Collateral Requirements and Asset Prices*

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Abstract

We consider an economy populated by investors with heterogeneous preferences and beliefs who receive non-pledgeable labor incomes. We study the effects of collateral constraints that require investors to maintain sufficient pledgeable capital to cover their liabilities. We show that these constraints inflate stock prices, give rise to clusters of stock return volatilities, and produce spikes and crashes in price-dividend ratios and volatilities. Furthermore, mere possibility of a crisis significantly decreases interest rates and increases Sharpe ratios. The stock price has large collateral premium over non-pledgeable incomes. Asset prices are in closed form, and investors survive in the long run.

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1. Introduction

Financial markets play a key role in facilitating risk sharing and efficient allocation of assets among investors. However, trading in financial assets often entails moral hazard due to investors' incentives to default on their risky positions. The moral hazard can be alleviated by collateralized trades whereby risky positions are backed by financial capital that can be seized in the event of default. The latter arrangement restores the functionality of the financial markets at a cost of restricting risk sharing among investors. In this paper, we develop a parsimonious model which sheds light on the economic effects of such restrictions on asset prices, their moments, and the distribution of consumption and wealth in the economy. Our analysis is facilitated by closed-form solutions of the model and the stationarity of equilibrium processes.

We consider a pure exchange economy with one consumption good produced by a tree, similar to Lucas (1978). The economy is populated by two representative investors with heterogeneous constant relative risk aversion (CRRA) preferences over consumption and heterogeneous beliefs about the growth rate of the output. Each investor receives a fraction of the tree's output as labor income and invests total wealth in financial assets such as bonds and stocks. The investors have limited liability and can re-enter the financial market following defaults on debt and short positions in financial assets. In the event of default the financial assets can be seized by counterparties but labor income cannot be expropriated. The arising moral hazard problem is resolved by requiring risky positions to be backed by collateral in such a way that each investor's total financial wealth stays positive at all times, and hence, investors can always pay back to counterparties. We label the latter constraint as collateral requirement.

The aggregate consumption growth rates are independent and identically distributed (i.i.d.) but may occasionally experience large negative transitory shocks during low-probability production crises in the economy. These shocks help us explore how mere anxiety about the possibility of a production crisis affects the economy by making collateral requirements binding. We solve the model in closed form for general risk aversions and beliefs, and explore the effects of collateral requirements on interest rates, Sharpe ratios, price-dividend ratios, stock return volatilities, and distributions of investors' consumption shares in the aggregate output. The advantage of closed-form model solutions over numerical solutions in the related literature (discussed below) is that they help us establish our qualitative results for general model parameters rather than for particular calibrations.

Our main results are as follows. First, we show that mere possibility of a large (albeit unpredictable) drop in the aggregate output next period decreases interest rates and increases Sharpe ratios in the current period when the irrational optimist is close to hitting the collateral constraint. The latter effect only occurs when production crises and collateral requirements are jointly present in the economy. Hence, the collateral requirements amplify the spillover of the production crisis to the financial market. The amplification effect arises because investors "fly to quality" by buying riskless bonds when there is a possibility of hitting the collateral constraint next period.

Next, we show that the collateral requirements increase stock price-dividend ratio relative to the unconstrained benchmark. The effects of constraints are stronger when investors are close to their default boundaries, which makes price-dividend ratio a U-shaped function of one of the investor's share of the aggregate consumption. The price-dividend ratio spikes upwards in response to small economic shocks near default boundaries giving rise to cycles of inflating and deflating stock prices in the economy.

Our intuition for the results on price-dividend ratios is as follows. Absent any frictions, the investors' consumption shares gradually approach zero or one and, accordingly, the economic impact of one of the investors vanishes in the long-run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015). The collateral requirements restrict financial losses and protect investors from losing their consumption shares. The result is that the consumption shares are bounded away from zero and one. Moreover, the constraints never bind simultaneously for both investors, and at each moment one of the investors is unconstrained. The unconstrained investor's marginal utility of consumption is proportional to the prices of Arrow-Debreu securities. This marginal utility is expected to be higher in the economy with constraints because the unconstrained investor's consumption is expected to be lower than in the unconstrained economy due to the upper bound on the consumption share, discussed above. Consequently, the prices of Arrow-Debreu securities, and hence, also the stock price, are higher in the constrained economy. The stock has an additional collateral value (in contrast to labor income), which further inflates its price.

