C) \geq \lambda\}$. It is obvious that $\hat{C} \leq \bar{C}$. Since $\beta^*(C) > 0$ for all $C < \bar{C}$ and $\bar{C} \to \bar{C}' > 0$ as $\lambda \to 0$, it follows that $\hat{C} \to 0$. Thus, for λ sufficiently small, it must be that

$\hat{C} < \bar{C}$ and there exists exactly one value solving the equation $\beta(C) = \lambda$, which completes the proof. □

D.2 Proof of Corollaries 2 and 3

Here, η is an arbitrary model parameter and define $\partial_\eta(\cdot) \equiv \frac{\partial(\cdot)}{\partial\eta}$. Throughout, let us consider the limit case $\theta \to 0$, so that shareholders cannot profitably deviate by paying out the entire cash-balance and the payout threshold satisfies the satisfies the smooth-pasting condition.

We start with an auxiliary lemma:

Lemma 5. *For $\tau = \inf\{t \geq 0 : M_t = 0\}$ the following holds:*

$$\frac{\partial v(C)}{\partial\eta} \propto \mathbb{E}\Bigg[\int_0^\tau e^{-(r+\delta)t}\Bigg(v'(C_t)\Big(\partial_\eta rC_t - \partial_\eta\frac{\rho r}{2}(\beta_t\sigma)^2dt - \partial_\eta\delta A\left(\frac{\varphi_t C_t}{1-\varphi_t}\right) + \partial_\eta\mu\Big)$$
$$+ \partial_\eta\frac{\sigma^2(1-\beta_t)^2}{2}v''(C_t) + \partial_\eta\delta L\Bigg)dt + \partial_\eta e^{-(r+\delta)\tau}L\Bigg|C_0 = C\Bigg]$$

Proof. Let η a model parameter and β, φ the optimal controls in optimum. Let $C \in [0, \bar{C}]$ and take the derivative

$$\frac{dv(C)}{d\eta} = \partial_\eta v(C) + \partial_{\bar{C}}v(C) \times \partial_\eta\bar{C} = \partial_\eta v(C),$$

where $\partial_{\bar{C}}v(C) = 0$ by means of the envelope theorem, provided the super-contact condition $v''(\bar{C}) = 0$ holds. Accordingly, differentiating (17) w.r.t. η yields:

$$\begin{aligned}\partial_\eta(r+\delta)v(C) = &-(r+\delta)v_\eta(C) + v'(C_t)\left[\partial_\eta(\mu + rC) - \partial_\eta\frac{\rho r}{2}(\beta\sigma)^2 - \partial_\eta\delta A\left(\frac{\varphi_t C_t}{1-\varphi_t}\right)\right] \\ &+ v_\eta'(C)\left[rC + \mu - \frac{\rho r}{2}(\beta\sigma)^2 - -\partial_\eta\delta A\left(\frac{\varphi_t C_t}{1-\varphi_t}\right)\right] \\ &+ \frac{\sigma^2(1-\beta)^2}{2}v_\eta''(C) + \partial_\eta\frac{\sigma^2(1-\beta)^2}{2}v''(C)\end{aligned}$$

where $\partial_\beta v(C) = \partial_\varphi v(C) = 0$ by the envelope theorem. The boundary conditions are $v_\eta'(\bar{C}) = v_\eta''(\bar{C}) = 0$ and $v_\eta(0) = \partial_\eta L$. Provided our smoothness conditions, we can interchange the order of differentiation, such that:

$$v_\eta'(C) \equiv \frac{\partial}{\partial\eta}\frac{\partial v(C)}{\partial C} = \frac{\partial}{\partial C}\frac{\partial v(C)}{\partial\eta} \text{ and } v_\eta''(C) \equiv \frac{\partial}{\partial\eta}\frac{\partial^2 v(C)}{\partial C^2} = \frac{\partial^2}{\partial C^2}\frac{\partial v(C)}{\partial\eta}.$$

Invoking the Feynman-Kac formula and integrating yields the desired expression. □

Next, note that

$$\beta_\eta \propto -v'(C)v_\eta''(C) + v''(C)v_\eta'(C), \tag{D.1}$$

so that $sign(\beta_\eta(C)) = sign(RA_\eta(C))$ for $RA(C) = -v''(C)/v'(C)$. Provided the super-contact condition $v''(\bar{C}) = 0$ holds, we evaluate the HJB-equation at the boundary $C = \bar{C}$:

$$(r+\delta)v(\bar{C}) = \left(r\bar{C} - \frac{\rho r}{2}(\lambda\sigma)^2 - \delta A\left(\frac{\varphi\bar{C}}{1-\varphi}\right) + \mu\right) \tag{D.2}$$

In the following, we derive our comparative statics for various model parameters. In each of the following subsections, we prove *all* claims regarding *one* particular parameters, so that one of the following subsections then proves one part of corollary 3 and 2 simultaneously. When deriving comparative statics for β with C close to zero, we implicitly assume $\beta(C) \geq \lambda$ does not bind for low values of C, unless otherwise mentioned.

D.2.1 Volatility: σ

Proof. To start with, invoke the implicit function theorem to differentiate (D.2), which yields

$$(r+\delta)v_\sigma(\bar{C}) + (\delta+r)\bar{C}_\sigma + \rho r\sigma\lambda^2 = r\bar{C}_\sigma - \frac{\delta\kappa\bar{C}_\sigma}{1-\kappa}A'\left(\frac{\kappa\bar{C}}{1-\kappa}\right)$$

Next,

$$v_\sigma(C) = \partial_\sigma v(C) = \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t}\Big(-\rho r\beta_t^2\sigma v'(C_t) + \sigma(1-\beta_t)^2 v''(C_t)\Big)dt\middle| C_0 = C\right]$$

As the integrand is almost everywhere negative, it follows that $v_\sigma(C) < 0$ and therefore $\bar{C}_\sigma > 0$, provided the smooth pasting condition holds and λ or ρ are sufficiently small.

Because zero is an absorbing state it must further be that $v'_\sigma(C) < 0$ in a neighbourhood of zero. Next, let us evaluate the HJB-equation at some value C, in order to obtain:

$$RA(C) = \frac{-v''(C)}{v'(C)} = \frac{2}{(1-\beta(C))^2\sigma^2} \times \underbrace{\left(\frac{-(r+\delta)v(C)}{v'(C)} + \left(\mu - \rho r\frac{\beta(C)^2\sigma^2}{2} + o(C)\right)\right)}_{\equiv\mathcal{E}>0}. \quad \text{(D.3)}$$

By the envelope theorem, we obtain:

$$\begin{aligned} RA_\sigma(C) = &- \frac{4}{(1-\beta(C))^2\sigma^3}\mathcal{E} - \frac{2\rho r\beta(C)^2}{(1-\beta(C))^2\sigma}v'(C) \\ &+ \frac{2}{(1-\beta(C))^2\sigma^2} \times (r+\delta)\frac{-v'(C)v_\sigma(C) + v(C)v'_\sigma C()}{(v'(C)^2} \end{aligned}$$

The first two terms are unambiguously negative. To sign the third term, note that $v_\sigma(C) = v_\sigma(0) + v'_\sigma(C)C + o(C^2) = o(C)$, as $v(0) = L$ is an identity. The third term is then also negative for C sufficiently small, as $v'_\sigma(C) < 0$ in a neighbourhood of zero. As a consequence, it must be that $RA_\sigma(C) < 0$ in a neighbourhood of zero and therefore $\beta_\sigma(C) < 0$, which completes the proof. □

