# Debt Maturity and the Dynamics of Leverage $^{*}$

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## Abstract

This paper shows that long debt maturities destroy equityholders' incentives to reduce leverage in response to poor firm performance. By contrast, a sufficiently short debt maturity commits equityholders to implement such leverage reductions. However, a short debt maturity also generates transactions costs associated with rolling over maturing bonds. We show that this tradeoff between higher expected transactions costs against the commitment to reduce leverage when the firm is doing poorly motivates an optimal maturity-structure of corporate debt. Since firms with high costs of financial distress benefit most from committing to leverage reductions, they have a stronger incentive to issue short-term debt. The debt maturity required to commit to future leverage reductions decreases with the volatility of the firm's cash flows. We also find that the equityholders' incentives to reduce debt is non-monotonic in the firm's leverage. If the firm is pushed towards bankruptcy by a persistent series of low cash flows, then equityholders resume issuing debt to refinance maturing bonds, even when debt maturities are short.

**Keywords:** debt maturity, optimal capital structure choice
**JEL:** G3, G32

# 1  Introduction

Significant progress has been made towards understanding firms' dynamic financing decisions. Major contributions to this literature model a firm's assets or cash flows as a stochastic process and assume that debt enjoys some benefit, such as a tax advantage, but generates dead weight costs associated with excessively high leverage, such as bankruptcy costs.[1] While these models have been successful in explaining firms' optimal target leverage ratios and their decisions to dynamically increase debt levels in response to increases in their asset values or cash flows, they have been much less successful in explaining leverage reductions. This is so since these models generally imply that equityholders never find it optimal to reduce dividends or issue equity to reduce debt. As shown by Admati, DeMarzo, Hellwig, and Pfleiderer (2015), equityholders not only lack any incentive to actively repurchase outstanding debt but frequently have incentives to increase debt even if this reduces total firm value. Thus, it is a feature of these models that debt reductions only occur following bankruptcy.[2]

This implication is in contrast to empirical evidence showing that firms frequently reduce debt even when bankruptcy has not yet occurred and no debt forgiveness has been negotiated.[3]

In this paper we develop a dynamic capital structure model where leverage reductions occur not only after bankruptcy or after renegotiations with debtholders. We show that such voluntary leverage reductions are closely related to the firm's debt maturities. Thus, we identify and analyze a largely unexplored aspect of debt maturity, namely its effect on future capital structure dynamics. We specifically address the following questions. How is debt maturity related to equityholders' dynamic leverage adjustments? How do firms optimally

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[1] See, for example, Fischer, Heinkel, and Zechner (1989), Leland (1994a), Leland and Toft (1996), Goldstein, Ju, and Leland (2001), Dangl and Zechner (2004), Strebulaev (2007).

[2] Some models consider debt renegotiations and derive partial debt forgiveness outside of bankruptcy (See, e.g. Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997), Mella-Barral (1999)), Christensen, Flor, Lando, and Miltersen (2014). Lahar (2015) considers multilateral bargaining and explicitly regards renegotiation breakdowns and subsequent inefficient liquidation. In contrast to these papers we focus on situations where coordination problems among bondholders prevent renegotiation solutions.

[3] Leary and Roberts (2005) report that a fraction of 28% of capital structure adjustments in their 1984 to 2001 dataset comprises active debt repurchases. Surveying 392 CFOs, Graham and Harvey (2001) report that 81% of firms in their sample use at least flexible target leverage ratios. If highly levered, firms tend to issue equity to maintain their target ratios. Hovakimian, Opler, and Titman (2001) find strong evidence that firms use (time varying) target leverage ratios. They find the deviation from this target as the dominant economic factor in determining whether a firm retires debt.

refinance expiring debt? What is the optimal debt maturity structure given its implications for dynamic capital structure adjustments and which firms are most likely to issue short-term debt? We address these questions in a framework that does not rely on information asymmetries or agency conflicts. In the model firms' equityholders are allowed to optimize the mix of debt and equity used to refinance maturing debt, but covenants do not allow them to increase the total face value of debt. If firms wish to do increase the face value of debt they must first repurchase the existing debt before they can make discrete capital structure adjustments. They are allowed to do that at any point in time.

We find that firms' equityholders may not wish to roll over maturing debt by issuing a new bond with the same face value. Instead, it may be optimal for them to issue a bond with lower face value, i.e. to at least partly use equity to repay the maturing debt. This happens for firms with sufficiently short debt maturities following a deterioration in the firm's profitability. In this situation issuing new debt to refinance maturing old debt is costly since the price of the new bonds reflect the increased default probability and the resulting increase in expected costs of financial distress. These costs would be effectively be borne by the equityholders and they may therefore optimally reduce the face value of the new bond issue.

If, by contrast, debt maturity is sufficiently long, then replacing maturing debt with equity always leads to a significant wealth transfer to the remaining bonds outstanding, since debt with a longer maturity is subject to more credit risk. This creates a more severe debt overhang problem and makes the use of equity to refinance maturing debt suboptimal. In this case the firm's equityholders always prefer to roll over debt at the maximum rate, i.e. to issue a new bond with a face value that corresponds to the face value of the maturing bonds. This result is in accordance with empirical evidence provided by Hovakimian, Opler, and Titman (2001), who find that long debt maturities seem to be major impediments to debt reductions.

We also find that shorter debt maturities lead to more pronounced debt reductions since the short maturities require the firm to refund a larger fraction of its debt during any given period of time. This implies that a firm which refinances part of the retired debt with equity will lower its debt level more quickly in response to a drop in its profitability.

Equityholders incentives to refund maturing debt with equity are non-monotonic in the firm's profitability and thus in firm value. For values around the initial cash flow level it is

optimal to roll over maturing debt by issuing new bonds with the same face value. If the firms profitability drops sufficiently, then the equityholders reduce the rollover rate, as explained above. However, if the firm's cash flows continue to deteriorate and the firm is pushed towards the default boundary, then equityholders find it optimal again to choose the maximum rollover rate. Since the firm is close to bankruptcy a reduction in leverage largely benefits the remaining bondholders, even if the maturity of the remaining debt is short. Thus, the resulting debt overhang problem implies that equityholders are no longer willing to contribute capital to reduce debt.

One way to understand this result intuitively is to recall that the equityholders effectively own a put option which gives them the right to sell the firms assets to its bondholders at an exercise price equal to the face value of the bonds. Reducing this exercise price by retiring some or all of the maturing debt leads to a particularly significant reduction in the value of this option when it is at or in the money, i.e. if the value of the firms assets is already close to or less than the face value of debt. In this case the equityholders are willing to roll over maturing debt, even if the new bonds can only be issued at a low price.

Hovakimian, Opler, and Titman (2001) present strong empirical support for this non-monotonicity in voluntary debt reductions. Interestingly, existing literature such as Welch (2004) has interpreted the fact that highly levered firms issue debt as evidence *against* the trade off theory of capital structure choice, since it moves the leverage ratio away from the optimal target ratio. Our analysis demonstrates that this behavior is in full accordance with a dynamic tradeoff paradigm once multiple debt issues and optimal financing of maturing debt are considered.

In our setting, debt maturity significantly influences the expected probability of bankruptcy. This is so since short debt maturities lead to more rapid debt reductions when the firm's profitability starts to decrease. Investors take this into account when they price the debt initially. This implies that firms' debt capacity generally *increases* as they choose shorter debt maturities. This result is also in contrast to existing literaturewhich unanimously predicts that short-term debt leads to early and inefficient default. The intuition is that equityholders incur the rollover cost. I.e. when a new bond issue with the same face value cannot fully refinance the maturing bond, equityholders must cover the remaining funding gap. For shorter matu-

rities, the firm must roll over a higher fraction of its debt and therefore equityholders face larger funding gaps. As a result they default sooner (see, e.g. Leland (1994b), Leland and Toft (1996), He and Xiong (2012a) or He and Milbradt (2014)). This aspect of short-term debt tends to reduce firms' debt capacities. In contrast to these papers we consider a new aspect of short-term debt, namely its effect on future leverage reductions. As we show, this implies that shortening debt maturity generally increases the firm's debt capacity and reduces the risk of bankruptcy ex ante.

Our analysis therefore generates a novel theory of debt maturity where, for reasonable parameter values, total firm value is maximized at an interior debt maturity. Since firms never engage in debt reductions for long debt maturities but still incur some transactions costs when debt is rolled over, total firm value is locally maximized for infinite-maturity debt. This saves transactions costs and prevents inefficient early default.[4] When shortening debt maturity sufficiently, however, firms start to engage in debt reductions when their profitability decreases, thereby reducing the probability of financial distress. In this maturity range, the firm's debt capacity increases with maturity reductions and total firm value starts to rise, until the transactions costs associated with refinancing maturing debt outweigh the benefits due to faster debt reductions along unfavorable cash flow paths.Thus, total firm value exhibits another local maximum at an interior value of debt maturity.[5] The exact location of this maximum depends on the parameters of the firm's cash flow process, such as its growth rate and its volatility, as well as on the transactions costs associated with rolling over debt and the magnitude of bankruptcy costs. For empirically reasonable model parameterizations we find that firm value is indeed maximized for interior debt maturities. Infinite-maturity debt maximizes firm value globally only if the costs of financial distress and/or the tax advantage of debt are very low and/or transactions costs for rolling over debt are high. In this case the benefit from increasing debt capacity and reducing the bankruptcy probability by committing to future leverage reductions via short debt is too low compared to the additional transactions costs associated with short-term debt.

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[4]Furthermore it is shown by Leland (1994b), Leland and Toft (1996), and Leland (1998) that the tax advantage of debt is maximized when issuing infinite-maturity debt. Hence, when finite-maturity debt does not imply more efficient downwards restructuring, it is dominated by debt with infinite maturity.

[5]Alternative rationales for short-term debt are based on agency costs originating from the 'asset substitution' problem, first introduced by Jensen and Meckling (1976).

Optimal debt maturity was first analyzed in tradeoff models by Leland (1994b), Leland and Toft (1996), and Leland (1998). Titman and Tsyplakov (2007) extend the analysis by endogenizing investment decisions. These papers have derived important modelling strategies allowing the analysis of debt maturity in a tractable continuous-time framework. They have also generated significant insights on the interplay between leverage and debt maturity. However, they cannot explain interior optimal debt maturities. In these models it would be optimal to issue perpetual debt.

Our model uses a similar modelling approach with one important difference. We allow firms to choose the mix of debt and equity to repay maturing debt, whereas firms in the above models *must* roll over maturing debt with new debt issues, keeping the face value of total debt constant. In contrast to these papers, we concentrate on debt maturity and its role in mitigating conflicts of interest between debtholders and equityholders on capital structure dynamics. [6]

Our paper relates to existing work on debt maturity which explores informational asymmetries. In this literature short debt maturities signal positive inside information, as demonstrated by the seminal work by Diamond (1991, 1993) and Flannery (1986, 1994). Other authors, such as Calomiris and Kahn (1991) and Diamond and Rajan (2001) have emphasized the disciplinary role of short term debt. Debt maturity has also been linked to the debt overhang or underinvestment problem. While the original work by Myers (1977) concludes that short-term debt mitigates these problems, Diamond and He (2014) show that maturing short-term debt can lead to more severe debt overhang than non-maturing long-term debt. More recently, Hackbarth and Mauer (2012) and Dockner, Maeland, and Miltersen (2016) analyze the effect of debt seniority on the underinvestment problem.

There is also an interesting related literature on the interaction between debt maturity, rollover risk, and capital structure. Examples are He and Xiong (2012a,b), He and Milbradt

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[6]Childs, Mauer, and Ott (2005) and Ju, Parrino, Poteshman, and Weisbach (2005) also explore debt maturity. However, in these models firms can only change their debt levels after the entire existing debt has matured. Also, at each point in time firms can only have one bond outstanding with a given maturity. In our model firms are allowed to change the debt level at any point in time. As a result, we are able to isolate the pure commitment effect of debt maturity on equityholders' willingness to adjust debt levels downwards after a decrease in profitability. Furthermore, firms in our model have many bonds with different maturities outstanding, as is frequently the case in practice. At any point in time firms retire only a fraction of outstanding bonds. Therefore, when some bonds mature and are refinanced with new debt or via equity, this influences the value of the existing bonds outstanding.

(2014), Cheng and Milbradt (2012) and Chen, Cui, He, and Milbradt (2015). In a similar vein, He and Xiong (2012a) and Acharya, Gale, and Yorulmazer (2011) analyze debt maturity when short-term debt can lead to early and inefficient asset liquidation.

Recently, optimal debt maturity adjustments over time have been analyzed. See, for example, Brunnermeier and Oehmke (2013), and He and Milbradt (2016). Finally, our paper is also related to Choi, Hackbarth, and Zechner (2015) who analyze rollover risk and the optimal dispersion of debt maturities. This paper focusses on an aspect of short-term debt which is not considered by any of the papers above, namely its effect on equityholders' future incentives to delever. While the existing literature assumes that the face value of debt is kept konstant at rollover dates, we allow firms to optimally choose the refinancing mix. This implies that a firm's leverage capacity increases if it chooses a capital structure which forces it to regularly roll over a non-trivial portion of its debt. This can only be achieved if the average debt maturity is not too long.

The remainder of the paper is structured as follows. Section 2 introduces the main building blocks of the model. The valuation of debt and equity claims and the optimal refinancing of expiring debt are derived in Section 3. Section 4.1 describes how the model is calibrated to mirror US tax code and analyzes optimal discrete capital structure adjustments and Section 5.2 provides numerical examples and comparative statics results. Section 6 concludes.