The dynamics of the price-dividend ratio determines the effect of constraints on volatilities. We show that collateral requirements dampen volatilities in bad times, when aggregate consumption is low, and amplify them in good times, when aggregate consumption is high. The latter effect makes collateral requirements a useful tool for curbing excessive volatility in bad times. The explanation is that the U-shape makes price-dividend ratio procyclical in good and countercyclical in bad times. As a result, the price-dividend ratio and the dividend move in the same direction in good times and in opposite directions in bad times. Because the stock price is the product of the price-dividend ratio and the dividend, stock return volatility increases in good times and decreases in bad times. We find that the volatility exhibits clustering and is sensitive to economic shocks when investors are close to hitting their constraints, which gives rise to spikes and crashes of volatility.

We also derive the distributions of investors' consumption shares in analytic form and show that they are stationary and non-degenerate (i.e., their support is a closed interval rather than a single point). The analysis of these distributions yields three important economic results. First, there is non-trivial time-variation of asset prices in the long run. Second, periods of binding collateral requirements are persistent. Third, all investors, including those with incorrect beliefs, survive in the long run and can have large economic impact in equilibrium. The survival is due to the fact that the constraints prevent investors from losing their consumption shares. The non-degeneracy arises because investors can accumulate labor income and re-enter the asset markets if shocks are favorable. The survival of investors in markets with frictions has been known before (e.g., Blume and Easley, 2006). However, the non-degeneracy of consumption share distributions and the persistence of the periods of binding constraints are more difficult to demonstrate and, to our best knowledge, are new to the literature.

Finally, we measure the collateral liquidity premium of the stock versus labor income due to the fact that the stock can be used as collateral. Because labor incomes are non-tradable, first, we derive shadow prices of claims to labor incomes such that exchanging marginal units of these claims for the consumption good at shadow prices does not affect investors' welfare. Then, we construct portfolios of stocks that replicate labor incomes. Such portfolios exist because labor incomes and stock dividends are proportional in our model. We define the collateral liquidity premium for the stock as the percentage difference in the value of the replicating portfolio and the shadow price. Absent any frictions, this premium is zero. The premium from the view of a particular investor widens close to that investor's default boundary and ranges from 0% to 35% in our calibration.

The paper develops new methodology for studying the effects of collateral requirements. This new methodology allows us to obtain closed-form equilibrium processes and prove their properties which previously could only be studied numerically. For example, we prove that collateral requirements always increase price-dividend ratios and generate spikes in asset prices, and lead to non-degeneracy and stationarity of consumption share distributions. Hence, collateralization emerges as a tractable way of inducing the stationary of equilibrium. Finally, the paper introduces a tractable discrete-time framework that makes exposition less technical and permits taking continuous-time limits. The tractability and stationarity make our model a convenient benchmark for asset pricing research that can be extended in various directions.

Related Literature. Closest to us are papers that study economies where investors have limited liability and face solvency constraints. Deaton (1990) considers a partial equilibrium model in which investors trade in a riskless asset with an exogenous interest rate and face a non-negativity constraint on their financial wealth. Detemple and Serrat (2003) also study the non-negative wealth constraint in a model where investors have heterogeneous beliefs and identical risk aversions. They show that the interest rates and Sharpe ratios are affected by the constraint only at the boundaries of the state-space. They do not solve for price-dividend ratios and volatilities as we do in this paper. Moreover, in our paper the constraint has an effect on interest rates and Sharpe ratios in the internal area of the state-space when there are rare production crises in the economy. Chien and Lustig (2010) study a similar constraint in an economy with a continuum of ex ante identical investors that receive non-pledgeable labor incomes affected by idiosyncratic shocks. Lustig and van Nieuwerburg (2005) study the role of housing collateral when labor income is non-pledgeable. The main difference of our paper from the latter two papers is that our investors are ex ante heterogeneous and are not affected by idiosyncratic shocks to labor income. The economic effects of heterogeneity in preferences and beliefs are different from the effects of ex-post heterogeneity in realized idiosyncratic labor income shocks in the above literature. For example, Krueger and Lustig (2010) show the irrelevance of market incompleteness induced by these income shocks for the risk premia.

Related works with similar constraints include Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008, 2014), and Blume et al (2015), among others. Compared to the latter models, our constraint is more tractable, investors are heterogeneous both in preferences and beliefs, and there are rare crises in the economy. Hence, we get new predictions on the effects of collateral requirements on asset prices. Cao (2017) shows that investors with incorrect beliefs have strictly positive shares of consumption in the long run (i.e., survive in the long-run) in economies with collateral constraints and stationary endowment processes bounded away from zero. In contrast to Cao (2017), our dividends follow a geometric Brownian motion, which is non-stationary and can become arbitrarily close to zero. Moreover, we not only show the survival of investors, but also derive consumption share distributions in closed form and prove that these distributions are stationary and non-degenerate, i.e., their support is a closed interval rather than a single point.