D.2.2 Moral Hazard: κ

Proof. Note that the incentive constraint $\varphi \geq \kappa$ binds everywhere, provided $\pi = 0$. Let us differentiate (D.2), to obtain

$$-(r+\delta)v_\kappa(\bar{C}) = \delta\left(\frac{\kappa\bar{C}_\kappa}{1-\kappa} + \frac{\bar{C}}{(1-\kappa)^2}\right)A'\left(\frac{\kappa\bar{C}}{1-\kappa}\right) + \delta\bar{C}_\kappa, \quad \text{(D.4)}$$

so that

$$\bar{C}_\kappa \propto -(r+\delta)v_\kappa(\bar{C}) - A'\left(\frac{\kappa\bar{C}}{1-\kappa}\right)\frac{\bar{C}}{(1-\kappa)^2}.$$

Moreover:

$$v_\kappa(C) = \partial_\kappa v(C) = -\mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t}\Big(A'(D_t)v'(C_t)\frac{C_t\delta}{(1-\kappa)^2}\Big)dt\middle| C_0 = C\right] \quad \text{(D.5)}$$

and therefore $v_\kappa(C) < 0$. Next, note that the integrand of (D.5) possesses derivative w.r.t. C:

$$
\begin{aligned}
&-A'\left(\frac{\kappa C}{1-\kappa}\right)v''(C)\frac{C}{(1-\kappa)^2} - \frac{C^2\kappa}{(1-\kappa)^3}\rho r e^{\rho r\kappa C/(1-\kappa)}v'(C) - \frac{A'(D)v'(C)}{(1-\kappa)^2} \\
&\propto -v''(C)C - \frac{C^2\kappa\rho r v'(C)}{1-\kappa} - v'(C) = -v'(C) + o(C)
\end{aligned}
$$

For $C \simeq 0$, the third term dominates and the integrand of (D.5) decreases in C. For $C > 0$, and κ sufficiently large, the second term dominates. Thus, there exists $\bar{\kappa} \in [0,1)$, such that the integrand of (D.5) decreases in κ for $\kappa \geq \bar{\kappa}$ for all $C \geq 0$. This readily implies that $-(r+\delta)v_\kappa(\bar{C}) < A'\left(\frac{\kappa\bar{C}}{1-\kappa}\right)\frac{\bar{C}}{(1-\kappa)^2}$ and it follows that $\bar{C}_\kappa < 0$ for $\kappa \geq \bar{\kappa}$.

Next, let us rewrite the HJB-equation:

$$
RA(C) = \frac{-v''(C)}{v'(C)} = \frac{2}{(1-\beta(C))^2\sigma^2} \times \underbrace{\left(\frac{-(r+\delta)v(C)}{v'(C)} + \left(\mu - \rho r\frac{\beta(C)^2\sigma^2}{2} + o(C)\right)\right)}_{\equiv \mathcal{E} > 0}.
$$

The envelope theorem yields then after some simplifications:

$$
\begin{aligned}
sign(RA_\kappa(C)) &= sign\Big((-v'(C)v_\kappa(C) + v(C)v'_\kappa(C) + o(C)\Big) \\
&= sign\Big(- v'(C)(v_\kappa(0) + v'_\kappa(C)C + o(C^2)) + v(C)v'_\kappa(C) + o(C)\Big)
\end{aligned}
$$

For C in a neighbourhood of zero, it is then immediate that $sign(RA_\kappa(C)) = sign(v(C)v'_\kappa(C))$. Since $v_\kappa(C) < 0$ and $v(0) = L$ is an identity independent of κ, it must also be that $v'_\kappa(C) < 0$, which implies $RA_\sigma(C)$ for $C \simeq 0$. Hence, $\beta_\kappa(C) < 0$ in a neighbourhood of zero, i.e., for $C \simeq 0$, which concludes the proof. □

D.2.3 Cash-Flow Rate: μ

Proof. Observe that

$$
v_\mu(C) = \partial_\mu v(C) \propto \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t}v'(C_t)dt \middle| C_0 = C\right] > 0
$$

and upon differentiating (D.2) it follows that

$$
\bar{C}_\mu \propto -(r+\delta)v_\mu(\bar{C}) - 1.
$$

Differentiating

$$
RA(C) = \frac{-v''(C)}{v'(C)} = \frac{2}{(1-\beta(C))^2\sigma^2} \times \underbrace{\left(\frac{-(r+\delta)v(C)}{v'(C)} + \left(\mu - \rho r\frac{\beta(C)^2\sigma^2}{2} + o(C)\right)\right)}_{\equiv \mathcal{E} > 0}
$$

w.r.t. μ yields after simplifications:

$$
sign(RA_\mu(C)) = sign\big(1 + v(C)v'_\mu(C) + o(C)\big).
$$

Since $v_\mu(C) > 0$, it is clear that $v'_\mu(C) > 0$ close to zero and therefore $RA(C)$ and $\beta(C)$ must increase in a neighbourhood of zero, i.e., for $C \simeq 0$, which concludes the proof. □

D.2.4 Risk-aversion: ρ

Proof. Note that

$$v_\rho(C) = \partial_\rho v(C) \propto \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t} v'(C_t)\left[-r/2(\beta_t\sigma)^2 - \delta A_\rho(D_t)\right] dt \middle| C_0 = C\right],$$

where $A_\rho(\cdot) = \partial_\rho A(\cdot) > 0$. Clearly, $v_\rho(C) < 0$. Differentiating (D.2) yields that

$$\bar{C}_\rho \propto -(r+\delta)v_\rho(\bar{C}) - \frac{r(\lambda\sigma)^2}{2} - \delta A_\rho(\bar{D}) = -(r+\delta)v_\rho(\bar{C}) - \frac{r(\lambda\sigma)^2}{2} - \delta$$

For λ and δ sufficiently small, it follows that $\bar{C}_\rho > 0$. Further, for ρ sufficiently large, the term A_ρ explodes for any argument and owing to $\bar{D} \geq D_t$ with the inequality being strict on a set with positive measure, it must be that $\bar{C}_\rho < 0$ for $\rho \geq \bar{\rho}$ for some value $\bar{\rho} > 0$ Moreover, in the limit case $\rho \to 0$, it is clear that all risk is shared with the agent, in that $\bar{C} \to 0$ for $\rho \to 0$. Hence, there exists $\underline{\rho} > 0$ with $\bar{C}_\rho < 0$ for $\rho < \underline{\rho}$.