## 2 The Model

Consider a firm that has debt outstanding with face value $B_t$ and a fixed coupon rate $i$. Coupon payments are tax deductible so that there is a tax advantage of debt. See Table 1 for the notation used throughout this paper. Following the modeling of finite maturity debt in Leland (1994b), Leland (1998), and Ericsson (2000), we assume that debt has no single explicit maturity date but that a constant fraction $m$ of the outstanding debt matures at any instant of time. Ignoring default and debt repurchase, the average maturity of a debt contract is then $1/m$ years.

The firm must repay maturing debt at par, and thus must maintain a flow of principal repayment $mB_t$. The retired portion of debt may be replaced by a new debt issue. However,

Table 1: Notation

| | |
|---|---|
| a firm's instantaneous free cash flow after corporate tax | $c_t$ |
| expected rate of change of $c_t$ | $\mu$ |
| risk adjusted drift of the cash flow process | $\hat{\mu}$ |
| riskless rate of interest | $r$ |
| instantaneous variance of the cash flow process | $c_t^2 \sigma^2$ |
| face value of debt | $B_t$ |
| debt retirement rate | $m$ |
| average debt maturity | $T = 1/m$ |
| debt roll over rate | $\delta$ |
| value of equity | $E$ |
| value of debt | $D$ |
| total value of the firm | $V$ |
| instantaneous coupon rate | $i$ |
| firm's inverse leverage ratio | $y_t$ |
| personal tax rate on ordinary income | $\tau_p$ |
| corporate tax rate | $\tau_c$ |
| proportional bankruptcy costs | $g$ |
| proportional transactions costs for rolling over debt | $k_i$ |
| proportional transactions costs for issuing debt after recapitalization | $k_r$ |
| proportional call premium | $\lambda$ |

we consider bond indentures ensuring that a new bond issue may not increase the total initial face value of debt, so that the rate $\delta_t$ at which the firm may issue new debt must satisfy $0 \leq \delta_t \leq m$. The new debt issue is associated with proportional transactions costs $k_i$, has the same priority as existing debt, and is amortized at the same constant rate $m$. This ensures that the entire debt of the firm is homogeneous and no distinction between early issues and later issues must be made. Although this modelling approach is a simplification it allows us to analyze the implications of debt maturity in the realistic setting in which firms have more than one debt issue outstanding and where the refinancing decision influences the value of the remaining bonds.

As discussed above, covenants prohibit the firm from issuing debt that would increase the total face value. The total amount of debt outstanding can therefore only be increased by repurchasing all outstanding debt contracts and subsequently issuing new bonds with higher face value. Again, proportional transactions costs $k_r$ are associated with the new bond issue.

The coupon rate of the new issue is set to ensure that it can be sold at par.

In contrast to existing firm models with finite average maturity, the firm is not required to roll over the entire amount of maturing debt. For certain leverage ratios, the firm may find it optimal to replace only part of the retired debt with new debt or it might entirely refrain from issuing new debt contracts. If the firm does not fully replace retired debt then the face value of debt outstanding shrinks at a rate $m - \delta_t$ which in turn may help the firm to avoid financial distress.

Debt covenants restrict the face value of debt issued in any given period to be less or equal to the face value of the retired debt. Therefore, after a phase of debt reduction the firm cannot return to the original debt level unless it eliminates the bond indenture by calling all outstanding debt.

If the firm's equityholders stop coupon payments and thereby trigger bankruptcy, all control rights over the firm's productive assets are handed over to debtholders who will then optimally relever the firm. Bankruptcy costs are assumed to be a certain fraction $g$ of the outstandimg face value of the firm's debt.

We assume that the firm's instantaneous free cash flow after corporate tax, $c_t$, follows a geometric Brownian motion given by

$$
\begin{aligned}
\frac{dc_t}{c_t} &= \mu dt + \sigma dW_t, \\
c_0 &= c(0),
\end{aligned}
\tag{1}
$$

where the expected instantaneous drift and the instantaneous variance of the cash flow process are defined by $c_t\mu$ and $c_t^2\sigma^2$ respectively, and $dW_t$ is the increment to a standard Wiener process.

Although there is considerable cross-country variation in the way corporate and personal income are taxed, many tax systems exhibit similar key features. First, the deductibility of interest expenses from taxable corporate income is frequently more generous than that of dividend payments. Second, the effective personal tax on equity income is frequently lower than that on debt income. The latter feature may be due to an outright favorable treatment of dividend income or due to the fact that a larger portion of equity income is generally realized in the form of capital gains, which are often treated more favorably than ordinary income. In

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addition capital gains income comes with a tax-timing option for the investor which is also contributes to the second feature of tax systems mentioned above.

We capture these two features by defining the firm's operating cash flow $c_t$ as after corporate tax and by allowing any coupon payments to be deductible at the constant statutory corporate tax rate $\tau_c$.[7] At the personal level, $\tau_p$ is interpreted broadly as the tax disadvantage of interest income over equity income. Therefore the appropriate discount rate to be applied to expected after-corporate-tax income from equity investment under the risk-neutral probability measure is given by $r(1-\tau_p)$, see Section 3. For a discussion of the calibration of the tax parameters and how they relate to the current US tax code we refer to Section 5.1.

At any point in time, equityholders can decide to adjust the amount of debt by a discrete amount to a new face value $B_t^*$. Alternatively, equityholders may maintain the current debt level and only decide on the rate $\delta_t \in [0,m]$ at which new debt is issued to roll-over (a fraction of) maturing debt. If $\delta_t = m$, then the firm issues new bonds with a face value exactly equal to the face value of the bonds retired at time $t$.[8] The dynamics of the face value of debt are therefore given by

$$
\begin{aligned}
\frac{dB_t}{B_t} &= \begin{cases} \dfrac{B_t^*}{B_t} - 1 & : \text{debt is increased from } B_t \text{ to } B_t^* \text{ at time } t, \\[2ex] -(m - \delta_t)\,dt & : \text{firm replaces maturing debt at a rate } \delta_t \in [0,m] \text{ at time } t \end{cases} \\[2ex]
B_0 &= B(0).
\end{aligned}
$$

(2)

We define $y_t$ as the inverse leverage ratio with respect to the unlevered firm value

$$
y_t = \frac{1}{B_t} \frac{c_t}{r(1-\tau_p) - \hat{\mu}},
$$

(3)

where $\tau_p$ is the personal income tax rate and $\hat{\mu}$ is the risk neutral drift rate of the free cash

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[7]Hence, we assume instant tax refunds for coupon payments and do not explicitly model any loss carry-back and loss carry-forward due to limited corporate taxable income.

[8]Depending on the market value of debt, the proceeds may be considerably less than what is required by repayment obligations even when $m = \delta_t$. In this case the remaining amount is financed by retained earnings or new equity. Alternatively, it may as well be the case that debt trades above par, then the net proceeds are paid out as a dividend to equityholders. I.e., we do not explicitly model the firm's cash holdings but assume that its residual cash flow is balanced by equityholders – either in the form of equity financing or as a cash dividend.

flow $c_t$.[9] Then the risk neutral dynamics of $y_t$ are

$$\frac{dy_t}{y_t} = \begin{cases} \dfrac{B_t}{B_t^*} - 1 & : \text{debt is increased from } B_t \text{ to } B_t^* \text{ at time } t, \\ (\hat{\mu} + (m - \delta_t))\,dt + \sigma\,dW_t & : \text{maturing debt is replaced at a rate } \delta_t \text{ at time } t \end{cases}$$
$$y_0 = y(c_0, B_0) = \frac{1}{B_0}\frac{c_0}{r(1-\tau_p) - \hat{\mu}}.$$

$$(4)$$

(See Appendix A.1 for the derivation of Equation (4).) A discrete adjustment of the debt level following a debt repurchase leads to an immediate jump in the inverse leverage ratio. When the face value of debt is maintained at a constant level (i.e., $\delta_t = m$), then the inverse leverage ratio follows a geometric Brownian motion with the same drift rate and volatility as the cash flow process $c_t$. When only part of the maturing debt is rolled over ($\delta_t < m$), then the drift rate of the inverse leverage ratio is $\hat{\mu} + (m - \delta_t) > \hat{\mu}$, i.e., due to the shrinking debt level, the firm's leverage ratio tends to fall, and thus, the inverse leverage ratio tends to rise.

Dynamic capital structure models with infinite maturity debt have utilized the fact that equity value and debt value are homogeneous of degree one in the face value of debt, $B$. This implies that all firm-relevant decisions can be made contingent on the leverage ratio $y$, hence $B$ serves as a scaling factor only. In the following, it is shown that this homogeneity can be preserved also in the case of finite-maturity debt, even if debt reduction leads to a gradually decreasing debt level. Therefore, all claims contingent on the cash flow $c_t$ are re-interpreted as claims contingent on the two state variables, debt level $B_t$ and inverse leverage ratio $y_t$. This formulation is the key to obtain closed form solutions for the optimal roll-over schedule $\delta_t$ and for the value of debt and equity of the firm.

# 3 Claim Valuation and Optimal Funding of Debt Repayment

In this section we derive the valuation equations for the firm's debt and equity as well as propositions on the optimal refinancing mix for maturing debt. Consider a firm which has

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[9]For a discussion of the effect of personal taxes on debt dynamics, see Hennessy and Whited (2005).

debt outstanding with face value $B_t$. Contingent on the choice of $\delta_t$, the firm's debt level changes at a rate $-(m-\delta_t)$ and, consequently, the drift rate of the inverse leverage ratio $y_t$ is $\hat{\mu} + (m-\delta_t)$. The required instantaneous pricipal repayment is $mB_t dt$, the after-tax coupon payment is $i(1-\tau_p)B_t dt$, and debtholders buy the new debt issues which cause an outflow to equityholders of $\delta_t D$. Therefore the value of debt, $D$, must satisfy the partial differential equation

$$\frac{1}{2}\sigma^2 y^2 \frac{\partial^2 D}{\partial y^2} + (\hat{\mu}+(m-\delta_t))y\frac{\partial D}{\partial y} + \frac{\partial D}{\partial t} + B_t(i(1-\tau_p)+m) - \delta_t D = r(1-\tau_p)D. \qquad (5)$$

Using the homogeneity with respect to the face value $B_t$, we can write $D = B_t \tilde{D}(y)$. The fact that the debt level changes at a rate of $-(m-\delta_t)$ then leads to $\partial D/\partial t = -(m-\delta_t)B_t\tilde{D}(y)$. Then the value of debt per unit of face value, $\tilde{D}(y)$, is not explicitly time dependent and satisfies the following differential equation

$$\frac{1}{2}\sigma^2 y^2 \frac{\partial^2 \tilde{D}}{\partial y^2} + (\hat{\mu}+(m-\delta_t))y\frac{\partial \tilde{D}}{\partial y} + (i(1-\tau_p)+m) = (r(1-\tau_p)+m)\tilde{D}. \qquad (6)$$

We next turn to the valuation of equity. Equityholders must provide a cash flow of $mB_t$ to service expiring debt contracts. Furthermore, debt requires coupon payments of $iB_t$ which are tax deductible. The tax-adjusted outflow to debtholders is therefore $(i(1-\tau_c)+m)B_t$. At the same time equityholders issue new debt at a rate $\delta_t$ to (partly) replace maturing debt. They receive the proceeds, i.e., the market value of the newly issued contracts, $\delta_t D(y,B)$, and have to bear proportional transactions costs $k_i$. The inflow from rolling over debt is therefore $\delta_t(1-k_i)D(y,B)$. Finally, equityholders receive the cash flow of the assets of the firm, $c = (r(1-\tau_p)-\hat{\mu})yB_t$.

Again using the homogeneity of the model with respect to the face value of debt we write $E = B_t\tilde{E}(y)$, where $\tilde{E}$ is the equity value per unit of face value of debt. While the individual debt contract amortizes at a constant rate $m$, the firm's total debt level changes at a rate $m-\delta_t$ depending on the firm's current re-issuing decision $\delta_t$. Consequently, the partial derivative of equity with respect to time is $\partial E/\partial t = -(m-\delta_t)B_t\tilde{E}(y)$. The value of equity thus satisfies

the following differential equation

$$\frac{1}{2}\sigma^2 y^2 \frac{\partial^2 \tilde{E}}{\partial y^2} + (\hat{\mu} + (m - \delta_t))y\frac{\partial \tilde{E}}{\partial y} - (i(1 - \tau_c) + m)$$
$$+ (1 - k_i)\delta_t \tilde{D}(y) + (r(1 - \tau_p) - \hat{\mu})y \; = \; (r(1 - \tau_p) + (m - \delta_t))\tilde{E}. \tag{7}$$

We are now able to derive the equilibrium roll-over rate for maturing debt, $\delta$. We hereby assume that the firm cannot ex-ante commit to a roll-over rate. Suppose that a firm announces a roll-over rate $\delta'$ and the market prices the bonds accordingly. As long as the partial derivative of equity value with respect to the roll-over rate is positive at $\delta'$, the equityholders have an incentive to re-enter the market and issue more debt. Rational investors must anticipate this incentive and price the new bonds, conjecturing a roll-over rate from which equityholders have no incentive to deviate, given the price of the bonds.