Kehoe and Levine (1993), Kocherlakota (1996), and Osambela (2015) study economies with participation constraints where investors are weakly better off not defaulting and are permanently excluded from securities markets if they default. Alvarez and Jermann (2000) show that such constraints can be implemented by imposing certain "not too tight" solvency portfolio constraints. Alvarez and Jermann (2001) find that such constraints help explain equity premia in the U.S. economy. They solve a simple example in closed form and develop a numerical method for the general case. In contrast to this literature, our investors have limited liability and can re-enter the market after a default.

Our constraint restricts borrowing and short-selling in equilibrium. Consequently, the paper is related to the literature on economic effects of borrowing, margin, short-sale and position limit constraints (e.g., Harrison and Kreps, 1978; Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Gromb and Vayanos, 2002, 2010; Pavlova and Rigobon, 2008; Brunnermeier and Pedersen, 2009; Gârleanu and Pedersen, 2011; Buss et al, 2013; Chabakauri, 2013, 2015; Rytchkov, 2014; Brumm et al, 2015), portfolio insurance (e.g., Basak, 1995) and VaR constraints (e.g., Basak and Shapiro, 2001). The constraints in this literature can increase or decrease stock prices depending on whether the investors' risk aversions are greater or less than one (e.g., Chabakauri, 2015). The equilibrium with these constraints and CRRA investors is characterized in terms of non-linear differential equations that can only be solved numerically (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014). In contrast to the above literature, we use a different methodology that allows us to find the equilibrium in closed form. Our economic results are also significantly different. In particular, our collateral requirements always increase stock prices irrespective of risk aversions and beliefs. They also generate new effects such as spikes and crashes of volatilities and stock prices, and clusters of volatility. Another innovation is that we allow for rare production crises which interact with the collateral constraints and have significant effects on equilibrium.

The paper is also related to macro-finance, financial intermediation, and banking literatures that study economies with frictions (Kiyotaki and Moore, 1997; Brunnermeier and

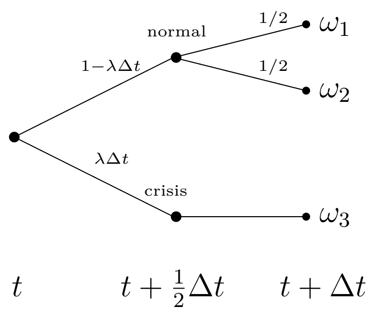

Figure 1
States of the Economy
After time t the economy moves to a normal state with probability $1 - \lambda\Delta t$ and to a crisis state with probability $\lambda\Delta t$. Conditional on being in a normal state the economy moves to either ω_1 or ω_2 with equal probabilities.

Sannikov, 2014; Kondor and Vayanos, 2015; Klimenko, Pfeil, Rochet, and De Nicolo, 2016) and to the literature on frictionless economies with heterogeneous investors (e.g., Basak, 2005; Chan and Kogan, 2002; Yan, 2008; Xiong and Yan (2009); Bhamra and Uppal, 2014; Borovička, 2015, among others).

2. Economic setup

We consider a pure-exchange infinite-horizon economy with one consumption good produced by an exogenous Lucas (1978) tree. The economy is populated by two representative heterogeneous investors A and B that hold shares in the tree and receive labor income each period. To facilitate the exposition, we start with a discrete-time economy with dates $t = 0, \Delta t, 2\Delta t, \ldots$, and later take a continuous-time limit.

At each point of time $t = 0, \Delta t, 2\Delta t, \ldots$ the economy is in one of the three states: ω_1, ω_2, and ω_3. With probability $1 - \lambda\Delta t$ the economy is either in state ω_1 or state ω_2, which we call *normal states*, and with probability $\lambda\Delta t$ in state ω_3, which we call the *crisis state*. Parameter $\lambda > 0$ is the *crisis intensity*. States ω_1 and ω_2 have probabilities 1/2 conditional on the economy being in a normal state. Figure 1 depicts the structure of uncertainty.