Taking C with $\beta(C) > \lambda$, differentiating (D.3) and doing some algebra, we get that

$$sign(\beta_\rho(C)) = sign(RA_\rho(C)) = sign\left(-r(\beta(C)\sigma)^2/2 + v(C)v_\rho'(C) + o(C)\right).$$

For C sufficiently close to zero, it follows that $v_\rho'(C) < 0$, so that $\beta(C)$ must decrease in ρ in a neighbourhood of zero, i.e., for $C \simeq 0$. Since $\beta(C) = \lambda$ for all C for high values of ρ, it follows that $\beta(C)$ is constant in ρ for large values of ρ or λ and decreases otherwise. □

D.2.5 Moral Hazard: λ

Proof. Observe that

$$v_\lambda(C) = \partial_\lambda v(C) = \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t}\left[-r\rho\lambda\sigma^2 v'(C_t) - (1-\lambda)\sigma^2 v''(C_t)\right] \mathbf{1}_{\beta_t=\lambda} dt \middle| C_0 = C\right],$$

Whenever

$$-r\rho\lambda v'(C) - (1-\lambda)v''(C) > 0,$$

it is clear that $\beta(C) > \lambda$, so that $v_\lambda(C) \leq 0$. Next, because β decreases it must be that also $v_\lambda(C)$ decreases, so that $v_\lambda'(C) < 0$. Implicitly differentiate (D.2) to obtain

$$\bar{C}_\lambda \propto -(r+\delta)v_\lambda(\bar{C}) - \rho r\lambda\sigma^2$$

For $\lambda = 0$, it follows that $\beta_t \geq \lambda$ for all t with equality if and only if $\bar{C} = C_t$, so that $v_\lambda(C) = 0$. Furthermore, for any $\varepsilon > 0$ there exists $\lambda \in o(\varepsilon)$ such that $\beta(C) = \lambda$ exactly for all $C \in (\bar{C} - \varepsilon, \bar{C}]$. On the interval $(\bar{C} - \varepsilon, \bar{C}]$, we have that $v'(C) = 1 + o(\varepsilon)$ and $v''(C) = o(\varepsilon)$. Thus,

$$v_\lambda(C) = \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t}\left[-r\rho\lambda\sigma^2\right] \mathbf{1}_{\{\beta_t=\lambda\}} dt \middle| C_0 = C\right] + o(\varepsilon),$$

so that there exists $\varepsilon > 0$, such that $\bar{C}_\lambda < 0$, which also means owing $\lambda \in o(\varepsilon)$, that $\bar{C}$ decreases in λ for λ sufficiently small. Taking the extreme case $\lambda = 1$, we immediately see that $\bar{C} = 0$, so that $\bar{C}$ must decrease in λ when λ is sufficiently large.

Next, we show the claim regarding β. First, assume that $\beta \geq \lambda$ does not bind in a neighbourhood of zero, which is the case for ρ or λ sufficiently low. Differentiating (D.3) and doing some algebra, we get that

$$sign(\beta_\lambda(C)) = sign(RA_\lambda(C)) = sign\left(v(C)v'_\lambda(C) + o(C)\right).$$

For C sufficiently close to zero, it follows that $v'_\lambda(C) < 0$, so that $\beta(C)$ must decrease in ρ in a neighbourhood of zero, i.e., for $C \simeq 0$, which concludes the proof. Second, assume that $\beta = \lambda$ everywhere, which is the case for ρ or λ sufficiently large. Under these circumstances, $\beta(C)$ mechanically increases in λ. □

D.2.6 Disaster Risk: δ

Proof. Differentiating boundary yields

$$\bar{C}_\delta \propto -(r+\delta)v_\delta(\bar{C}) - v(\bar{C}) - A(\bar{D}).$$

Next, observe that

$$v_\delta(C) = \partial_\delta v(C) \propto -\mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t}\Big((v(C_t) - L) + v'(C_t)(D_t)\Big)dt\middle| C_0 = C\right] - e^{-(r+\delta)\tau}L,$$

so that $v_\delta(C) < 0$, which readily implies $v'_\delta(C) < 0$ for C close to zero. Let us wlog assume $L = 0$. It is clear that

$$(r+\delta)\mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t}v(C_t)dt\middle| C_0 = C\right] < v(\bar{C}),$$

as $C = 0$ is an absorbing state, that is reached at an $\mathbb{P}$ almost surely finite time, and v is monotonically increasing. Next, let us consider the derivative of $A(D)v'(C)$ with respect to C:

$$\frac{\kappa}{1-\kappa}v'(C)e^{\rho r D} + v''(C)A(D).$$

For κ sufficiently large, the first term must dominate owing to $v' \geq 1$. Under this condition:

$$\frac{A(\bar{D})}{r+\delta} \geq E\left[\int_0^\infty e^{-(r+\delta)t}v'(C_t)A(D_t)dt\right] \text{ for } \bar{D} = \frac{\kappa\bar{C}}{1-\kappa}.$$

From there, it follows that for κ sufficiently large the payout boundary must decrease in δ, i.e., $\bar{C}_\delta < 0$. Furthermore, we know that for sufficiently large κ, the absolute value of the integrand increases and since the integrand is negative, this means that $v'_\delta(C) < 0$.

To prove the claim regarding β we differentiate (D.3) and simplify, to get:

$$sign(RA_\delta(C)) = sign\Big((r+\delta)v(C)v'_\delta(C) - v(C)v'(C) + o(C)\Big).$$

For $C \simeq 0$, it follows that $sign(RA_\delta(C)) < 0$, so that $\beta(C)$ decreases in δ for $C \simeq 0$, provided a loose IC-condition $\beta(C) > \lambda$. □

D.2.7 Commitment θ

Proof. It is evident that $\frac{\partial\overline{C}}{\partial\theta} = 0$ as well as $\frac{\partial v(C)}{\partial\theta} = 0$, whenever $v(\overline{C}) > \frac{\theta\overline{C}}{1-\kappa} + L$. The latter inequality is always satisfied if $\theta < 1-\kappa$. Let us therefore consider the case $\theta \geq 1-\kappa$ and $v(\overline{C}) = \frac{\theta\overline{C}}{1-\kappa} + L$. Differentiating this identity wrt. θ yields:

$$v'(\overline{C})\frac{\partial\overline{C}}{\partial\theta} + v_\theta(\overline{C}) = \frac{\overline{C}}{1-\kappa} + \frac{\theta}{1-\kappa}\frac{\partial\overline{C}}{\partial\theta}.$$

Since θ affects firm value only through the boundary conditions: $v_\theta(\overline{C}) = 0$. Owing to $v'(\overline{C}) = 1$:

$$\frac{\partial \overline{C}}{\partial \theta} = \frac{\overline{C}}{1-\kappa} \times \left(1 - \frac{\theta}{1-\kappa}\right)^{-1} \leq 0$$

with the inequality being strict if $\theta > 1-\kappa$. We take the risk-aversion:

$$RA(C) = \frac{-v''(C)}{v'(C)} = \frac{2}{(1-\beta(C))^2\sigma^2} \times \underbrace{\left(\frac{-(r+\delta)v(C)}{v'(C)} + \left(\mu - \rho r \frac{\beta(C)^2\sigma^2}{2} + o(C)\right)\right)}_{\equiv \mathcal{E} > 0}.$$

Close to zero $v(C) \simeq L$. Moreover, it must be that $\frac{dv(C)}{d\theta}) < 0$ for C in a neighbourhood of zero, while $\frac{dv(0)}{d\theta} = 0$ as the identity $v(0) = L$ holds. Therefore: $\frac{dv'(C)}{d\theta} < 0$ in neighbourhood of zero, so that $RA(C)$ and therefore also $\beta(0)$ decrease in θ for $C \simeq 0$. □

D.3 Stock-Return Volatility

The formula for the stock-returns follows upon invoking Ito's Lemma:

$$\begin{aligned}
dR_t &= \frac{dDiv_t + dv(C_{t-})}{v(C_{t-})} \\
&= \frac{dDiv_t + \mathcal{L}v(C_{t-}) + v'(C_{t-})\sigma_{Ct} dZ_t + \big[v(C^*) - \Delta - v(C_{t-})\big] d\Pi_t \mathbf{1}_{\{C_{t-} < C^*\}}}{v(C_{t-})} \\
&= r + \delta + \mathbf{1}_{\{C_{t-} < C^*\}} \left(\pi - \frac{\pi(v(C^*) - \Delta_t)}{v(C_{t-})}\right) + \frac{dDiv_t}{v(C_{t-})} \\
&+ \underbrace{\frac{v'(C_{t-})}{v(C_{t-})} \times \sigma(1-\beta_t)}_{\equiv \Sigma_t = \Sigma(C_{t-})} dZ_t + \frac{\big[v(C^*) - \Delta - v(C_{t-})\big] d\Pi_t}{v(C_{t-})} \mathbf{1}_{\{C_{t-} < C^*\}},
\end{aligned}$$

where we used the HJB-equation under the optimal controls:

$$(r+\delta)v(C_{t-}) = dDiv_t + \mathcal{L}v(C_{t-}) + \pi\big[v(C^*) - \Delta - v(C_{t-})\big]\mathbf{1}_{\{C_{t-} < C^*\}}.$$