Since it follows from the two Hamilton-Jacoby-Bellman equations (6) and (7) that there is no explicit time dependence, the optimal debt roll-over rate depends only on the current leverage of the firm, i.e., $\delta_t = \delta(y)$. The optimal roll-over schedule $\delta^*(y)$ is therefore determined as a rational expectations equilibrium (i.e., a Markovian Nash-equilibrium) of the game between equityholders (setting the roll over rate $\delta$) and the market (valuing equity and debt).[10] To derive the equilibrium, the following corollary will be useful.

**Corollary 1.** *The partial derivative of equity with respect to the debt roll-over rate $\delta$ is given by*

$$\frac{\partial \tilde{E}}{\partial \delta} = \frac{K_1 - (r(1 - \tau_p) + m)K_2}{(r(1 - \tau_p) + (m - \delta))^2},$$

*where $K_1$ and $K_2$ are given by*

$$K_1 \;=\; \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 \tilde{E}}{\partial y^2} + (\hat{\mu} + m)y\frac{\partial \tilde{E}}{\partial y}$$
$$- (i(1 - \tau_c) + m) + (r(1 - \tau_p) - \hat{\mu})y,$$
$$K_2 \;=\; y\frac{\partial \tilde{E}}{\partial y} - (1 - k_i)\tilde{D}(y).$$

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[10]For a game theoretic analysis of a trading environment in which buyers or sellers cannot commit to a single trade, see DeMarzo and Bizer (1993). For a comprehensive discussion of differential games, see Dockner, Jørgensen, Van Long, and Sorger (2000)

*The partial derivative of debt with respect to the debt roll over rate δ is given by*

$$\frac{\partial \tilde{D}}{\partial \delta} = -\frac{y}{r(1-\tau_p)+m}\frac{\partial \tilde{D}}{\partial y}.$$

(See Appendix A.2 for the proof of Corollary 1.)

Corollary 1 implies that the sign of the partial derivative of equity with respect to the roll-over rate depends on the value of debt per unit of face value, $\tilde{D}(y)$. For sufficiently large values of debt it is positive whereas it is negative for sufficiently low values. The partial derivative is zero for a critical value $\tilde{D}^I$. These results imply the following proposition.

**Proposition 1.** *Equityholders are indifferent to changes in the debt roll over rate* $\delta(y)$ *if and only if the value of debt per unit of face value satisfies*

$$\begin{aligned} \tilde{D}(y) &= \frac{1}{1-k_i}\left(y\frac{\partial \tilde{E}}{\partial y}(y) - \tilde{E}(y)\right) \\ &=: \tilde{D}^I(y). \end{aligned}$$

*If and only if $\tilde{D}(y) > \tilde{D}^I(y)$, the firm optimally rolls over debt at $\delta^* = m$. If and only if $\tilde{D}(y) < \tilde{D}^I(y)$ the firm optimally finances debt repayments entirely with equity, i.e., $\delta^* = 0$.*

(See Appendix A.3 for the proof of Proposition 1.)

This result is quite intuitive. Suppose the firm issues one additional unit of debt $dB$ then it will receive the proceeds of this issue (net transactions costs). In addition to the proceeds there will be a change in equity value because the issue influences both B and *y*. Equityholders find it optimal to go ahead with this debt issue only if the sum of these effects is positive, i.e.,

$$\begin{aligned} 0 &< (1-k_i)\tilde{D}(y)dB + dE \\ &= (1-k_i)\tilde{D}(y)dB + \frac{\partial E}{\partial B}dB + \frac{\partial E}{\partial y}\frac{dy}{dB}dB \\ &= \left((1-k_i)\tilde{D}(y) + \tilde{E}(y) - y\frac{\partial \tilde{E}}{\partial y}(y)\right)dB, \end{aligned} \tag{8}$$

which is equivalent to the statement in Proposition 1.

On first inspection one may conclude from Proposition 1 that the optimal solution for δ is characterized by a 'bang-bang' solution, i.e., either full re-issuance of no re-issuance.
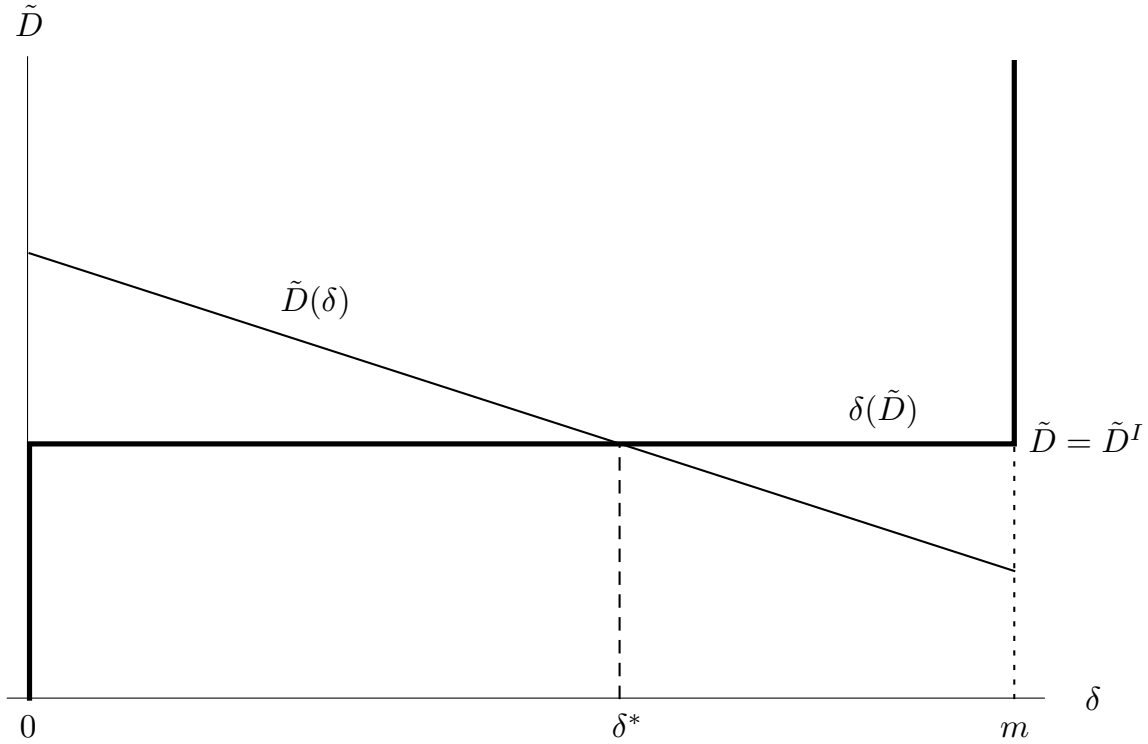
Figure 1: The shape of the response functions $\delta(\tilde{D})$ and $\tilde{D}(\delta)$ in the case of an interior equilibrium. The equilibrium debt roll over rate is $\delta^*$

This first intuition is, however, not correct since the value of debt per unit of face value, $\tilde{D}(y)$ reflects the roll-over rate $\delta^*$. In many situations it will not be optimal to fully roll-over maturing debt, since this would imply a $\tilde{D}(y)$ less than $\tilde{D}^I$. At the same time it will not be optimal to set the roll-over rate to zero, since this would imply a debt value larger than $\tilde{D}^I$, thus implying a positive partial derivative of equity value with respect to the roll-over rate. In all these cases there exists an interior equilibrium which implies that $\tilde{D} = \tilde{D}^I$.

This situation represents a differential game between equityholders, who determine the roll over rate $\delta^*$ and the market, which determines the value of debt and equity. For a given value of $\tilde{D}$, the best response of equityholders is characterized by Proposition 1. The best response of the capital market to a given roll-over rate $\delta$ is to price debt at the value given by Equation (6). Therefore, the response curve is a straight line with slope $\partial \tilde{D}/\partial \delta = -y \, \partial \tilde{D}/\partial y \frac{1}{r(1-\tau_p)+m}$. Figure 1 illustrates the typical shape of the response functions $\delta(\tilde{D})$ and $\tilde{D}(\delta)$ in the case of an interior equilibrium.

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This interior equilibrium with $0 < \delta^* < m$ is characterized by the following equilibrium conditions on $\tilde{E}$, $\tilde{D}$, and $\delta^*$.

**Proposition 2.** *In an interior equilibrium for $\delta$ the value of equity, debt, and the roll over rate must satisfy*

$$
\begin{aligned}
\tilde{E} &= \frac{K_1}{(r(1-\tau_p)+m)}, \\
\tilde{D}(y) &= \tilde{D}^I(y), \\
0 < \delta^* &= \frac{1}{y\dfrac{\partial \tilde{D}}{\partial y}} \left[ \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 \tilde{D}}{\partial y^2} + (\hat{\mu}+m)y\frac{\partial \tilde{D}}{\partial y} \right. \\
&\qquad \left. +(i(1-\tau_p)+m) - (r(1-\tau_p)+m)\tilde{D} \right] < m.
\end{aligned}
\tag{9}
$$

*Furthermore, the existence of an interior equilibrium requires*

$$
\begin{aligned}
\frac{\partial^2 \tilde{E}}{\partial y^2} &> 0, \\
\frac{\partial \tilde{D}}{\partial y} &> 0.
\end{aligned}
$$

(See Appendix A.4 for the proof of Proposition 2.)

The following Proposition gives the analytic solutions for debt and equity for all possible roll-over rates. For $\delta = m$ and for $\delta = 0$, analytic solutions are straightforward. However, a closed-form solution can also be obtained for the case of an interior equilibrium since the valuation equations for equity and debt in Proposition 2 do not explicitly depend on the equilibrium roll-over rate, $\delta^*$.

**Proposition 3.** *In a region where the firm fully rolls over its debt, i.e., $\delta = m$, the value of*

*equity and debt are given by*

$$\tilde{E}(y) = E_1 y^{\beta_{m1}} + E_2 y^{\beta_{m2}} - \frac{i(1-\tau_c)+m}{r(1-\tau_p)}$$

$$+ m(1-k_i)\left[\frac{1}{r(1-\tau_p)}\frac{i(1-\tau_p)+m}{(r(1-\tau_p)+m)}\right.$$

$$+ \frac{D_1 y^{\gamma_1}}{r(1-\tau_p)-\hat{\mu}\gamma_1 - \frac{1}{2}\sigma^2\gamma_1(\gamma_1-1)}$$

$$\left. + \frac{D_2 y^{\gamma_2}}{r(1-\tau_p)-\hat{\mu}\gamma_2 - \frac{1}{2}\sigma^2\gamma_2(\gamma_2-1)}\right] + y,$$

$$\tilde{D}(y) = D_1 y^{\gamma_1} + D_2 y^{\gamma_2} + \frac{i(1-\tau_p)+m}{r(1-\tau_p)+m}$$

*In a region where the firm rolls over its debt at an interior optimum* $\delta^*$*, the value of equity and debt are given by*

$$\tilde{E}(y) = E_1 y^{\beta_{01}} + E_2 y^{\beta_{02}} - \frac{i(1-\tau_c)+m}{r(1-\tau_p)+m} + y,$$

$$\tilde{D}(y) = \tilde{D}^I(y).$$

*In a region where the firm funds repayment of retiring debt entirely with equity, i.e., where* $\delta = 0$*, the value of equity and debt are given by*

$$\tilde{E}(y) = E_1 y^{\beta_{01}} + E_2 y^{\beta_{02}} - \frac{i(1-\tau_c)+m}{r(1-\tau_p)+m} + y,$$

$$\tilde{D}(y) = D_1 y^{\beta_{01}} + D_2 y^{\beta_{02}} + \frac{i(1-\tau_p)+m}{r(1-\tau_p)+m}.$$

*The exponents* $\beta$ *and* $\gamma$ *are the characteristic roots of the homogeneous differential equations given by*

$$\beta_{m1,m2} = \frac{1}{2} - \frac{\hat{\mu}}{\sigma^2} \pm \sqrt{(\frac{1}{2} - \frac{\hat{\mu}}{\sigma^2})^2 + \frac{2(r(1-\tau_p))}{\sigma^2}},$$

$$\beta_{01,02} = \frac{1}{2} - \frac{\hat{\mu}+m}{\sigma^2} \pm \sqrt{(\frac{1}{2} - \frac{\hat{\mu}+m}{\sigma^2})^2 + \frac{2(r(1-\tau_p)+m)}{\sigma^2}},$$

$$\gamma_{1,2} = \frac{1}{2} - \frac{\hat{\mu}}{\sigma^2} \pm \sqrt{(\frac{1}{2} - \frac{\hat{\mu}}{\sigma^2})^2 + \frac{2(r(1-\tau_p)+m)}{\sigma^2}}.$$

*The constants* $E_{1,2}$ *and* $D_{1,2}$ *have to be determined separately for each of the regions by*

*proper boundary conditions (see below).*

(See Appendix A.5 for the proof of Proposition 3.)

In equilibrium, the financing strategy of the firm and the corresponding valuation given by Proposition 3 are in accordance with the optimality conditions stated by Proposition 1.

In addition to choosing the optimal roll-over rate of maturing debt, equityholders have the possibilities to adjust the firm's capital structure. They may declare bankruptcy, thereby effectively creating an all-equity financed firm which can be relevered optimally. Alternatively they can repurchase all existing debt to eliminate the existing bond indenture, thereby creating the possibility to subsequently increase the total face value of debt. In the next chapter we analyze these discrete financial restructurings.