2.1. Aggregate output and securities markets

At date t the tree produces $D_t\Delta t$ units of aggregate output, where D_t follows a process

$$\Delta D_t = D_t[\mu_D\Delta t + \sigma_D\Delta w_t + J_D\Delta j_t], \tag{1}$$

where $\mu_D \geq 0$, $\sigma_D > 0$, and $J_D \leq 0$ are output growth mean, volatility, and drop during a crisis, respectively, and $\Delta D_t = D_{t+\Delta t} - D_t$ is the change in output. Processes w_t and j_t are discrete-time analogues of a Brownian motion and Poisson processes, respectively.[1] These processes follow dynamics $w_{t+\Delta t} = w_t + \Delta w_t$ and $j_{t+\Delta t} = j_t + \Delta j_t$, where increments Δw_t and Δj_t are i.i.d. random variables given by:

$$\Delta w_t = \begin{cases} +\sqrt{\Delta t}, & \text{in state } \omega_1, \\ -\sqrt{\Delta t}, & \text{in state } \omega_2, \\ 0, & \text{in state } \omega_3, \end{cases} \qquad \Delta j_t = \begin{cases} 0, & \text{in state } \omega_1, \\ 0, & \text{in state } \omega_2, \\ 1, & \text{in state } \omega_3. \end{cases} \tag{2}$$

It can be easily verified that $\mathbb{E}_t[\Delta w_t|\text{normal}] = 0$ and $\text{var}_t[\Delta w_t|\text{normal}] = \Delta t$, similar to a Brownian motion, where $\mathbb{E}_t[\cdot]$ and $\text{var}_t[\cdot]$ are expectation and variance conditional on time-t information. Parameters μ_D, σ_D, and J_D are such that $D_t > 0$ for all t.

The economy is populated by two representative price-taking investors A and B. Each investor stands for a continuum of identical investors of unit mass. Fractions l_A and l_B of the aggregate output $D_t \Delta t$ are paid to investors A and B as their labor incomes, respectively. Labor incomes are non-tradable. Fractions l_A and l_B can be also interpreted as non-tradable shares in the aggregate output such as holdings of illiquid assets. The remaining fraction $1 - l_A - l_B$ is paid as a dividend to the shareholders.

The investors can trade three securities at each date t: 1) a riskless bond in zero net supply, which pays one unit of consumption at date $t + \Delta t$; 2) one stock in net supply of one unit, which is a claim to the stream of dividends $(1 - l_A - l_B) D_t \Delta t$; 3) a one-period insurance contract in zero net supply, which pays one unit of consumption in the crisis state ω_3 and zero otherwise. Absent any frictions the market is complete. Bond, stock, and insurance prices B_t, S_t, and P_t, respectively, are determined in equilibrium.

2.2. Investor heterogeneity and optimization problems

The investors have heterogeneous CRRA preferences over consumption, given by

$$u_i(c) = \begin{cases} \dfrac{c^{1-\gamma_i}}{1-\gamma_i}, & \text{if } \gamma_i \neq 1, \\ \ln(c), & \text{if } \gamma_i = 1, \end{cases} \tag{3}$$

[1] Chabakauri (2014) shows that process (1) converges to a continuous-time Lévy process as $\Delta t \to 0$.

where $i = A, B$. The investors agree on time-t asset prices and the aggregate output but disagree on the probabilities of states. Investor A is rational and has correct probabilities

$$\pi_A(\omega_1) = \frac{1-\lambda\Delta t}{2}, \quad \pi_A(\omega_2) = \frac{1-\lambda\Delta t}{2}, \quad \pi_A(\omega_3) = \lambda\Delta t, \tag{4}$$

where λ is such that probabilities (4) are positive. Investor B has biased probabilities

$$\pi_B(\omega_1) = \frac{1-\lambda_B\Delta t}{2}(1+\delta\sqrt{\Delta t}), \quad \pi_B(\omega_2) = \frac{1-\lambda_B\Delta t}{2}(1-\delta\sqrt{\Delta t}), \quad \pi_B(\omega_3) = \lambda_B\Delta t, \tag{5}$$

where crisis intensity λ_B and disagreement parameter δ are such that probabilities (5) are positive. It is immediate to verify that $\pi_B(\omega_1) + \pi_B(\omega_2) + \pi_B(\omega_3) = 1$, and hence, $\pi_B(\omega)$ is a probability measure. Throughout the paper, by $\mathbb{E}_t^i[\cdot]$ and $\text{var}_t^i[\cdot]$ we denote conditional expectations and variances under the probability measure of investor i.

It can be easily verified that time-t conditional expected output growth rate in normal times under the beliefs of investor B is given by:

$$\mathbb{E}_t^B\left[\frac{\Delta D_t}{D_t}\Big|\text{normal}\right] = (\mu_D + \delta\sigma_D)\Delta t, \tag{6}$$

Therefore, parameter δ measures the extent of the investor disagreement about the expected output growth during normal times. For tractability, we assume that investor B does not update probabilities over time. We also assume that investor B is weakly less risk averse and more optimistic than investor A: $\gamma_A \geq \gamma_B$, $\lambda \geq \lambda_B$ and $\delta \geq 0$. The assumption that the less risk averse investor is also more optimistic is imposed to simplify the exposition and does not affect the qualitative results in the paper.[2] We allow for the heterogeneity in both risk aversions and beliefs for generality. Main qualitative results do not change if we keep only one source of heterogeneity.