Let us now derive an expression for the average stock-return volatility during the firm's lifetime:

$$\bar{\Sigma}(C) = \mathbb{E}\big[\Sigma_T \big| T < \tau \wedge C_{0-} = C\big] = \frac{\mathbb{E}\big[\int_0^\tau \Sigma_t dt \big| C_{0-} = C\big]}{\mathbb{E}\big[\tau \big| C_{0-} = C\big]} \equiv \frac{\mathcal{V}(C)}{\mathcal{T}(C)}.$$

First, it can be derived that $\mathcal{V}$ solves:

$$\begin{aligned}
&\delta\mathcal{V}(C) = \Sigma(C) + \mathcal{L}\mathcal{V}(C) + \pi\big[\mathcal{V}(C^*) - \mathcal{V}(C))\big]\mathbf{1}_{\{C_{t-} < C^*\}} \\
&\Longleftrightarrow \mathcal{V}''(C) = \frac{2}{\sigma_C^2} \times \Big(\delta\mathcal{V}(C) - \Sigma(C) - \mu_C \mathcal{V}'(C) - \pi\big[\mathcal{V}(C^*) - \mathcal{V}(C))\big]\mathbf{1}_{\{C_{t-} < C^*\}}\Big).
\end{aligned}$$

Second, observe that $\int_0^\tau 1dt = \tau$, so that $\mathcal{T}$ solves:

$$\begin{aligned}
&\delta\mathcal{T}(C) = 1 + \mathcal{L}\mathcal{T}(C) + \pi\big[\mathcal{T}(C^*) - \mathcal{T}(C))\big]\mathbf{1}_{\{C_{t-} < C^*\}} \\
&\Longleftrightarrow \mathcal{T}''(C) = \frac{2}{\sigma_C^2} \times \Big(\delta\mathcal{T}(C) - \Sigma(C) - \mu_C \mathcal{T}'(C) - \pi\big[\mathcal{T}(C^*) - \mathcal{T}(C))\big]\mathbf{1}_{\{C_{t-} < C^*\}}\Big)
\end{aligned}$$

The boundary conditions are then given by:

$$\mathcal{V}(0) = \mathcal{T}(0) = 0 \text{ and } \mathcal{V}'(\bar{C}) = \mathcal{T}'(\bar{C}) = 0.$$

Likewise, we can calculate average β, i.e., $\bar{\beta}$ through:

$$\bar{\beta}(C) = \frac{\mathbb{E}\left[\int_0^\tau \Sigma_t dt \middle| C_{0^-} = C\right]}{\mathbb{E}\left[\tau \middle| C_{0^-} = C\right]} \equiv \frac{\mathcal{B}(C)}{\mathcal{T}(C)},$$

where $\mathcal{B}$ solves:

$$\delta\mathcal{B}(C) = \beta(C) + \mathcal{L}\mathcal{B}(C) + \pi\left[\mathcal{B}(C^*) - \mathcal{B}(C)\right]\mathbf{1}_{\{C_{t^-} < C^*\}}$$
$$\iff \mathcal{B}''(C) = \frac{2}{\sigma_C^2} \times \left(\delta\mathcal{B}(C) - \Sigma(C) - \mu_C \mathcal{B}'(C) - \pi\left[\mathcal{B}(C^*) - \mathcal{B}(C))\right]\mathbf{1}_{\{C_{t^-} < C^*\}}\right),$$

subject to $\mathcal{B}(0) = \mathcal{B}'(\bar{C}) = 0$.

D.3.1 Proof of Corollary 4

i) *Proof.* To start with, for all C with $\beta(C)$ we rewrite:

$$\Sigma(C) = (1-\beta)\sigma\frac{v'(C)}{v(C)} = \sigma\rho r \times \frac{(v'(C))^2}{v(C)\big(\rho r v'(C) - v''(C)\big)},$$

so that

$$\begin{aligned}\Sigma'(C) &\propto 2v(C)\big(\rho r v'(C) - v''(C)\big)v'(C)v''(C) \\ &- (v'(C))^2 \times \Big[v'(C)\big(\rho r v'(C) - v''(C)\big) + v(C)\big(\rho r v''(C) - v'''(C)\big)\Big] \\ &= -(v'(C))^3 \times \big(\rho r v'(C) - v''(C)\big) + o\big(v(C)\big)\end{aligned}$$

It follows then that $\Sigma'(C) < 0$ in a neighbourhood of zero, provided the scrap value $L \geq 0$ is sufficiently low.

Next, note that in a neighbourhood of $\bar{C}$, we have that $\beta(C) = \lambda$, provided $\lambda > 0$, in which case it is clear that

$$\Sigma(C) = (1-\lambda)\sigma\frac{v'(C)}{v(C)}$$

decreases in this neighbourhood of $\bar{C}$.

□

ii-1) *Proof.* Note that in the limit $\lambda \to 1$, the firm value converges to,

$$\frac{\mu - \rho r/2\sigma^2}{r+\delta} + M_{0^-},$$

where all cash (the firm is born with) is paid out immediately as dividends to shareholders and continuation value from time 0 onwards is deterministic, as the agent absorbs all cash-flow risk. Hence, for $\lambda \to 1$, it follows that $\Sigma(C) \to 0$ for any C, so that by continuity, there exists $\bar{\lambda} \in (0,1)$, so that $\Sigma(C)$ decreases in λ for $\lambda > \bar{\lambda}$, thereby concluding the proof. □

ii-2) *Proof.* Fix $\lambda \in (0,1)$. For all $\varepsilon > 0$ we can pick $\rho > 0$ small enough such that $\beta(C) = \lambda$ exactly for all $C \in (\bar{C} - \varepsilon, \bar{C}]$. On the interval $(\bar{C} - \varepsilon, \bar{C}]$, we have that $v'(C) = 1 + o(\varepsilon)$ and

$v''(C) = o(\varepsilon)$. As a consequence:

$$v_\lambda(C) = \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t} \left[-r\rho\lambda\sigma^2\right] \mathbf{1}_{\beta_t=\lambda} dt \middle| C_0 = C \right] + o(\varepsilon) < \frac{-r\rho\lambda\sigma^2}{r+\delta} + o(\varepsilon).$$

On the interval $(\bar{C} - \varepsilon, \bar{C}]$:

$$\Sigma(C) = \frac{1+o(\varepsilon)}{v(C)} \times (1-\lambda)\sigma$$
$$\implies \Sigma_\lambda(C) \propto o(\varepsilon) - v(C) - v_\lambda(C)(1-\lambda) > o(\varepsilon) + \frac{-r\rho\lambda\sigma^2(1-\lambda)}{r+\delta} - v(C).$$