# 4   Capital Structure Strategy

While the decision regarding the financing mix used to repay expiring debt is done continuously, we allow equityholders to discretely reorganize the firm's capital structure through a buyback of all outstanding debt (possibly at a call premium) to implement an optimal target leverage ratio by issuing new debt. Firms will make use of this option only in good states of nature, when cash flows have grown high and tax shields are non-optimally utilized. In bad states of nature, firms will potentially issue equity to finance the repayment of expiring debt. They will, however, never actively repurchase debt over and above the contractually required amount, as shown by Admati, DeMarzo, Hellwig, and Pfleiderer (2015). Thus, if cash flows deteriorate fast and seriously, equityholders will find it optimal to default on their contracted obligations.

We restrict  our analysis to time-invariant barrier strategies in which initial total firm owners choose a starting capital structure $\acute{y}$ and the average maturity $m$ to maximize total firm value, fully anticipating equityholders strategy. Implementing a time-invariant barrier strategy, equityholders decide to default on their contractual obligation to repay retiring debt at par when the inverse leverage ratio $y$ first hits a lower boundary $\underline{y}$, which they can freely choose. They decide to repurchase its entire debt when $y$ first hits an upper boundary $\bar{y}$. After that, equityholders are owners of an unlevered firm. Due to homogeneity with respect to the

debt level, they will find it optimal to reestablish again the initial capital structure $\acute{y}$, $m$. The market, i.e., debtholders, price bonds in anticipation of total firm owners' / equityholders' strategy choice.

Furthermore, we allow firms' equityholders to reduce debt roll-over below the maximum i.e., $\delta < m$, within an internal interval as well as near the upper recapitalization threshold $\bar{y}$. Thus, equityholders may choose two time-invariant intervals, $[\tilde{y}_1, \tilde{y}_2]$ and $y \in [\tilde{y}_3, \bar{y}]$, where $\tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq \bar{y}$. These two ranges are based on the following intuition. First, for low cash flows, the price of newly issued debt reflects a high bankruptcy probability. Thus, it may be too costly for equityholders to roll over debt at the maximum allowable rate and lower rollover rates may be potentially optimal in a range $[\tilde{y}_1, \tilde{y}_2]$. Now consider a firm that is close to the upward recapitalization threshold. The equityholders of such a firm may also be reluctant to roll over debt at the maximum possible rate. This is so, because new debt issues close to the reorganization trigger are likely to be recalled only a short time thereafter, thereby causing transaction costs that are high compared to the tax shield these issues create over their short expected lifetime. Typically we therefore have $\partial \tilde{D}/\partial y < 0$ for $y$ close to debt repurchasing, in which case we do not have an internal optimum for $\delta$ but it is optimal to entirely stop reissuing new debt, i.e., $\delta = 0$, see Proposition 2.

Equityholders are allowed to choose $\tilde{y}_1 = \tilde{y}_2$ and / or $\tilde{y}_3 = \bar{y}$, in which case the model resembles a simple Leland (1994b) model in which equityholders always hold the face value of debt constant by fully rolling over debt.

To summarize, we determine a rational expectations equilibrium in time-invariant thresholds characterized by

$$(\underline{y} \leq \tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq \bar{y}, \delta(y) \; ; \acute{y}, m),$$

$$
\begin{aligned}
\text{with:} \quad & \underline{y} \;\; \leq \;\; \acute{y} \leq \bar{y}, \\
& \delta(y) \;\; = \;\; \delta^*(y) \quad \text{for } y \in [\tilde{y}_1, \tilde{y}_2], \\
& \delta(y) \;\; = \;\; 0 \qquad \text{for } y \in [\tilde{y}_3, \bar{y}], \\
& \delta(y) \;\; = \;\; m \qquad \text{elswhere.}
\end{aligned}
\tag{10}
$$

The critical restructuring thresholds $\underline{y}$, $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$, and $\bar{y}$ as well as the roll over schedule $\delta(y)$

are set to maximize equity value. The initial capital structure $\acute{y}$ and $m$ is set to maximize firm value. Maximization is based on the market value of debt, which, in equilibrium, is priced by debtholders in anticipation of the stated dynamic capital structure strategy.

## 4.1 Discrete Restructuring

When the firm decides to discretely (re-) structure its debt, the following boundary conditions apply. Debt is assumed to be sold at par so that the coupon rate $i$ is determined endogenously by

$$\text{choose } i \text{ such that} \qquad D(\acute{y}, B) = B. \tag{11}$$

In the case of default, equity is worthless. If equityholders repurchase the entire debt at $\bar{y}$, thereby paying a call premium of $\lambda$ times the face value, they receive an all-equity firm which they immediately relever to achieve the inverse leverage ratio $\acute{y}$.[11] This leads to the conditions

$$E(\underline{y}, B) = 0, \tag{12}$$
$$E(\bar{y}, B) = \left[ V(\acute{y}, B\frac{\bar{y}}{\acute{y}}) - k_r B\frac{\bar{y}}{\acute{y}} \right] - (1+\lambda)B. \tag{13}$$

Condition (11) is already used in (13).

As discussed above, debtholders take control over the productive assets of the firm after bankruptcy. They have to incur bankruptcy costs and transactions costs due to relevering the firm. When debt is called by equityholders, debtholders receive the face value plus a proportional call premium $\lambda$. This implies

$$D(\underline{y}, B) = \max \left\{ \left[ V(\acute{y}, B\underline{y}\frac{1}{\acute{y}}) - k_r B\underline{y}\frac{1}{\acute{y}} \right] - gB, \ 0 \right\}, \tag{14}$$
$$D(\bar{y}, B) = (1+\lambda)B. \tag{15}$$

The inverse leverage ratio $y$ is a diffusion that can freely move inside the interval $[\underline{y}, \bar{y}]$.

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[11]As introduced in Section 2 we differentiate between transactions costs $k_i$ for rolling over debt and transactions costs $k_r$ for placing a discrete portion of debt in the case of a recapitalization of the firm.

Thus, to ensure consistent expectation formation under the equivalent martingale measure, both equity and debt must be continuous and smooth in the entire interval $[\underline{y}, \overline{y}]$, independent of the segmentation into sub-regions induced by the choice of $\delta(y)$.

The first order conditions of optimality at the upper and the lower reorganization threshold follow from the 'smooth pasting' condition (see Dixit (1993) for a discussion of these optimality conditions)

$$\frac{\partial E}{\partial y}(\underline{y}, B) = 0, \tag{16}$$

$$\frac{\partial E}{\partial y}(\overline{y}, B) = \frac{1}{\acute{y}}\left[E(\acute{y}, B) + B(1 - k_r)\right]. \tag{17}$$

Recognizing that the optimal values of $\underline{y}$, $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$, $\overline{y}$, and $\delta(y)$ are functions of $\acute{y}$ and $m$, the initial firm value $V$ can be written as $V(y, B; \acute{y}, m)$. Taking into account transaction costs the maximand is

$$\max_{\acute{y}, m}(V(y, B; \acute{y}, m) - k_r B|_{y = \acute{y}}), \tag{18}$$

with the first order conditions

$$\frac{\partial V}{\partial m}(\acute{y}, B; \acute{y}, m) = 0, \tag{19}$$

$$\frac{\partial V}{\partial y}(\acute{y}, B; \acute{y}, m) + \frac{\partial V}{\partial \acute{y}}(\acute{y}, B; \acute{y}, m) - \frac{1}{\acute{y}}(V(\acute{y}, B; \acute{y}, m) - k_r B) = 0. \tag{20}$$

# 5  Debt Maturity, Capital Structure Dynamics and Firm Value

In this section we analyze how debt maturity affects equityholders' debt roll-over rates, firms' debt capacity and total firm value. Comparative static results with respect to firm characteristics like cash flow growth, cash flow volatility, bankruptcy cost will also be discussed. We will do this with the help of a model that is calibrated to the US tax code, the base-case parameters are listed in Table 2.

## 5.1 Calibration

For valuation of equity and debt we employ a simplified way in which we treat taxation on the personal level with a single tax rate $\tau_p$. This personal tax rate represents the general tax-disadvantage of income from interest bearing investments over income from holding equity capital, see Sections 2 for details on our assumption about the taxation on the corporate and the personal level. In the US, interest income is treated as ordinary income so we align our calibration to the maximum tax rate on wage income. This is currently at 39.6% for high income earners on top of which a 3.8% Medicare surtax on investment income applies, so we use a personal tax rate on interest income of 43.4%. High income earners pay 20% tax on dividend income plus 3.8% for Medicare which totals to 23.8%. As a total disadvantage of interest income over income from equity investment we use, hence, $\tau_p = 19.6\%$.

On corporate level we assume that income is taxed at a constant statutory rate $\tau_c$, which we calibrate to empirical effective marginal tax rates. For this purpose we use two sources of information. First, we use marginal tax rates from COMPUSTAT MTR database, which employs the nonparametric estimation method introduced by Blouin, Core, and Guay (2010) that explicitly takes care of mean reverting tendencies present in corporate income. We merge the MTR database to COMPUSTAT firm characteristics to calculate total-asset-weighted average marginal tax rates after interest expense over the available horizon from 1994 to 2012, which yields 30.6%. Average marginal tax rates peak in 1993 (33.0%) and are lowest in 2010 (22.0%).

Second, as a robustness check we analyze John Graham's file of simulated tax rates.[12] The average marginal tax rate after interest expense over the last 20 years, i.e., from 1994 to 2013, is estimated to 25.9%. Again, average simulated marginal tax rates in the sample period are lowest in 2010 (18.7%) and highest in 1995 (30.7%). The total-assets-weighted average marginal tax rate before interest expenses over the stated period is 33.1%.

We address the variations in marginal tax rates via comparative statics that we present in

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[12]We want to thank John Graham for providing us with his comprehensive set of simulated marginal tax rates covering the range from 1980 to 2013 and for his advice on calibrating our model to the US tax code. Please see Graham (1996a) and Graham (1996b) for details on the applied simulation procedure. Graham and Mills (2008) use federal government tax return data and show that simulated marginal tax rates provided in the file are close approximations.

comparison to the base case results.

Empirical papers on corporate bankruptcy costs in recent years considerably changed researcher's view of the magnitude of these costs. Early papers estimate bankruptcy costs investigating sets of defaulted firms and estimated these costs to be only a few percent of the firm's asset value.[13] More recently, researchers realized that a subset of defaulted firms is not necessarily representative of the entire population of firms. They argue that low-destress cost firms are overrepresented in this sample and, thus, existing estimates of bankruptcy costs might be massively downward biased. Reindl, Stoughton, and Zechner (2015) infer implied distress costs from market prices of equity and prices of put options employing a dynamic capital structure model. They show that estimated bankruptcy costs vary considerably across industries from below 10% to well over 60% with typical values in the range between 20% to 30%. In our calibration we refer to Glover (2014), who estimates parameters of a structural trade-off model of the firm with time-varying macroeconomic conditions by employing simulated methods of moments. He estimates the mean firm's costs of default with 45% and the median firm's cost with 37% of asset value. Our model specifies bankruptcy costs as a fraction $g$ of the face value of debt. Thus, aiming for a base-case parametrization that resembles median bankruptcy costs, we select $g$ sucht that a firm with optimally chosen debt maturity experiences bankruptcy costs of 37% of its asset value. This leads to a base-case parameter of $g = 34.39\%$. Variations in bankruptcy cost (e.g., over industries) are again treated via comparative statics.

## 5.2 Results

We start by analyzing the effect of average debt maturity on the firm's optimal refinancing decision. If not otherwise mentioned, base case parameters listed in Table 2 are used. We start exploring firms which have issued debt with long maturities. Panels (a) and (b) of Figure 2 illustrate the partial derivative $\frac{\partial \tilde{E}}{\partial \delta}(y)$ and the optimal roll over rate normalized by the retirement rate $\delta/m$ over the inverse leverage ratio $y$ (which in our model is proportional to the firm's cash flow level), both for $m = 0.0\dot{3}$ ($T = 30$ years). As can be seen, the partial

---

[13]See the following papers for studies on default costs, estimated averages are in parenthesis: Warner (1977) (5.3%), Ang, Chua, and McConnell (1982) (mean 7.5%, median 1.7%), Weiss (1990) (3.1%), Altman (1984) (6.0%), Andrade and Kaplan (1998) (10% to max. 23%).
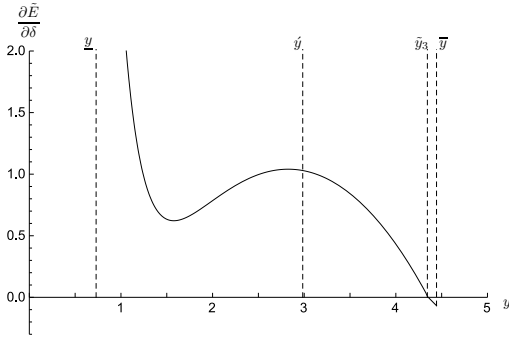
Table 2: Base case parameters

| parameter | | |
|---|---:|---|
| riskless rate of interest $r$ | 5 | % |
| personal tax rate $\tau_p$ | 19.6 | % |
| corporate tax rate $\tau_c$ | 30.6 | % |
| standard deviation $\sigma$ | 13 | % |
| risk adjusted drift $\hat{\mu}$ | 0 | % |
| bankruptcy cost $g$ | 34.39 | % |
| transactions costs for rolling over debt $k_i$ | 0.5 | % |
| transactions costs after recapitalization $k_r$ | 1 | % |
| call premium $\lambda$ | 0 | % |

derivative of equity with respect to $\delta$ is always positive, except for the small region near the upper restructuring threshold $\bar{y}$, where it is negative. Consequently, equityholders will not engage in voluntary debt reductions when the firm's cash flow decreases but instead the firm always fully rolls over all debt by setting $\delta = m$. Only immediately before calling the bonds to subsequently issue more debt, i.e. in the region $y \in [\tilde{y}_3, \bar{y}]$ does it become optimal for equityholders to use equity to repay maturing debt. The intuition for this latter result is straightforward. In this leverage region, it is optimal to use retained earnings to finance principal repayments since it would be inefficient to incur transactions costs for a new bond issue, knowing that the bond will be called in the near future with high probability.