At date 0 the investors have certain endowments of financial assets. The total time-t disposable wealth of investor i is given by $W_{it} + l_i D_t \Delta t$, where W_{it} is the financial wealth, defined as the time-t value of all positions in financial assets acquired at the previous date, and $l_i D_t \Delta t$ is the labor income. At date t, investor i allocates wealth to $c_{it}\Delta t$ units of consumption, b_{it} units of bond, and a portfolio of risky assets $n_{it} = (n_{i,St}, n_{i,Pt})$, where $n_{i,St}$ and $n_{i,Pt}$ are units of stock and insurance, respectively. The bond and the risky assets are pledgeable, i.e., can be used as collateral, but the labor income is not.

[2] Assuming that the less risk averse investor is more optimistic makes our main state variable $s_t = c_{At}^*/D_t$ (introduced in Section 2.3 below) countercyclical, which facilitates the analysis of the results. If this assumption is relaxed, the qualitative results remain the same, but additional analysis is required to determine whether the state variable s is counter- or pro-cyclical.

In a frictionless economy, the financial wealth W_{it} can become negative when investors take risky positions backed by their future labor income. However, we assume that labor income is not pledgeable, and the investors can default when their financial wealth becomes negative. The investors also have limited liability and can re-enter the market after default, which gives rise to a moral hazard problem, similar to the related literature (e.g., Chien and Lustig, 2010; Geanakoplos, 2009). This problem is addressed here by requiring the investors to keep their next-period financial wealth $W_{i,t+\Delta t}$ positive at all times, so that their pledgeable capital is sufficient to cover all liabilities such as debt and short positions.

Investor $i = A, B$ maximizes expected discounted utility with time discount ρ

$$\max_{c_{it},b_{it},n_{it}} \mathbb{E}_t^i \left[\sum_{\tau=t}^{\infty} e^{-\rho\tau} u_i(c_{i\tau}) \Delta t \right], \tag{7}$$

subject to the self-financing budget constraints, given by

$$W_{it} + l_i D_t \Delta t = c_{it} \Delta t + b_{it} B_t + n_{it} (S_t, P_t)^\top, \tag{8}$$

$$W_{i,t+\Delta t} = b_{it} + n_{it} \Big(S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t,\ \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}} \Big)^\top, \tag{9}$$

and the collateral constraint:

$$W_{i,t+\Delta t} \geq 0, \tag{10}$$

where $W_{i,t+\Delta t}$ is the financial wealth at date $t + \Delta t$ given by equation (9). Constraint (10) requires investors to cross-collateralize their positions in such a way that losses on one position are always offset by gains on the other positions.

Remark 1 (Partially pledgeable labor income). Our model can be easily extended to economies where fraction $k_i \in [0, 1]$ of investor i's labor income can be pledged. The requirement to keep next-period pledgeable wealth is then given by:

$$W_{i,t+\Delta t} + \underbrace{\frac{k_i l_i}{1 - l_A - l_B} \Big(S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t} \Delta t \Big)}_{\text{measure of pledgeable labor income}} \geq 0. \tag{11}$$

The second term in constraint (11) measures the value of the pledgeable income. Let $k_i l_i D_t \Delta t$ be the pledgeable income of investor i. This income is proportional to stock dividends $(1 - l_A - l_B) D_t \Delta t$, and hence, can be replicated by a portfolio of $\widehat{n}_i = k_i l_i / (1 - l_A - l_B)$ units of stock with cum-dividend value $\widehat{n}_i (S_t + (1 - l_A - l_B) D_t \Delta t)$. The investors can circumvent the non-tradability of pledgeable income by shorting stocks against this income. Hence, the claims to pledgeable income are, effectively, tradable and have the same value as the replicating portfolio. The requirement to have positive pledgeable wealth then becomes $W_{i,t+\Delta t} + \widehat{n}_i\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big) \geq 0$, which is equivalent to constraint (11). Lemma A.1 in the Appendix shows that models with $k_i \neq 0$ reduce to models with $k_i = 0$ by a change of variable. Hence, the economic implications of our baseline model with constraint (10) and the model with a more general constraint (11) are the same.