Note that we can pick ρ or λ arbitrarily small, so as to achieve $\Sigma_\lambda(C) < 0$, which concludes the proof. □

iii) *Proof.* For $\lambda > 0$, there exists $\varepsilon > 0$, so that on $(\bar{C} - \varepsilon, \bar{C}]$:

$$\Sigma(C) = (1-\lambda)\sigma \frac{v'(C)}{v(C)} + o(\varepsilon) = (1-\lambda)\sigma \frac{1+o(\varepsilon)}{v(C)} + o(\varepsilon),$$

so that

$$\frac{\partial \Sigma(C)}{\partial \kappa} \propto o(\varepsilon) - v_\kappa(C)(1+o(\varepsilon)) > 0$$

for $\varepsilon > 0$ sufficiently small, thereby concluding the proof. □

D.4 Proof of Corollary 6

We split the proof in three parts. The first part proves the claims regarding $\overline{C}$. The second part proves the claims regarding $\beta(0)$ and the third part the claim regarding C^*. We will not introduce additional notation, so that C^* is a constant under limited commitment w.r.t. refinancing strategy and a function of C under full commitment w.r.t. the refinancing strategy/

D.4.1 Part 1

Proof. First, obtain

$$v_\pi(C) = \partial_\pi v(C) = \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t} \Big(v'(C_{t^-}) \frac{1-e^{-\rho r \Gamma_t}}{\rho r} + \underbrace{\left[v(C_{t^-} + \Delta_t - \Gamma_t) - v(C_{t^-}) - \Delta_t\right]}_{>0} \Big) dt \middle| C_{0^-} = C \right],$$

from where it is obvious that $v_\pi(C) > 0$ for any $C > 0$. Continuity and the identity $v(0) = L$ imply then that $v'_\pi(C)$ for C in a neighbourhood of zero. Let us differentiate the HJB-equation at the boundary w.r.t. π (i.e., (D.3)), which yields:

$$0 = (r+\delta)v_\pi(\bar{C}) + \delta \bar{C}_\pi + \frac{\delta\kappa\bar{C}_\pi}{1-\kappa} A'\left(\frac{\kappa\bar{C}}{1-\kappa}\right) \implies \bar{C}_\pi \propto -v_\pi(\bar{C}) < 0.$$

Note that the argument did not make use of any assumed commitment structure, so that the claim holds true regardless of the commitment structure. □

D.4.2 Part 2

Proof. Second, denoting the fixed value $C^* = C_{t^-} + \Delta_t - \Gamma_t$, let us rewrite the HJB-equation:

$$RA(C) = \frac{-v''(C)}{v'(C)} = \frac{2}{(1-\beta(C))^2\sigma^2} \times \underbrace{\left(\frac{-(r+\delta)v(C)}{v'(C)} + \mu_C + \pi[v(C^*) - v(C) - \Delta] \right)}_{\equiv \mathcal{E} > 0}.$$

One can show that

$$\frac{\partial}{\partial \pi} \frac{-(r+\delta)v(C)}{v'(C)} \propto v(C)v'_\pi(C) + o(C),$$

which is strictly positive for C in a neighbourhood of zero.

Let us assume now limited commitment w.r.t. to the refinancing strategy. Then:

$$\begin{aligned} &\frac{\partial}{\partial \pi}\Big(\mu_C + \pi[v(C^*) - v(C) - \Delta]\Big) \\ &= \frac{1 - e^{-\rho r \Gamma_t}}{\rho r} + \big[v(C^*) - v(C) - \Delta\big] + \pi \left[v'(C^*)\frac{\partial C^*}{\partial \pi} - v_\pi(C) - \frac{\partial \Delta}{\partial \pi} \right] \end{aligned}$$

Utilizing $v'(C^*) = \frac{1}{1-\kappa}$, $\frac{\partial \Delta}{\partial C^*} = \frac{1}{1-\kappa}$ and $\frac{\partial \Delta}{\partial \pi} = \frac{\partial \Delta}{\partial C^*} \times \frac{\partial C^*}{\partial \pi}$, the above expression simplifies to:

$$\frac{1 - e^{-\rho r \Gamma_t}}{\rho r} + \big[v(C^*) - v(C) - \Delta\big] - \underbrace{\pi v_\pi(C)}_{=o(C)}.$$

Thus, for C sufficiently close to zero, the above expression is positive, which implies that $RA(C)$ decreases in π for $C \simeq 0$. Provided a loose IC-condition $\beta \geq \lambda$ in a neighbourhood of zero, also $\beta(C)$ increases in π for C close to zero.

Last, we assume full commitment to a refinancing strategy is possible. Then, the envelope theorem applies, so that:

$$\begin{aligned} &\frac{\partial}{\partial \pi}\Big(\mu_C + \pi[v(C^*) - v(C) - \Delta]\Big) \\ &= \frac{1 - e^{-\rho r \Gamma_t}}{\rho r} + \big[v(C^*) - v(C) - \Delta\big] > 0, \end{aligned}$$

so that $RA(C)$ increases in π close to zeros and so does $\beta(C)$. This concludes the proof of the second part. □

D.4.3 Part 3

Proof. Third, we show the claim regarding C^*. First, note that for $C > C^*$, we can write

$$v_\pi(C) = \mathbb{E}e^{-(r+\delta)\tau^*} v_\pi(C^*) < v_\pi(C^*)$$

for $\tau^* = \inf\{t \geq 0 : C_{t^-} = C^*\}$. Hence, $v'_\pi(C) < 0$ for $C \in [C^*, \bar{C}]$, since there is no refinancing in this region and $v_\pi(C) > 0$. By continuity, it even follows that $v'_\pi(C) < 0$ for $C \in [C^* - \varepsilon, \bar{C}]$ for some $\varepsilon > 0$. We differentiate the identity $v'(C^*) = \frac{1}{1-\kappa}$, which yields:

$$v'_\pi(C^*) + v''(C^*)\frac{\partial C^*}{\partial \pi} = 0 \Longrightarrow \frac{\partial C^*}{\partial \pi} = \frac{v'_\pi(C^*)}{-v''(C^*)} < 0,$$

thereby concluding the proof. □

D.5 Proof of Corollary 5

Proof. To prove part i), assume to the contrary there exist $C_1 < C_2$ with $\Delta(C_1) \leq \Delta(C_2)$. This clearly implies $C^*(C_2) > C^*(C_1)$, so that $v'(C^*(C_2)) < v'(C^*(C_1))$ by concavity. Likewise: $v'(C_2) < v'(C_1)$. Wlog, we may assume $\overline{C} > C^*(C_2)$, as otherwise the claim is trivial. However, it is easy to verify that (**??**) cannot hold for both C_1 and C_2, contradicting the optimality of the hypothesized strategy.

Part ii) follows immediately from the fact that $C^*(C) \geq C$ by definition. Thus, either $C^*(C) = \overline{C}\ \forall\ C \in [\overline{C} - \varepsilon, \overline{C}]$ for appropriate $\varepsilon > 0$, in which case the claim is trivially true, or there exist $\varepsilon > 0$, $C < \overline{C}$ with $C^*(C) < \overline{C}\ \forall\ C \in [\overline{C} - \varepsilon, \overline{C}]$, in which case the claim follows by continuity and $\lim_{C \to \overline{C}} C^*(C) = \overline{C}$.