Panels (c) and (d) of Figure 2 show the partial derivative $\frac{\partial \tilde{E}}{\partial \delta}(y)$ and the optimal roll over rate $\delta / m$ for the critical debt maturity of $T = 23.86$. years ($m = 0.04192$). This is the lowest average maturity for which there is no voluntary debt reduction, given the base case parameterization. We see that the partial derivative of equity with respect to the roll-over rate, $\frac{\partial \tilde{E}}{\partial \delta}(y)$, touches zero between $\underline{y}$ and $\acute{y}$. That is, there is one point between $\underline{y}$ and $\acute{y}$ at which equityholders are indifferent between rolling over all debt and refraining from issuing debt to replace retired debt. Thus, shortening the debt maturity from $T = 30$ to $T = 23.86$ considerably weakens equityholders' incentives to always fully roll-over maturing debt. We illustrate below that if the average maturity is less than the critical value of $T = 23.86$, there exists a region where equityholders choose an interior roll-over rate, $\delta^*$.
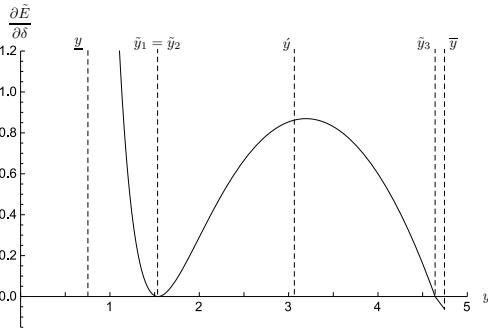
The example plotted in Figure 3 considers an even shorter debt maturity. Now $m$ is set
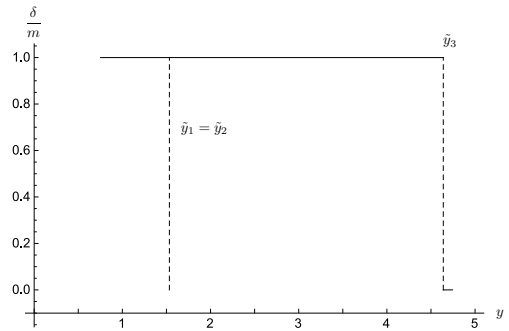
(a) $\frac{\partial \tilde{E}}{\partial \delta}(y)$ for $m = 0.0333$ ($T = 30$)
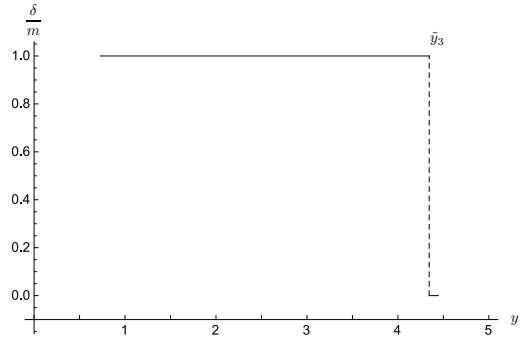


(b) $\frac{\delta(y)}{m}$ for $m = 0.0333$ ($T = 30$)



(c) $\frac{\partial \tilde{E}}{\partial \delta}(y)$ for $m = 0.04192$ ($T = 23.86$)



(d) $\frac{\delta(y)}{m}$ for $m = 0.04192$ ($T = 23.86$)

Figure 2: (a) and (b): The partial derivative of equity value $\tilde{E}$ with respect to the roll over $\delta$, together with the implied optimal roll-over rate $\frac{\delta}{m}$ for base-case parameters and debt maturity of $T = 30$ years, which is so long that equityholders will not engage in debt reductions in bad states of the firm.

(c): Debt with a critical maturity of $T^{\text{crit}} = 23.86$ years lowers $\frac{\partial \tilde{E}}{\partial \delta}$ in bad states of the firm such that it touches the critical value of zero at $\tilde{y}_1 = \tilde{y}_2$, at wich the firm is indifferent to the particular choice of $\delta$.

(d): Optimal choice of the roll-over rate $\delta/m$ for debt with critical maturity of $T^{\text{crit}} = 23.86$. At $\tilde{y}_1 = \tilde{y}_2$, the firm is indifferent with respect to the choice of $\delta \in [0, m]$.

to 0.2350 which corresponds to an average debt maturity of $T = 4.255$.[14] We find that there exists a region $[\tilde{y}_1, \tilde{y}_2]$ between the bankruptcy threshold $\underline{y}$ and the initial inverse leverage ratio $\acute{y}$ where equityholders find it optimal to reduce $\delta$ below $m$ to voluntarily reduce the debt level. Figure 4 shows the partial derivative of equity with respect to $\delta$. For $y \in [\tilde{y}_1, \tilde{y}_2]$, this derivative vanishes, thus, the choice how to fund debt repayments results in an interior equilibrium. Intuitively, equityholders find the prices of new bonds too low, since they would

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[14] As can be seen from below, these values correspond to the optimal maturity choice of a base case firm.
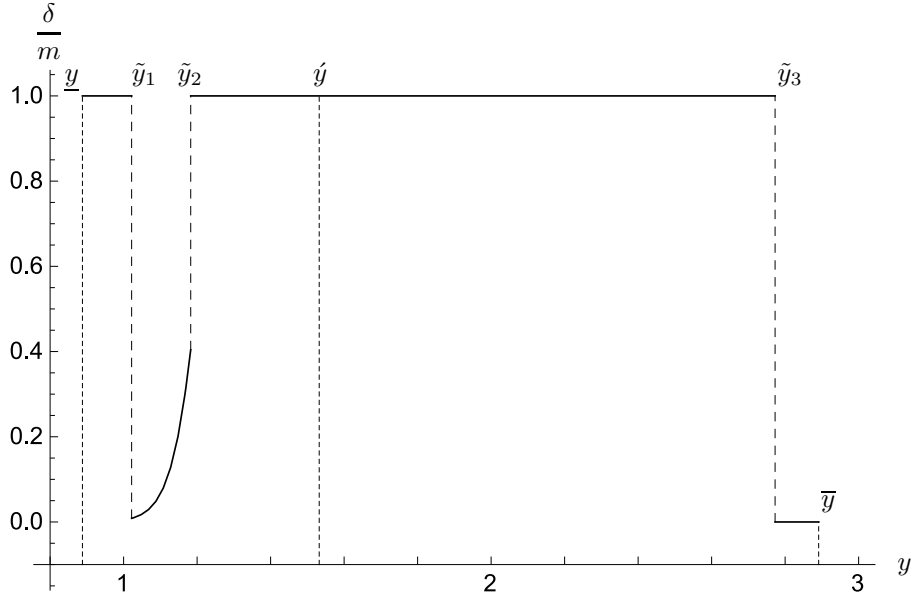
Figure 3: Endogenously determined roll over rate $\delta/m$ for base-case parameters and optimally-chosen debt maturity $m = 0.2350$ ($T = 4.255$). Optimally chosen, debt maturity is sufficiently short to create an incentive for debt reductions in the range $[\tilde{y}_1, \tilde{y}_2]$. There, only part of expiring debt is rolled over.

reflect excessively high leverage and future costs of financial distress. It is in their own interest to partly use equity to refund maturing debt, despite the fact that it implicitly also benefits the remaining bondholders. For all parameter values we have used, voluntary debt reduction was associated with an interior choice of $\delta$. I.e., we could not find a case where equityholders stopped issuing debt completely and funded debt repayment exclusively with equity. However, we do not have an analytic proof that this is a general result.

Interestingly, the firm's willingness to use equity to repay debt is non-monotonic in the inverse leverage ratio, $y$. When the firm approaches bankruptcy, i.e. for $y < \tilde{y}_1$ equityholders terminate their effort to reduce debt. In this region they once again fully roll over debt and exploit existing debtholders by re-issuing the expired debt. The intuition for this result is the following. When pushed very close to bankruptcy, equityholders are no longer willing to make additional voluntary equity investments in the firm. On the contrary, they would rather issue new debt at the maximum rate tolerated by debt covenants, even if this can only be done at unfavorable terms, i.e. at high credit spreads.
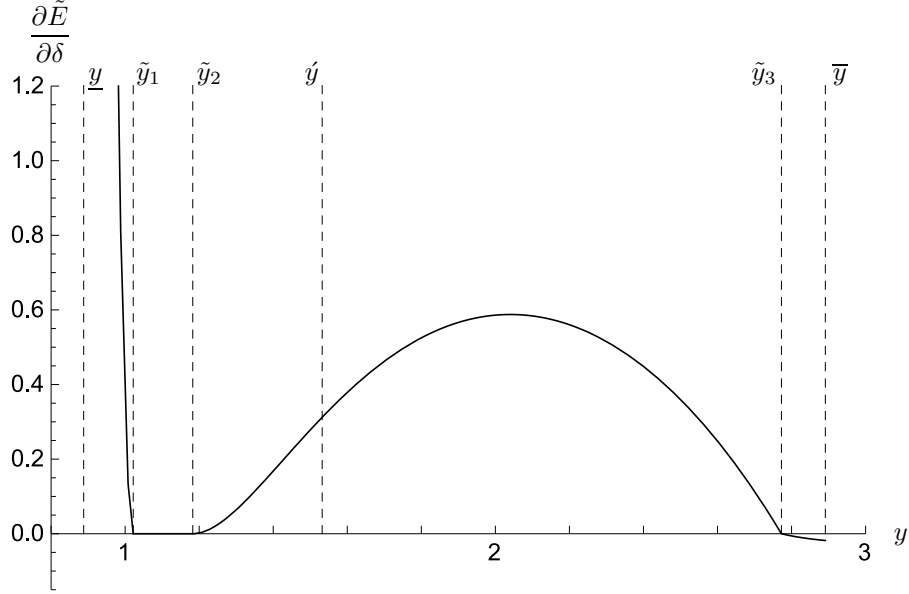
Figure 4: The partial derivative $\frac{\partial \tilde{E}}{\partial \delta}(y)$ for base-case parameters and optimally-chosen debt maturity $m = 0.2350$ ($T = 4.255$). In $[\tilde{y}_1, \tilde{y}_2]$ the derivative vanishes, i.e., the firm is indifferent towards the choice of $\delta$. This is the requirement for an interior optimum.

To summarize, there are four main insights that the above numerical analysis provides. First, for sufficiently long maturities, we find that equityholders never use retained earnings or equity issues to repay maturing debt except immediately before a discrete leverage increase. This result changes if the average debt maturity is shortened sufficiently. In this case there exists a range of leverage ratios strictly above the initial optimum for which equityholders find it optimal to partly use retained earnings of equity to repay maturing debt. This is in accordance with the empirical findings of Hovakimian, Opler, and Titman (2001) who report that long-term debt is an impediment to movements toward the target leverage ratio.

Second, at the initial leverage ratio $\acute{y}$ the firm always holds its debt level constant and fully rolls over maturing debt, $\delta = m$. This follows directly from the optimality of the initial leverage ratio. Since the initial issue of debt is associated with proportional transactions costs $k_r$, equityholders would not incur these costs if they would immediately find it optimal to reduce debt by repaying debt with equity.

Third, near the restructuring threshold $\overline{y}$ the firm entirely refrains from issuing debt. This is so because approaching $\overline{y}$ is associated with the repurchase of all debt in order to reestablish

the optimal initial capital structure. Therefore, near this threshold, equityholders do not find it optimal to incur costs $k_i$ for rolling over contracts which will (with high probability) be repurchased only after a short period. With $k_i \to 0$ this region of $\delta = 0$ vanishes.

Fourth, if loss-given-default is less than 100%, near the bankruptcy threshold $\underline{y}$ the firm fully rolls over all expiring debt, $\delta = m$. Thus, even with short-term debt outstanding, equityholders resume issuing debt if the leverage ratio is sufficiently high. In this case the equityholders are not longer willing to invest in debt reductions to keep their equity option alive. This latter result can be derived analytically.

**Proposition 4.** *If loss-given-default is less than 100%, it is optimal to roll over debt at the maximum rate $\delta = m$ in a neighborhood above the bankruptcy threshold $\underline{y}$.*

(See Appendix A.6 for the proof of Proposition 4.)

**Bankruptcy costs, corporate taxes and critical debt maturity:** We find that bankruptcy costs as well as the magnitude of the tax shield of debt financing represent the main determinants for the critical average maturity below which equityholders find it optimal to engage in voluntary debt reductions. The lower the bankruptcy costs the shorter the maturity required to give sufficient incentives for voluntary debt reductions. Figure 5 plots the critical average maturity over bankruptcy costs for two different levels of corporate tax, $\tau_c$. Lower line represents our base case where corporate tax rates are calibrated to average marginal tax rates provided by COMPUSTAT MTR database ($\tau_c = 30.6\%$, see Section 5.1).