2.3. Equilibrium

Definition. *An equilibrium is a set of asset prices $\{B_t, S_t, P_t\}$ and of consumption and portfolio policies $\{c^*_{it}, b^*_{it}, n^*_{it}\}_{i\in\{A,B\}}$ that solve optimization problem (7) for each investor, given processes $\{B_t, S_t, P_t\}$, and consumption and securities markets clear:*

$$c^*_{At} + c^*_{Bt} = D_t, \quad b^*_{At} + b^*_{Bt} = 0, \quad n^*_{A,St} + n^*_{B,St} = 1, \quad n^*_{A,Pt} + n^*_{B,Pt} = 0. \tag{12}$$

In addition to asset prices, we derive price-dividend and wealth-aggregate consumption ratios $\Psi = S/\Big((1 - l_A - l_B)D\Big)$ and $\Phi_i = W^*_i/D$, respectively. We also derive annualized Δt-period riskless interest rates r_t, stock mean-returns μ_t and volatilities σ_t in normal times, and the percentage change of the stock price in the crisis state, denoted by J_t.

We derive the equilibrium in terms of state variable v_t given by the log-ratio of marginal utilities of investors evaluated at their shares of the aggregate consumption c^*_{it}/D_t:

$$v_t = \ln\left(\frac{(c^*_{At}/D_t)^{-\gamma_A}}{(c^*_{Bt}/D_t)^{-\gamma_B}}\right). \tag{13}$$

Substituting consumption shares of investors A and B, denoted by $s_t = c^*_{At}/D_t$ and $1-s_t = c^*_{Bt}/D_t$, into equation (13), we express v_t as a function of s_t:

$$v_t = \gamma_B \ln(1 - s_t) - \gamma_A \ln(s_t). \tag{14}$$

Variable v_t is a decreasing function s_t, and hence, s_t is an alternative state variable.

We assume that the process for the aggregate consumption is such that the investors' value functions evaluated at aggregate consumption are finite:

$$\mathbb{E}_0\left[\sum_{\tau=0}^{\infty} e^{-\rho\tau} u_i(D_\tau)\Delta t\right] < +\infty. \tag{15}$$

3. Characterization of equilibrium

First, we derive the investors' state price densities (SPD) ξ_{At} and ξ_{Bt}. Then, we find asset prices from the following standard equations of asset pricing (e.g., Duffie (2001, p.23)):

$$B_t = \mathbb{E}_t^i\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\Big], \tag{16}$$

$$S_t = \mathbb{E}_t^i\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big)\Big], \tag{17}$$

$$P_t = \mathbb{E}_t^i\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\mathbf{1}_{\{\omega_{t+\Delta t}=\omega_3\}}\Big], \tag{18}$$

where $i = A, B$. The state price density ξ_{it} exists for each investor i due to the absence of arbitrage opportunities in our economy.[3] The investors can eliminate arbitrage because strategies with zero investment and non-negative payoffs are feasible given constraints (8)–(10). To ensure the uniqueness of SPD ξ_{it} for each investor i, we assume and later verify that the matrix of asset payoffs is invertible in equilibrium. The SPDs ξ_{At} and ξ_{Bt} differ due to heterogeneity in beliefs and are linked by the *change of measure equation*[4]

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = \frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\pi_A(\omega_{t+\Delta t})}{\pi_B(\omega_{t+\Delta t})}. \tag{19}$$

We find the SPDs from the first order conditions in terms of investors' marginal utilities of consumption and Lagrange multipliers for collateral requirements (10). First, we rewrite the budget equations (8)–(9) in a static form that expresses the current wealth in terms of current consumption and the expected discounted future wealth (e.g., Cox and Huang, 1989). Then, we solve investor optimizations by dynamic programming and the method of Lagrange multipliers. Lemma 1 below reports the results.

Lemma 1 (Dynamic programming and the first order condition).

1) *Let $V_i(W_{it}, v_t; l_i)$ denote the value function of investor i, where v_t is the state variable. Then, the value function solves the following Hamilton-Jacobi-Bellman equation:*

$$V_i(W_{it}, v_t; l_i) = \max_{c_{it}} \Big\{u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i[V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)]\Big\}, \tag{20}$$

subject to the static budget and collateral constraints:

$$W_{it} + l_i D_t \Delta t = c_{it}\Delta t + \mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}W_{i,t+\Delta t}\right], \tag{21}$$

[3] The proof of existence of the SPD in arbitrage-free economies can be found in Duffie (2001, p.4).