For Part iii), we can wlog focus on the case where $C^*(C) < \overline{C}$ throughout. We implicitly differentiate (32), in order to obtain:

$$v''(C^*)\frac{\partial C^*}{\partial C} = \frac{\kappa e^{-\rho r \frac{\kappa}{1-\kappa}[C^* - C]}}{1-\kappa}\left[-v''(c) + \frac{\rho r \kappa}{1-\kappa} v'(C)\left(\frac{\partial C^*}{\partial C} - C\right)\right],$$

which can be solved for:

$$\frac{\partial C^*}{\partial C} \propto v''(C) + \frac{\rho r \kappa}{1-\kappa} C = v''(C) + o(\rho\kappa),$$

so that C^* decreases for small C, provided ρ or κ are sufficiently low. $\square$

E Further comparative statics

In Figures 6, 7, and 8 we present the full numerical comparative statics of the baseline model without refinancing.

Changing ρ. Next, varying the agent's CARA coefficient ρ makes hedging via labor contracts more expensive as agents require higher risk-premia for variability in their certainty equivalent wages Y_t. In response, as Column 3 in Figure 8 shows, the firm reduces its usage of pay-performance sensitivity, reducing avg β, and instead increases its average cash-holdings, raising $\overline{C}$. On the other hand, moral hazard has more bite for larger ρ, which in turn implies that overall firm value decreases in ρ. As a result, liquidation gets less inefficient, which calls for less hedging of liquidity risks. This leads to the non-monotonic behavior of $\overline{C}$ in ρ. Again, numerically there is only a very mild reduction in $\beta(0)$.

Changing μ, κ and δ As Column 2 in Figure 8 show, the comparative statics w.r.t. μ exhibit non-monotonicity of $\overline{C}$. As pointed out in Décamps et al. (2011), this already occurred in a model absent IC considerations, i.e., $\lambda = 0$. Intuitively, for low μ, the project is not worth a lot as a going concern, and thus it is better to drain cash quickly in terms of dividends. As μ starts increasing, the project value increases, making shareholders more willing to accumulate cash and delay dividend payments. This is the first effect. A second effect is highlighted for very high μ: Here, the optimal payout boundary $\overline{C}$ declines. Intuitively, negative cash-flow shocks can be more easily overcome by the drift, and the need to hold expensive cash balances shrinks. Another way to express this second effect is that all else equal, a higher μ leads to more of the probability mass to be close to $\overline{C}$, and thus average cash-holdings to increase. Lowering average cash-holdings thus requires decreasing $\overline{C}$. The (scaled) $\beta(0)$ and avg β both inherit the non-monotonicity of $\overline{C}$.

Last, δ and κ essentially determine endogenously arising carry cost of cash. Not surprisingly, we find that increases in either κ and λ make cash-holdings more costly, thereby reducing $\overline{C}$. As a

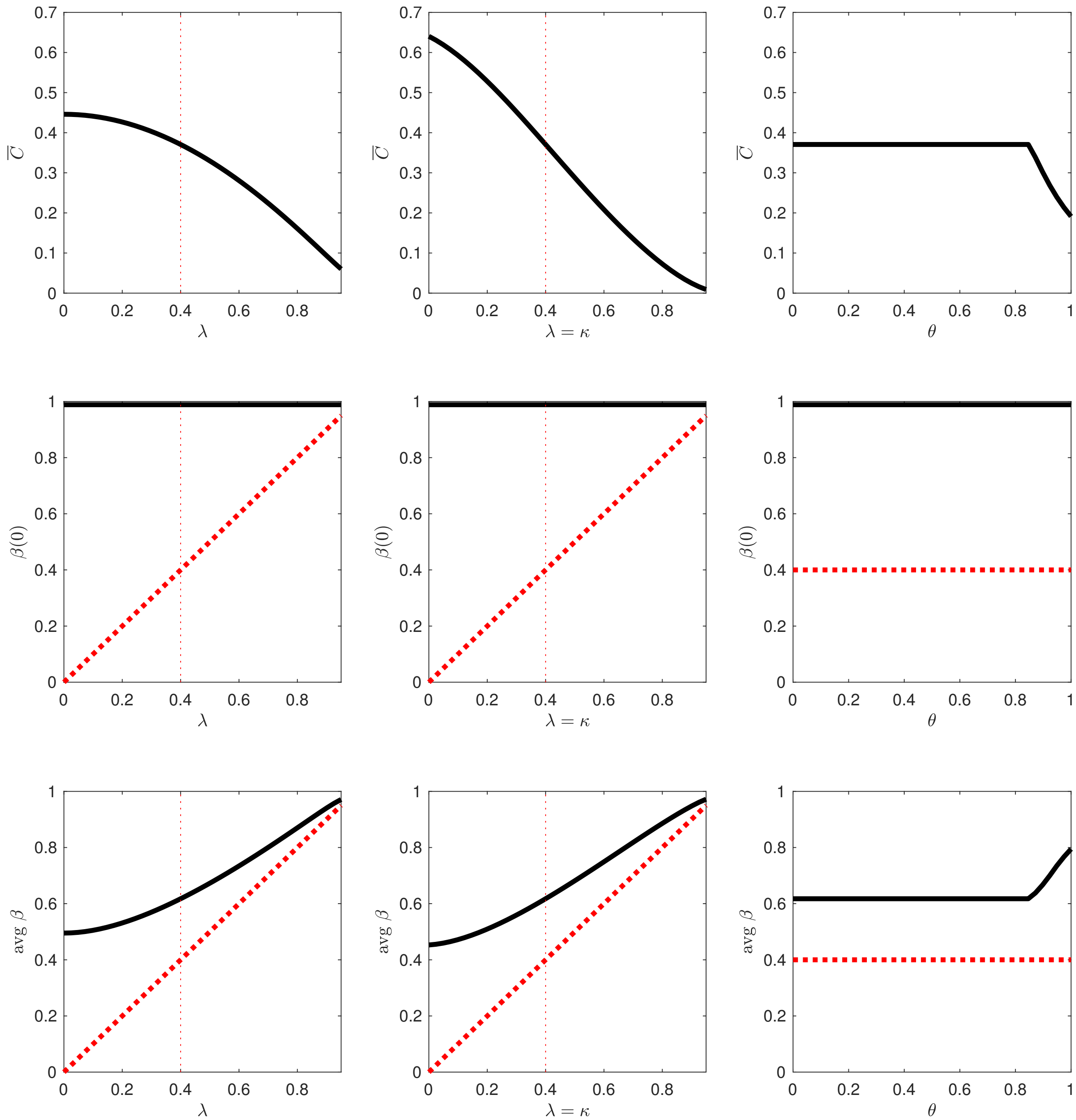

Figure 6: **Comparative statics** w.r.t. λ (Column 1), w.r.t. $\lambda = \kappa$ (Column 2), w.r.t. θ (Column 3), top row $\overline{C}$, middle row $\beta(0)$, bottom row (σ-scaled) avg β. The solid black lines depict the object described on the y-axis, the dashed red line depicts the IC constraint (19), the thin vertical dashed red line depicts the parameter value in our benchmark.