The upper line shows critical debt maturities over bankruptcy costs when using lower average effective corporate tax rates implied by the marginal-tax rate data provided by John Graham ($\tau_c = 25.9\%$,). It is evident, that in case of higher tax shields it requires shorter debt to induce sufficient incentives for equityholders to engage in active debt reduction when the firm's conditions deteriorate. This result is quite intuitive, since actively replacing retired debt with equity reduces the firms tax shields and, hence, providing larger tax shields reduces the incentive to substitute equity for debt.

With base-case parametrization, i.e., $\tau_c = 30.6\%$, $g = 34.39\%$, debt maturity below the critical maturity of 23.86 years induces debt reduction in bad times. With lower tax shields, using $\tau_c = 25.9\%$ estimated from marginal tax rates provided by John Graham, the critical
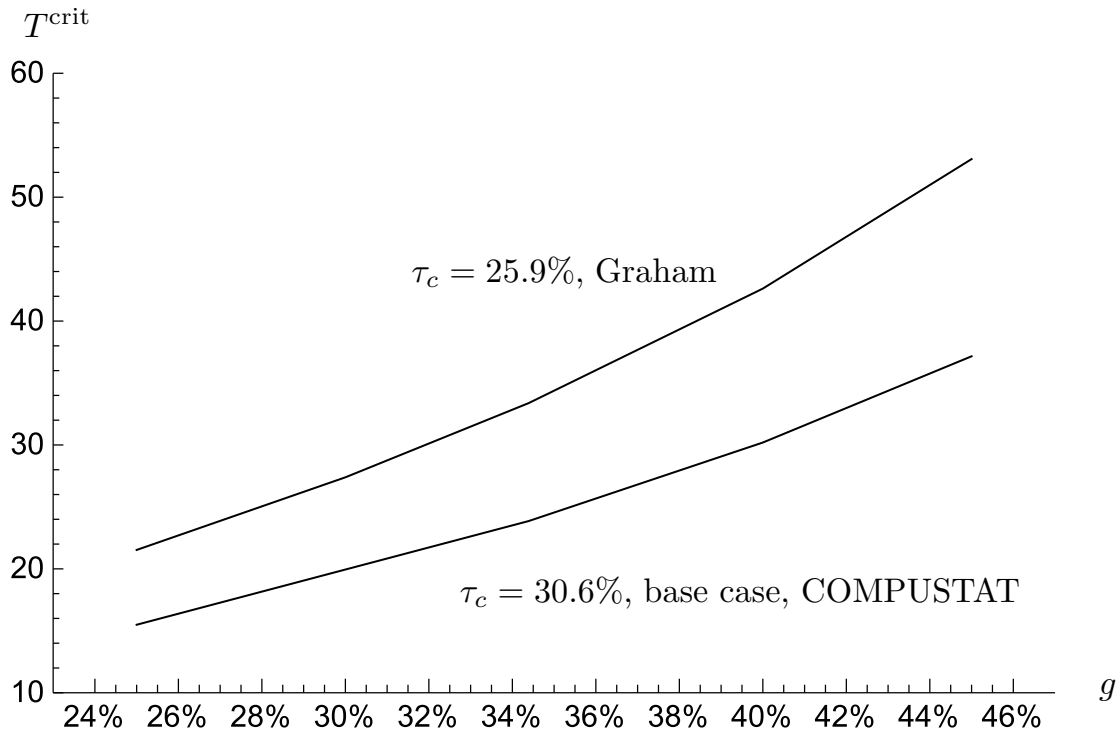
Figure 5: Critical average debt maturity below which the commitment to debt reductions in bad times is credible as a function of bankruptcy costs. Critical maturities are plotted for the base case parametrization $\tau_c = 30.6\%$, which is the average marginal tax rate estimated from COMPUSTAT MTR database, which employs the approach of Blouin, Core, and Guay (2010). Additionally, critical maturity for the average tax rate from John Graham's database, i.e., $\tau_c = 25.9\%$, is also plotted. See Section 5.1 for more details.

debt maturity at $g = 34.39\%$ is 33.4 years. Bankruptcy costs as low as $g = 25\%$ require average maturities of 15.49 years and 21.53 years when using corporate tax rates of 30.6% and 25.8% respectively. Bankruptcy costs as high as $g = 45\%$ induce debt reduction for average maturities below 37.16 years and 53.07 years respectively. Thus, with lower bankruptcy costs, it needs shorter-term debt to induce debt reductions.

**Debt maturity and firm value:** We next consider the effect of debt maturity on firm value and the illustrate potential benefit of a short debt maturity with base-case parameters.[15] Results for different parametrizations reported below. Figure 6 displays the initial total-firm value divided by the value of the unlevered assets as a function of the debt maturity, $m$.

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[15] A novel rationale for the existence and significance of bankruptcy costs is provided in Berk, Stanton, and Zechner (2010).
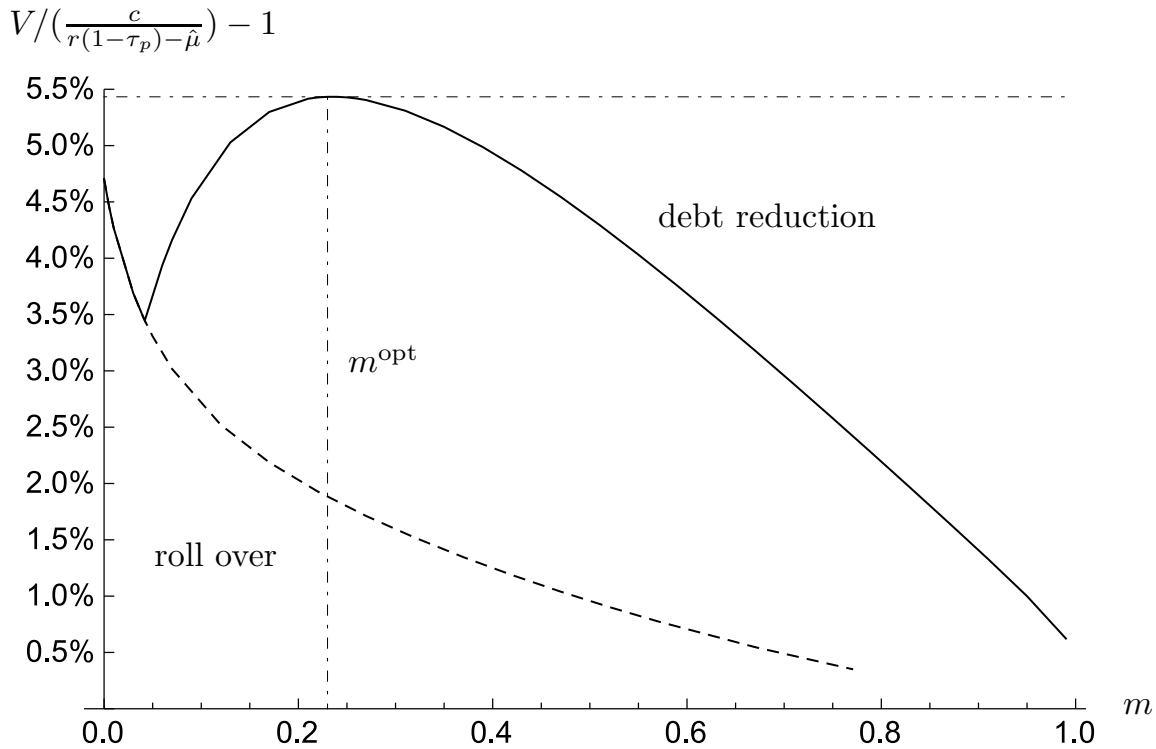
Figure 6: The optimal initial total-firm value divided by the value of the unlevered assets plotted against the retirement rate *m* for high bankruptcy costs. The dotted line shows the corresponding firm value for a firm that has to keep the debt level constant and therefore rolls over all expiring debt. The relation between the maturity structure of debt and firm value is non-monotonous. Firms with high bankruptcy costs prefer short-term debt.

The figure also displays the relative value of a reference firm (dotted line) which is assumed to always fully roll-over maturing debt with new debt issues.[16] For the reference firm, total firm value is maximized by choosing the longest possible maturity for its debt, as reported in Leland (1994b) and Leland and Toft (1996). By contrast, if the firm can engage in debt reductions, the relationship between total firm value and the maturity structure of debt is not longer monotonic.[17] This is so because debt with sufficiently short maturity induces more efficient capital structure adjustments by equityholders when the firm's cash flows decrease, thereby lowering probability of default and, hence, expected bankruptcy costs.

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[16]This is modelled as in Leland (1994b). In addition, we also allow the firm to increase its debt by repurchasing all debt outstanding and to issue a higher amount of debt.

[17]Empirical evidence for this nonmonotonicity is provided by Guedes and Opler (1996) who report that investment grade firms seem to be indifferent between issuing debt at the long end of the maturity spectrum and issuing debt at the short end of the spectrum.

As illustrated in Figure 6, the beneficial effect of shorter debt maturity on future capital structure dynamics outweights the disadvantages due to higher transactions costs from rolling over maturing debt. In the base case, overall firm value is maximized at a debt maturity of $\approx 4.26$ years.

**Debt capacity:** The commitment effect of debt maturity also has a significant effect on the optimal initial leverage ratio. In contrast to existing results in the finance literature we find that shorter debt maturities lead to higher debt capacities.

This effect is illustrated by Figure 7, which plots the initial optimal leverage as a function of $m$ for the base case firm. Unlike firms which must roll over all maturing debt, firms that choose the roll-over rate optimally actually *increase* their debt capacity as they shorten their debt maturities. The optimal initial leverage increases from approximately 39% percent for perpetual bonds and reaches its maximum with 80% at an average debt maturity of approximately 1.5 years. At the firm-maximizing debt maturity of 4.26 years, the firm's debt capacity is approx. 65%. For very short maturities, debt capacity decreases, since transactions costs for rolling over debt with very short maturity make this sort of debt unfavorable.

## 5.3 Comparative Statics

In this section we explore the effect of various model parameters on firm value, optimal debt maturity and dynamic capital structure policy. **Bankruptcy costs:** We first focus on the role of bankruptcy costs. The key role of bankruptcy costs for the commitment to debt reductions was already discussed above. Figure 8 plots the tax advantage of debt, i.e., the extend to which the initial firm value exceeds the unlevered firm value, for different levels of bankruptcy costs. Several effects can be seen: (i) lower bankruptcy costs require a shorter debt maturity in order to induce voluntary debt reductions, (ii) lower bankruptcy reduce the maximum attainable tax advantage of debt, (iii) lowering bankruptcy costs moves the optimal finite maturity towards shorter maturities, (iv) for very low bankruptcy costs it becomes relatively more advantageous to issue console bonds.

The most surprising effect is that higher bankruptcy costs imply higher firm values. Higher bankruptcy costs make it easier for equityholders to credibly commit to debt reductions. The resulting decrease in the expected probability of bankruptcy more than offsets the
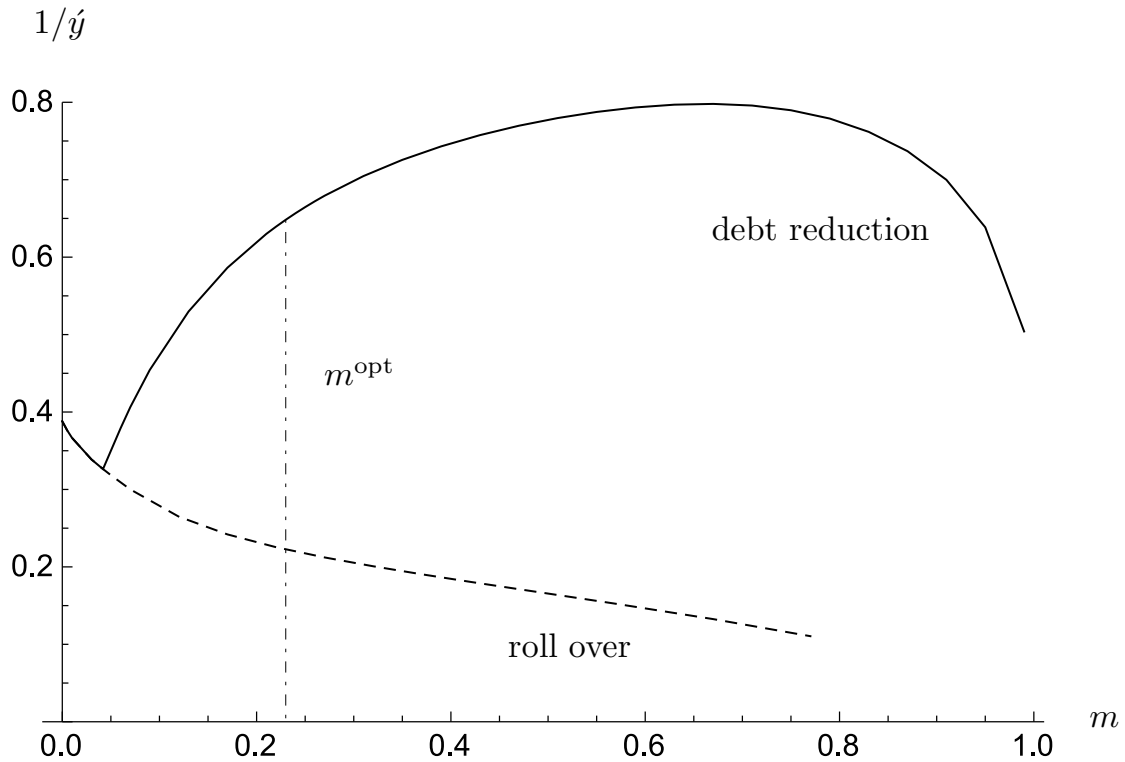
Figure 7: Optimal initial leverage ratios $1/\acute{y}$ plotted over the retirement rate $m$. Without allowing for downward restructuring, debt capacity decreases when moving from long to short-term debt. For firms that explicitly consider debt reduction, debt capacity increases once maturity is sufficiently short in order to commit to debt reductions to avoid financial distress. Only at the very short end, transaction costs lead to a deterioration in debt capacity.

effect of the increased costs given a default.