[4] Three equations (16)–(18) can be rewritten as equations for three unknowns $\pi_i(\omega_k)\xi_{i,t+\Delta t}(\omega_k)/\xi_{it}$, where $k = 1, 2, 3$ and i is set to either A or B. The solution of these equations is unique when the matrix of asset payoffs is invertible, and hence, $\pi_B(\omega_{t+\Delta t})\xi_{B,t+\Delta t}/\xi_{Bt} = \pi_A(\omega_{t+\Delta t})\xi_{A,t+\Delta t}/\xi_{At}$ for all states.

$$W_{i,t+\Delta t} \geq 0. \tag{22}$$

2) *Value function* $V_i(W_{it}, v_t; l_i)$ *is a concave function of wealth* W_{it}*.*

3) *The SPDs* ξ_{it} *and optimal consumptions* c^*_{it} *satisfy the first order conditions*

$$\frac{\xi_{i,t+\Delta t}}{\xi_{it}} = e^{-\rho\Delta t}\frac{(c^*_{i,t+\Delta t})^{-\gamma_i} + \ell_{i,t+\Delta t}}{(c^*_{it})^{-\gamma_i}}, \tag{23}$$

where $\ell_{i,t+\Delta t} \geq 0$ *is the Lagrange multiplier for collateral requirement* (22) *satisfying the complementary slackness condition* $\ell_{i,t+\Delta t}W^*_{i,t+\Delta t} = 0$*.*

We use Lemma 1 to derive the dynamics of state variable v_t. First, suppose constraints do not bind. In this case, Lagrange multipliers $\ell_{i,t+\Delta t}$ vanish and the first order conditions (23) are the same as in the unconstrained economy. The dynamics of the state variable v_t in the unconstrained region of the state-space is then the same as in the unconstrained economy, and is found in closed form below, similar to Chabakauri (2015). Next, let $\overline{v}$ and $\underline{v}$ be the values of the state variable v_t when constraints (10) of investors A and B bind, respectively. We guess and verify below that state variable v_t stays within boundaries $\underline{v} \leq v_t \leq \overline{v}$. Intuitively, binding collateral requirements restrict the investors' losses of wealth and consumption, which traps the state variable in the interval $[\underline{v}, \overline{v}]$. The boundaries $\overline{v}$ and $\underline{v}$ are found from the condition that the constraints bind: $W_{i,t+\Delta t} = 0$. Dividing these constraints by $D_{t+\Delta t}$, we obtain equations for $\overline{v}$ and $\underline{v}$:

$$\Phi_A(\overline{v}) = 0, \quad \Phi_B(\underline{v}) = 0. \tag{24}$$

Proposition 1 below reports the dynamics of v_t, and Appendix contains the proof.

Proposition 1 (Closed-form dynamics of state variable v_t).
Given the boundaries $\overline{v}$ *and* $\underline{v}$*, the equilibrium dynamics of state variable* v_t *is given by:*

$$v_{t+\Delta t} = \max\Big\{\,\underline{v}; \min\{\,\overline{v}; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}\Big\}, \tag{25}$$

where drift μ_v*, volatility* σ_v*, and jump* J_v *are given in closed form by:*

$$\mu_v = \frac{1}{2\Delta t}\left((\gamma_A - \gamma_B)\ln[(1+\mu_D\Delta t)^2 - \sigma_D^2\Delta t] + \ln\left(\frac{1-\lambda_B\Delta t}{1-\lambda\Delta t}\right)^2 + \ln(1-\delta^2\Delta t)\right), \tag{26}$$

$$\sigma_v = \frac{1}{2\sqrt{\Delta t}}\left((\gamma_A - \gamma_B)\ln\left(\frac{1+\mu_D\Delta t + \sigma_D\sqrt{\Delta t}}{1+\mu_D\Delta t - \sigma_D\sqrt{\Delta t}}\right) + \ln\left(\frac{1+\delta\sqrt{\Delta t}}{1-\delta\sqrt{\Delta t}}\right)\right), \tag{27}$$

$$J_v = (\gamma_A - \gamma_B)\ln(1+\mu_D\Delta t + J_D) + \ln\left(\frac{\lambda_B}{\lambda}\right) - \mu_v\Delta t. \tag{28}$$

Boundaries $\overline{v}$ *and* $\underline{v}$ *are reflecting when* Δt *is sufficiently small; that is,* v_t *does not stay at the boundaries forever:* $\text{Prob}(\overline{v} > v_{t+\Delta t}|v_t = \overline{v}) > 0$ *and* $\text{Prob}(v_{t+\Delta t} > \underline{v}|v_t = \underline{v}) > 0$.