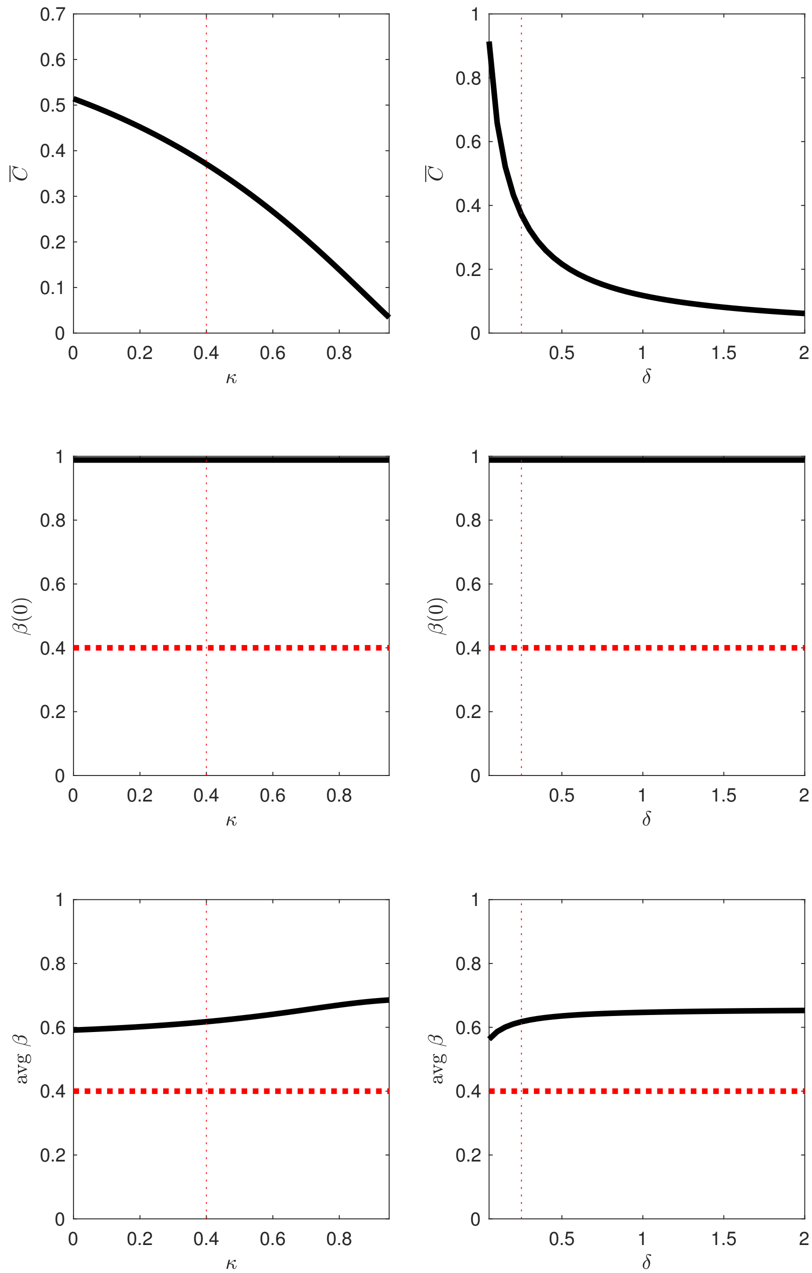

Figure 7: **Comparative statics** w.r.t. κ (Column 1), w.r.t. δ (Column 2), top row $\overline{C}$, middle row $\beta(0)$, bottom row avg β. The solid black lines depict the object described on the y-axis, the dashed red line depicts the IC constraint (19), the thin vertical dashed red line depicts the parameter value in our benchmark.

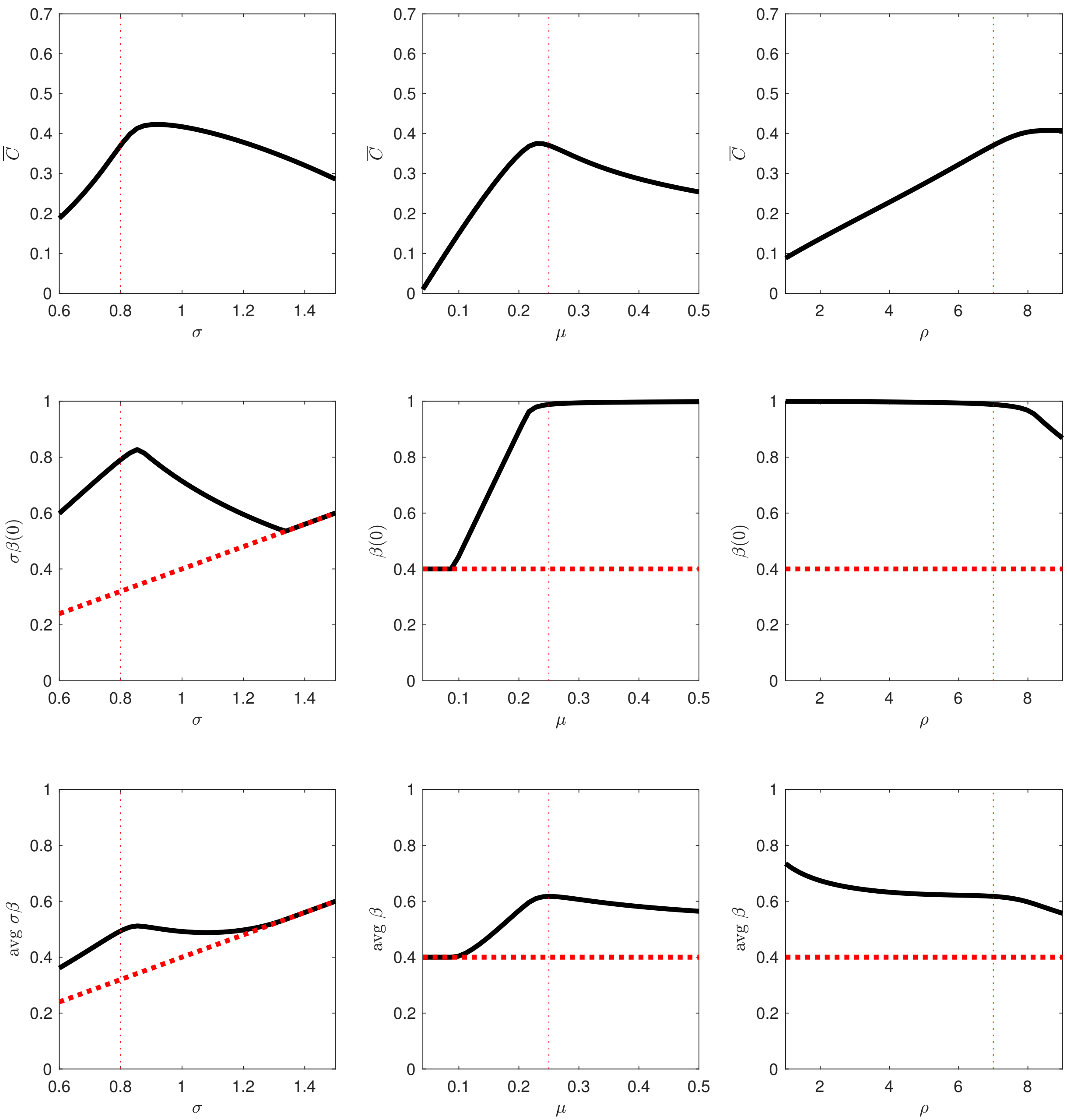

Figure 8: **Comparative statics** w.r.t. σ (Column 1), w.r.t. μ (Column 2), w.r.t. ρ (Column 3), top row $\overline{C}$, middle row $\beta(0)$, bottom row (σ-scaled) avg β. The solid black lines depict the object described on the y-axis, the dashed red line depicts the IC constraint (19), the thin vertical dashed red line depicts the parameter value in our benchmark.

result, firm value decreases, which makes liquidation less inefficient and therefore also curbs hedging through labour markets.

F Numerical solution

We utilize a shooting method to solve for the value function. We shoot from $\overline{C}$ towards $C = 0$, iterating on the condition $v(0) = L$.