The costs associated with rolling over debt are another key determinant of firm value when debt with finite maturity is issued. Figure 9 illustrates the effect on firm value for different values of $k_i$. When moving to lower values of $k_i$ we observe that (i) firms with shorter-term debt gain relatively more and (ii) the local maximum of total firm value for finite debt maturity moves towards shorter maturities.

Figure 10 shows how changes in cash flow characteristics affect total firm value. Panel 10a plots total firm value as a function of debt maturity for several values of cash flow volatility $\sigma$. Moving to higher volatility (i) results in lower firm value, (ii) requires shorter debt maturity to induce debt reductions, (iii) moves the local maximum of total firm value towards shorter maturity debt. High cash flow volatilities reduce the firm's debt capacity but
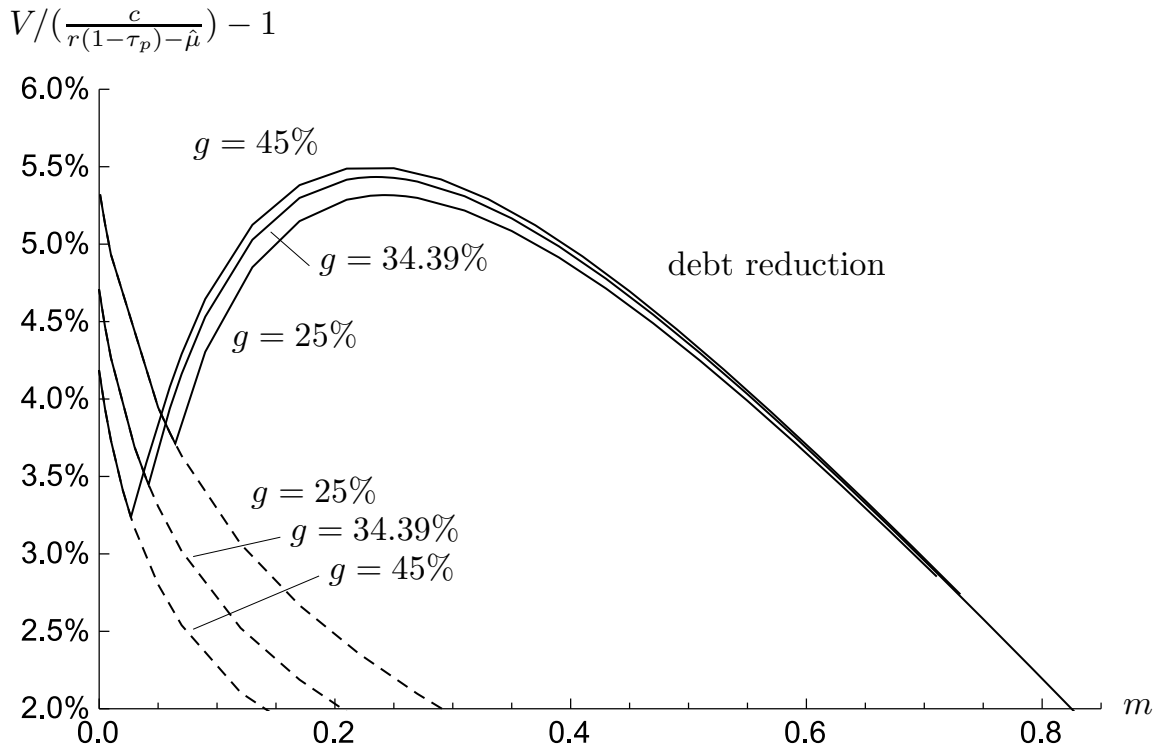
Figure 8: Total firm value (as a multiple of the unlevered firm value) plotted over the retirement rate *m* for different levels of bankruptcy costs.

increases the option value for equityholders and thus make them more reluctant to default on their debt obligations. This makes the commitment effect of short-term debt relatively less advantageous and requires short debt maturities to induce voluntary debt reductions.

Panel 10b plots total firm value as a function of the debt maturity for several values of the risk-adjusted cash flow growth rate $\hat{\mu}$. Moving to higher growth simply causes an approximately parallel shift of the firm value towards higher levels.

**Bankruptcy costs and debt capacity:** Next, Figure 11 plots the firm's optimal initial leverage ratio, which we refer to as the firm's debt capacity, for different debt maturities and for different levels of bankruptcy costs. It is again counterintuitive that high bankruptcy costs are associated with a higher debt capacity. As discussed above, the higher bankruptcy costs effectively ensure that equityholders can commit to aggressive debt reductions when cash flows decrease. This results in a reduced bankruptcy probability which more then offsets the higher bankruptcy costs conditional on default.
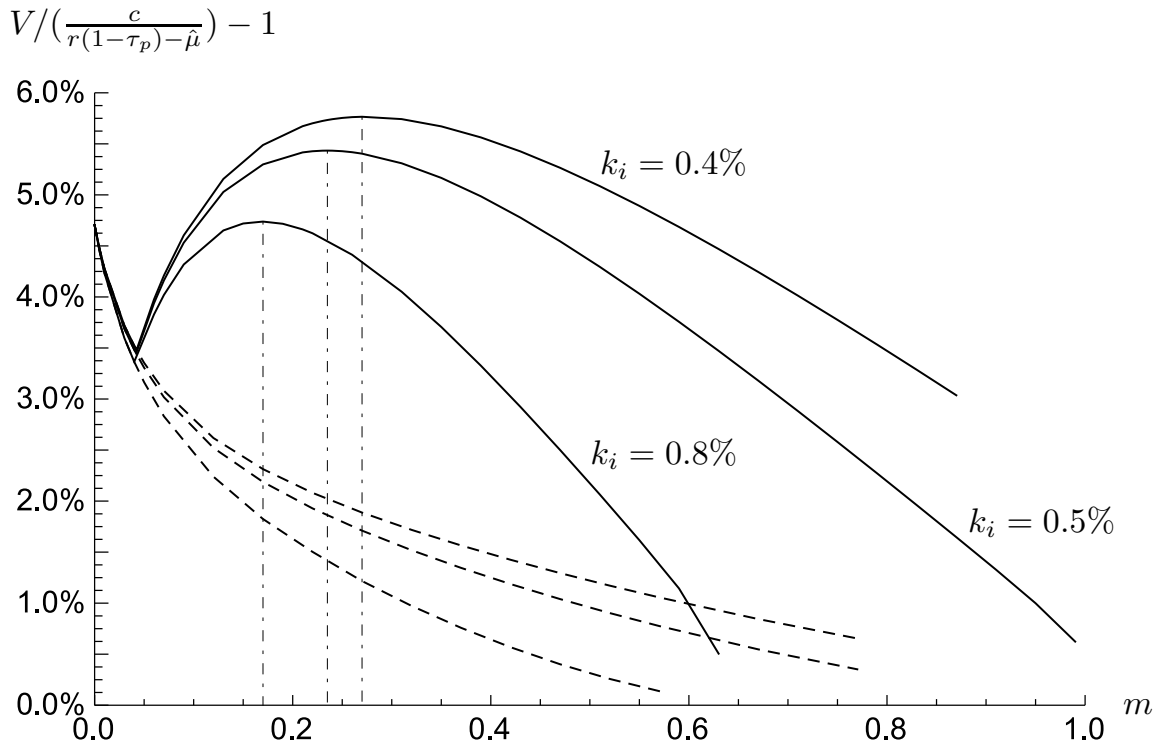
Figure 9: Total firm value (as a multiple of the unlevered firm value) plotted over the retirement rate $m$ for different costs $k_i$ associated with rolling over.
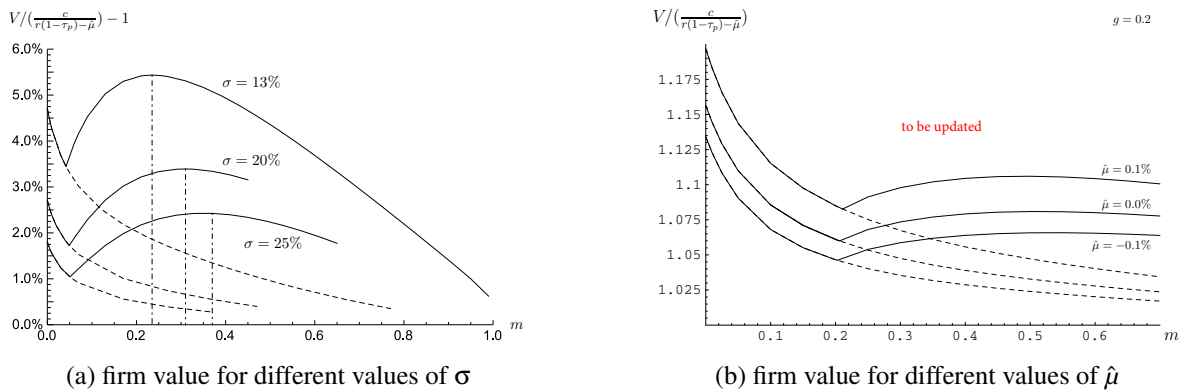


(a) firm value for different values of $\sigma$



(b) firm value for different values of $\hat{\mu}$

Figure 10: Total firm value (as a multiple of the unlevered firm value) plotted over the retirement rate $m$ for different values of $\sigma$ and $\hat{\mu}$.

**Firm value and corporate tax rates:** Finally, Figure 12 shows the tax advantage of debt over the retirement rate $m$ for the base case firm with $\tau_c = 30.6\%$ from simulated data provided by COMPUSTAT MTR database, following the approach of Blouin, Core, and Guay
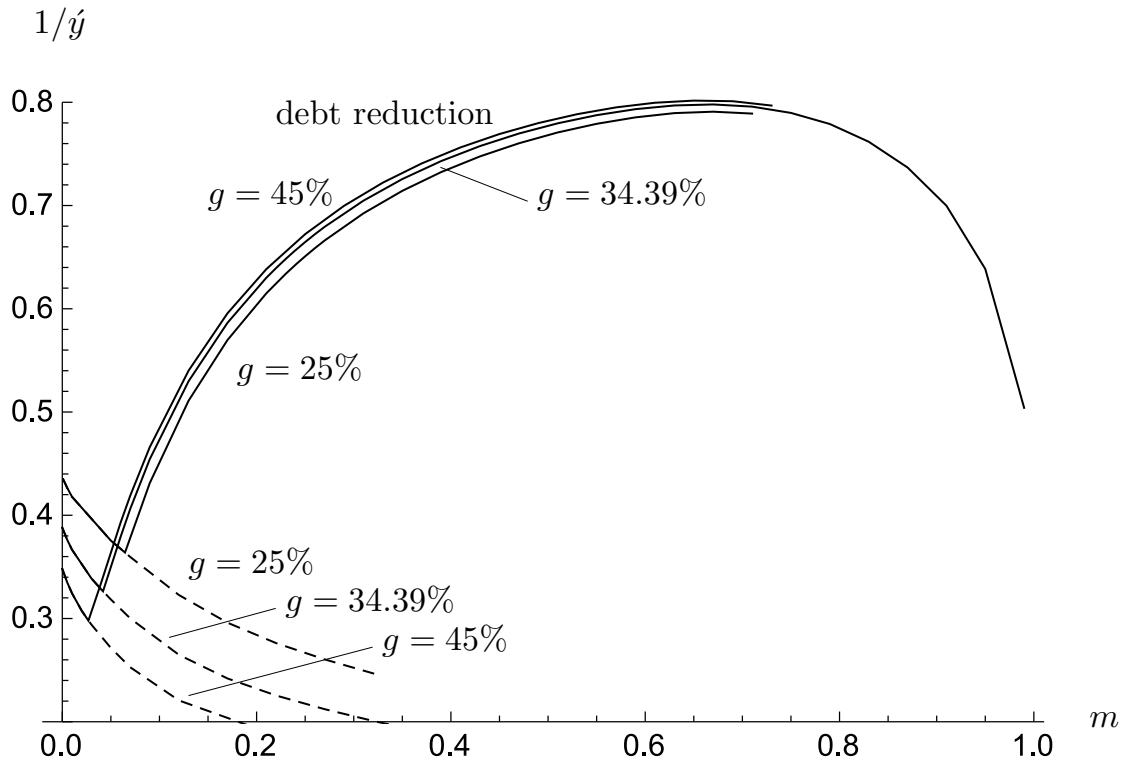
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Figure 11: Optimal initial leverage ratios $1/\acute{y}$ plotted over the retirement rate $m$. High bankruptcy costs lead to high debt capacity if using short-term debt.