First, define

$$B(C) := \frac{\theta}{1-\kappa}C + L$$
$$D(C) := rC + \mu - \delta A\left(\frac{\kappa\overline{C}}{1-\kappa}\right)$$

Next, write the ODE with optimized $\varphi = \kappa$ as

$$(r+\delta)\,v(C) = v'(C)\left[D(C) - \rho r\frac{\sigma^2}{2}\beta(C)^2\right] + \frac{\sigma^2}{2}(1-\beta(C))^2\,v''(C)$$

Suppose for the moment that $\beta^*(C) = \frac{-v''(C)}{\rho r v'(C) - v''(C)} > \lambda$. Then, after plugging in for $\beta^*(C)$ and cancelling out terms, we have the non-linear ODE

$$\begin{aligned}(r+\delta)\,v(C) &= v'(C)\,D(C) + \rho r\frac{\sigma^2}{2}\frac{v'(C)\,v''(C)}{\rho r v'(C) - v''(C)}\\ &= v'(C) - \rho r\frac{\sigma^2}{2}v'(C)\,\beta^*(C)\end{aligned}$$

whereas when $\beta^*(C) < \lambda$, we have the linear ODE

$$(r+\delta)\,v(C) = v'(C)\left[D(C) - \rho r\frac{\sigma^2}{2}\lambda^2\right] + \frac{\sigma^2}{2}(1-\lambda)^2\,v''(C)$$

Next, let us consider the boundary conditions. Note that we have $v(0) = L$ and $v'\left(\overline{C}\right) = 1$ in any scenario. We have to consider two scenarios:

1. Suppose first that $v\left(\overline{C}^*\right) > B\left(\overline{C}^*\right)$, where $\overline{C}^*$ is defined by $v''\left(\overline{C}^*\right) = 0$. Then, at $\overline{C} = \overline{C}^*$ we have

   $$(r+\delta)\,v\left(\overline{C}\right) = 1 \times \left[D(C) - \rho r\frac{\sigma^2}{2}\lambda^2\right] + \frac{\sigma^2}{2}(1-\beta(C))^2 \times 0$$

   which implies that

   $$v\left(\overline{C}\right) = \frac{D(C) - \rho r\frac{\sigma^2}{2}\lambda^2}{r+\delta}$$

   and we initialize the shooting algorithm at $\overline{C}$ with

   $$\begin{pmatrix} v \\ v' \\ v'' \end{pmatrix}\left(\overline{C}\right) = \begin{pmatrix} \frac{D\left(\overline{C}\right) - \rho r\frac{\sigma^2}{2}\lambda^2}{r+\delta} \\ 1 \\ 0 \end{pmatrix}$$

2. Suppose next that $v\left(\overline{C}^*\right) = \frac{D\left(\overline{C}^*\right) - \rho r\frac{\sigma^2}{2}\lambda^2}{r+\delta} < B\left(\overline{C}^*\right)$, which implies that the payout bound-

ary cannot be chosen via the super-contact condition. Then, we need to initialize the shooting algorithm at $\overline{C}$ with

$$\begin{pmatrix} v \\ v' \\ v'' \end{pmatrix}\left(\overline{C}\right) = \begin{pmatrix} B\left(\overline{C}\right) \\ 1 \\ v''\left(\overline{C}\right) \end{pmatrix}$$

where $v''\left(\overline{C}\right)$ is given by the continuous function

$$v''\left(\overline{C}\right) = \begin{cases} \frac{(r+\delta)B\left(\overline{C}\right)-D\left(\overline{C}\right)+\rho r\frac{\sigma^2}{2}\lambda^2}{\frac{\sigma^2}{2}(1-\lambda)^2} & ,\overline{C} \geq \overline{C}_\lambda \\ \frac{\rho r\left[(r+\delta)B\left(\overline{C}\right)-D\left(\overline{C}\right)\right]}{\rho r\frac{\sigma^2}{2}+(r+\delta)B\left(\overline{C}\right)-D\left(\overline{C}\right)} & ,\overline{C} < \overline{C}_\lambda \end{cases}$$

and where the constant $\overline{C}_\lambda$ solves

$$\frac{-v''\left(\overline{C}_\lambda\right)}{\rho r - v''\left(\overline{C}_\lambda\right)} = \lambda \iff -\frac{\lambda\rho r}{1-\lambda} = v''(\overline{C}_\lambda) = \frac{(r+\delta)\,B\left(\overline{C}_\lambda\right) - D\left(\overline{C}_\lambda\right) + \rho r\frac{\sigma^2}{2}\lambda^2}{\frac{\sigma^2}{2}\left(1-\lambda\right)^2}$$

The derivation is straightforward. Note that $v\left(\overline{C}\right) = B\left(\overline{C}\right)$ as well as $v'(\overline{C}) = 1$ by assumption. There are two cases w.r.t. $\beta^*\left(C\right)$:

(a) When $\beta^*\left(\overline{C}\right) = \frac{-v''\left(\overline{C}\right)}{\rho r - v''\left(\overline{C}\right)} < \lambda \iff v''\left(C\right) > -\frac{\lambda\rho r}{1-\lambda}$, we have

$$(r+\delta)\,B\left(\overline{C}\right) = \left[D\left(\overline{C}\right) - \rho r\frac{\sigma^2}{2}\lambda^2\right] + \frac{\sigma^2}{2}\left(1-\lambda\right)^2 v''\left(\overline{C}\right).$$

(b) Next, for $\beta^*\left(\overline{C}\right) = \frac{-v''\left(\overline{C}\right)}{\rho r - v''\left(\overline{C}\right)} > \lambda \iff v''\left(C\right) < -\frac{\lambda\rho r}{1-\lambda}$, we have

$$(r+\delta)\,B\left(\overline{C}\right) = D\left(\overline{C}\right) + \rho r\frac{\sigma^2}{2}\frac{v''\left(\overline{C}\right)}{\rho r - v''\left(\overline{C}\right)}.$$

As $v'''(C) > 0$, the partition on $\overline{C} \gtrless \overline{C}_\lambda$ results.

G Steady-state KFE

To evaluate the average β of a firm w.r.t. to a density implied by the process C, we want to derive the steady-state density induced by resetting all liquidating firms to $C = \overline{C}$. Let us write the dynamics of C on the equilibrium path as

$$dC_t = \mu_C(C_t)dt + \sigma_C(C_t)dZ_t,$$

and define $s_C(C) := (\sigma_C(C))^2$. Then, the stationary density $f(\cdot)$ on $(0, \overline{C})$ solves

$$0 = \frac{1}{2}\partial_{CC}[s_C(C)f(C)] - \partial_C[\mu_C(C)f(C)] - \delta f(C)$$

The boundary conditions are given by

$$f(0) = 0 \tag{G.1}$$

as well as

$$0 = \frac{1}{2}\partial_C[s_C(C)f(C)]_{C=\overline{C}} - \mu_C(\overline{C})f(\overline{C}) + \delta - \frac{1}{2}s_C(0)f'(0) \tag{G.2}$$

Here, the first two terms are the traditional reflection boundary conditions, the third term is the inflow from the (state-independent) Poisson defaults at rate δ, and the fourth term is the inflow from the liquidity defaults at $C = 0$.

Recall that along the equilibrium path (assuming $\varphi = \kappa$) on $(0, \overline{C})$ we have

$$dC_t = \left[rC_{t^-} - \frac{\rho r}{2}(\beta(C_{t^-})\sigma)^2 - \delta A \left(\frac{\kappa}{1-\kappa} C_{t^-} \right) + \mu \right] dt + [1 - \beta(C_{t^-})]\, \sigma dZ_t - C_{t^-} dN_t. \quad \text{(G.3)}$$