(2010). As a comparison we plot the tax advantage when corporate taxes are estimated from John Graham's marginal tax rate data, $\tau_c = 25.9\%$. Higher tax shields caused by higher corporate tax rates lead to (i) lower optimal debt maturity and (ii) a higher tax advantage at the optimal debt maturity. While it is intuitive that higher corporate tax rates lead to higher total firm value if debt is used optimally, the fact that higher tax rates reduce optimal maturity is less clear. As a direct consequence, higher tax rates make debt reduction less desirable, because reducing debt diminishes the associated tax shield. A secondary effect is that debt capacity increases with shorter debt, and higher debt capacity ex ante allows the firm to use debt more aggressively, which increases the debt tax shield. From Figure 12 we see that the latter effect dominates the direct effect and, over all, higher corporate tax rates reduce optimal debt maturity, from an optimal average maturity of 6.25 years for $\tau_c = 25.9\%$ to 4.26 years in the base case with an effective corporate tax rate of $\tau_c = 30.6\%$.
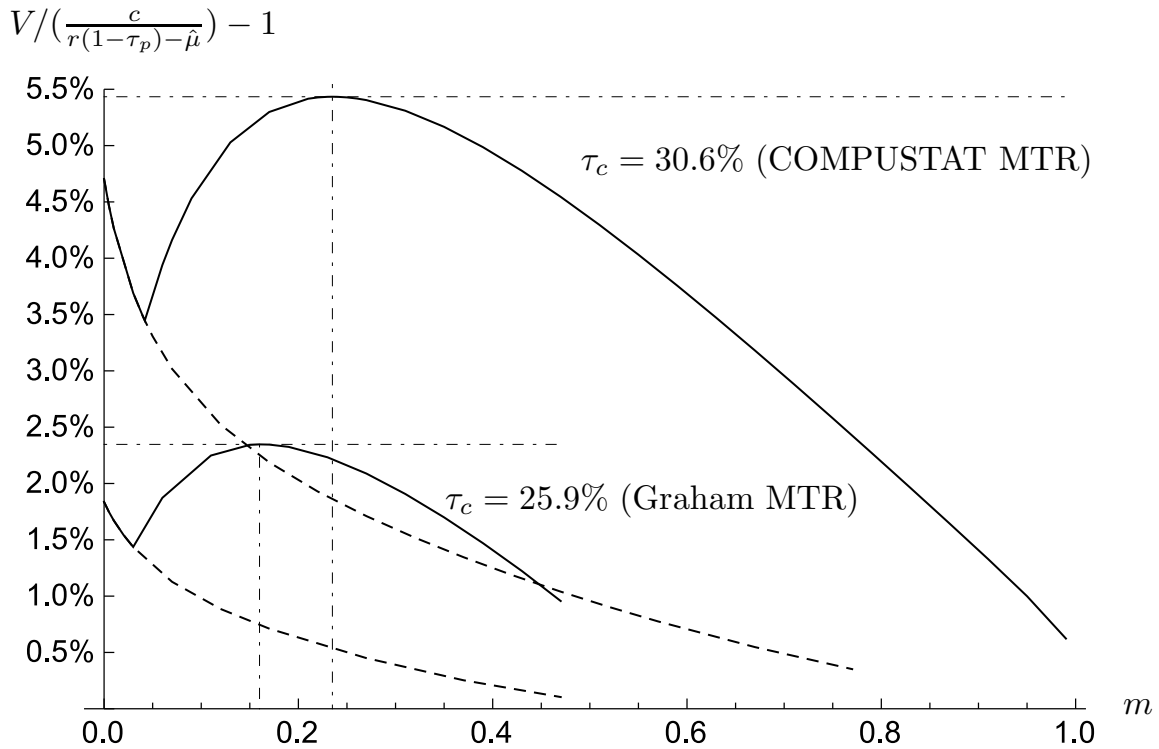
Figure 12: Tax advantage of debt for different levels auf corporate tax, $\tau_c$, plotted over the retirement rate $m$. High corporate tax rates lead to higher tax advantages and lower optimal maturity.

# 6 Conclusions

This paper explores the effects of debt maturity on subsequent dynamic capital structure adjustment. We demonstrate that long debt maturities destroy equityholders' incentives to engage in future voluntary debt reductions. By contrast, short debt maturities serve as a commitment to lower leverage in times when the firm's profitability decreases. This positive effect of short debt maturities must be balanced against the increased transactions costs associated with the higher frequency of rolling over maturing bonds. The resulting tradeoff generates a new theory of optimal debt maturity.

We find that the equityholders' incentives to engage in debt reductions is non-monotonic in the firm's leverage. After intermediate deteriorations in the firm's profitability, equityholders find it in their own best interest to repay maturing debt at least partly with equity.

However, if the firm's profitability drops so far that it is pushed close to bankruptcy, then equityholders resume issuing new debt and gamble for resurrection.

Ex ante, the debt capacity of the firm increases if it uses debt with sufficiently short maturity. We find that high costs of bankruptcy induce a stronger incentive to use short-term debt since this reduces the expected probability of bankruptcy for given debt level. Higher tax shields caused by a higher corporate tax rate also makes shorter-term debt more advantageous, since increased debt capacity associated with using short-term debt allows for a better utilization of debt tax shields. Additional comparative statics analyses reveal that increased cash flow risk reduces the optimal debt maturity, whereas the growth rate of the cash flow process and the transactions costs of rolling over debt have the opposite effect.

All our main results are in accordance with findings of existing empirical studies which confirm that firms readjust their capital structure if they are highly levered and that firms with a high portion of long-term debt are more reluctant to reduce debt in financial distress compared to firms with a high portion of short-term debt. Other empirical predictions of our theory, such as the effects of growth and firm risk on firms' leverage adjustments in financial distress remain to be tested.

# A   Appendix

## A.1   Derivation of Equation 4

The inverse leverage ratio with respect to the unlevered firm value, $y_t$, depends on two state variables, the cash flow of the firm's productive assets, $c_t$, and the current face value of debt, $B_t$. Thus one can write $y_t = y(c_t, B_t)$. If the debt level is adjusted by repurchasing all existing debt with face value $B_t$ and issuing new debt with face value $B_t^*$, the leverage ratio immediately jumps to the new value, i.e., in this case we have

$$dy_t = \left(\frac{1}{B_t^*} - \frac{1}{B_t}\right)\frac{c_t}{r(1-\tau_p)-\hat{\mu}}, \tag{21}$$

and therefore

$$\frac{dy_t}{y_t} = \frac{B_t}{B_t^*} - 1. \tag{22}$$

In the absence of a discrete adjustment, the inverse leverage ratio, $y_t$, follows a diffusion and its dynamics can be determined using a Taylor-series expansion and Itô's Lemma

$$dy_t = \frac{\partial y}{\partial c} dc + \frac{\partial y}{\partial B} dB + \frac{1}{2} \left( \frac{\partial^2 y}{\partial c^2} (dc)^2 + \frac{\partial^2 y}{\partial B^2} (dB)^2 \right) + \frac{\partial^2 y}{\partial c \, \partial B} dc \, dB. \tag{23}$$

Neglecting all terms that are $o(dt)$ gives

$$
\begin{aligned}
dy_t &= \frac{1}{B_t} \frac{1}{r(1-\tau_p)-\hat{\mu}} c_t \left( \mu \, dt + \sigma \, dW_t \right) \\
&\quad - \frac{1}{B_t^2} \frac{c_t}{r(1-\tau_p)-\hat{\mu}} \left( -(m-\delta)B_t \, dt \right) \\
&= y_t \left( (\hat{\mu} + (m-\delta)) \, dt + \sigma \, dW_t \right)
\end{aligned}
\tag{24}
$$

for $0 \le \delta \le m$.

## A.2  Proof of Corollary 1

From Equation (7) it follows that $\tilde{E}$ can be written as

$$\tilde{E} = \frac{K_1 - \delta K_2}{r(1-\tau_p) + (m-\delta)}, \tag{25}$$

hence, the partial derivative of $\tilde{E}$ with respect to $\delta$ is given by the expression in Corollary 1.

The partial derivative of $\tilde{D}$ with respect to $\delta$ can be directly determined from Equation (6) to be equal to $-y\frac{\partial \tilde{D}}{\partial y}\frac{1}{r(1-\tau_p)+m}$.

## A.3  Proof of Proposition 1

Consider the expression for $\frac{\partial \tilde{E}}{\partial \delta}$ from Corollary 1. Since $\delta \le m$ it follows that the denominator in this expression is always strictly positive, the sign of the partial derivative equals the sign

of $K_1 - (r(1-\tau_p)+m)K_2$. From (7) it follows that

$$K_1 = (r(1-\tau_p)+m-\delta)\tilde{E} + \delta K_2, \tag{26}$$

so we have

$$K_1 - (r(1-\tau_p)+m)K_2 = (r(1-\tau_p)+m-\delta)(\tilde{E}-K_2). \tag{27}$$

Since $(r(1-\tau_p)+m-\delta) > 0$ the sign of the partial derivative $\frac{\partial \tilde{E}}{\partial \delta}$ equals the sign of

$$\tilde{E}-K_2 = \tilde{E} - (y\frac{\partial \tilde{E}}{\partial y} - (1-k_i)\tilde{D}). \tag{28}$$

Consequently, the firm is indifferent with respect to $\delta$ if and only if $\tilde{D}$ satisfies

$$\tilde{D}(y) = \frac{1}{1-k_i}\left(y\frac{\partial \tilde{E}}{\partial y}(y) - \tilde{E}(y)\right). \tag{29}$$

$\frac{\partial \tilde{E}}{\partial \delta} > 0$ if and only if the value of $\tilde{D}$ exceeds the value of the right-hand-side expression and it is optimal to choose $\delta = m$. $\frac{\partial \tilde{E}}{\partial \delta} < 0$ if and only if the value of $\tilde{D}$ is lower than the value of the right-hand-side expression and it is optimal to choose $\delta = 0$.

## A.4   Proof of Proposition 2

In a region of internal optimum for the roll-over rate $0 < \delta^* < m$ we require $\tilde{D} = 1/(1-k_i)[y\frac{\partial \tilde{E}}{\partial y} - \tilde{E}]$. From Proposition 1 we know that under this condition we have $\frac{\partial \tilde{E}}{\partial \delta} = 0$, thus, the value of equity determined by valuation equation (7) is independent of the particular choice of $\delta$. For simplicity, we substitute $\delta = 0$ into (7) to receive the expression for $\tilde{E}$ stated in Proposition 2.

The equilibrium roll-over rate $\delta^*$ is then determined by solving Equation (6) for $\delta$. Since equityholders are indifferent with respect to the choice of $\delta$ the particular choice $\delta^*$ does not change the valuation of equity.

The local response function of the value of debt to a re-issuance rate $\delta$ has the slope $\partial \tilde{D}/\partial \delta = -y \, \partial \tilde{D}/\partial y \frac{1}{r(1-\tau_p)+m}$, (see Equation (6)). From Figure 1 we can conclude that

the equilibrium is stable only if $\tilde{D}(\delta)$ is downward sloping, i.e., $\partial\tilde{D}/\partial\delta < 0$ which requires $\partial\tilde{D}/\partial y > 0$. The latter condition simply requires that the value of debt per unit of face value increases as the inverse leverage ratio increases, i.e. leverage decreases.

In the case of an internal equilibrium we have $\tilde{D}(y) = \tilde{D}^I(y)$, hence $\frac{(1-k_i)}{y}(\partial\tilde{D}/\partial y) = (\partial^2\tilde{E}/\partial y^2) > 0$. which concludes the proof.

## A.5   Proof of Proposition 3

In regions where $\delta = m$ or $\delta = 0$, the value function for $\tilde{D}$ and $\tilde{E}$ are the general solutions of the second-order ordinary differential equations (6) and (7) which can be proved by substituting the solution into the equation. In a region of an interior equilibrium $0 < \delta^* < m$ we know from Proposition 2 that $\tilde{D} = \tilde{D}^I$. The value of equity must be the solution of the Hamilton-Jacobi-Bellman equation (7) with $\delta^*$ from Proposition 2 substituted for $\delta$. However, since we know that in an internal equilibrium the value of equity is invariant with respect to the choice of $\delta$ we solve (7) for $\delta = 0$ and argue that this solution must hold for every $0 \le \delta \le m$, and in particular for $\delta = \delta^*$. Substitution of this solution together with the equilibrium conditions of Proposition 2 into (7) constitutes an alternative proof.

## A.6   Proof of Proposition 4

Suppose loss-given-default is less than 100%. This is the case, if $g < 1$ and optimal bankruptcy occurs at a level $\underline{y}$ such that the value of the remaining assets exceeds bankruptcy costs. Then it follows from boundary condition (14) that $\tilde{D}(\underline{y}) > 0$. However, for the value of equity and its partial derivative with respect to the inverse leverage ratio it follows from boundary condition (12) and optimality condition (16) that

$$\lim_{y \to \underline{y}} \tilde{E}(y) = \lim_{y \to \underline{y}} \frac{\partial\tilde{E}(y)}{\partial y} = 0.$$

Therefore, in a neighborhood of $\underline{y}$ it is true that

$$\tilde{D}(y) > \frac{1}{1-k_i}\left(y\frac{\partial\tilde{E}}{\partial y}(y) - \tilde{E}(y)\right).$$

According to Proposition 1 this implies that $\delta = m$ is the optimal strategy.

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