Equation (25) reveals the exact structure of the state variable and sheds light on the equilibrium effects of the collateral requirement. The equation demonstrates that the constraint does not alter the dynamics of the state variable when the constraint is not binding, and all its effects are due to imposing low and upper bounds on process v_t. To our best knowledge, this paper is the first to provide closed-form dynamics of the state variable in an economy with collateral requirements. This dynamics helps us build a theory of collateral requirements (10). In particular, we use dynamics (10) to prove the existence of equilibrium and stationarity of equilibrium processes, to derive asset prices in closed-form in Section 3.1, and to study the effects of collateralization on asset prices.

Our next step is to find the SPD and asset prices. Proposition 2 reports the results.

Proposition 2 (Characterization of equilibrium in discrete time).

1) *The state price density under the beliefs of investor A is given by:*

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\exp\Big(\max\{0; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\}\Big), \quad (29)$$

where investor A's time-t consumption share $s(v_t)$ *solves equation* (14).

2) *The price-dividend ratio* Ψ *and wealth-aggregate consumption ratios* Φ_i *are functions of the state variable* v*, and satisfy equations:*

$$\Psi(v_t) = \mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{D_{t+\Delta t}}{D_t}\Big(\Psi(v_{t+\Delta t}) + \Delta t\Big)\right], \quad (30)$$

$$\Phi_i(v_t) = \mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{D_{t+\Delta t}}{D_t}\Phi_i(v_{t+\Delta t})\right] + \Big(\mathbf{1}_{\{i=A\}}s(v_t) + \mathbf{1}_{\{i=B\}}(1-s(v_t)) - l_i\Big)\Delta t, \quad (31)$$

Boundaries $\underline{v}$ *and* $\overline{v}$ *solve equations* (24). *The matrix of asset payoffs is invertible if and only if* $\sigma_t(1 + r_t\Delta t) \neq 0$*, where* σ_t *and* r_t *are the stock return volatility in normal times and the interest rate, respectively.*

3) *The price-dividend ratio in the constrained economy is higher than in the unconstrained economy with pledgeable income for the same value of state variable* v_t *in the two economies.*

Equation (29) captures the effect of collateralization on the SPD in our economy. It shows that the change in the SPD, $\xi_{t+\Delta t}/\xi_t$, can be decomposed into two terms. The first term, $e^{-\rho\Delta t}(s(v_{t+\Delta t})D_{t+\Delta t})^{-\gamma_A}/(s(v_t)D_t)^{-\gamma_A}$, given by the ratio of marginal utilities of investor A at times $t + \Delta t$ and t, is the change in SPD in the frictionless economy. The second term captures the effect of the friction on the SPD, and is only activated when the constraint of investor A is binding. Similar and equivalent representation of SPD can be obtained in terms of the marginal utilities of investor B. Using the SPD we then find the price-dividend and wealth-consumption ratios (30)–(31).

Proposition 2 also demonstrates that imposing collateral requirements inflates the price-dividend ratio of the stock for all model parameters for which the equilibrium exists. This result is in contrast to the effects of borrowing, margin, and restricted participation constraints in the related literature (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014), which increase or decrease the stock prices depending on the investors' elasticities of intertemporal substitution. We discuss the detailed intuition and further economic differences between our constraint and the constraints in the literature in Section 4.1.

The related literature discussed in the introduction evaluates the effects of frictions in multi-period economies numerically for particular calibrations. In contrast to this literature, aided by the closed-form dynamics of the state variable (25) and the SPD (29), we provide a rigorous proof that constraint (10) increases stock price-dividend ratios for general risk aversions and beliefs. We demonstrate that the SPD in the economy with frictions exceeds the SPD in the frictionless economy for a given value of v_t. Hence, for the same level of output D_t, the stock price at date t is higher in the constrained economy.

3.1. Closed-form solution in a continuous-time limit

Next, we take continuous-time limit $\Delta t \to 0$ and derive the equilibrium in closed form. Taking the limit allows rewriting equations (30) and (31) for the price-dividend and wealth-consumption ratios, Ψ_t and Φ_{it}, as differential-difference equations. For tractability, we derive ratios Ψ_t and Φ_{it} in terms of a transformed ratio $\widehat{\Psi}(v;\theta)$, which satisfies a simpler equation reported in Lemma 2 below.

Lemma 2 (Differential-difference equation). *In the limit $\Delta t \to 0$, the price-dividend ratio Ψ and wealth-aggregate consumption ratios Φ_i are given by:*

$$\Psi(v) = \widehat{\Psi}(v; -\gamma_A) s(v)^{\gamma_A}, \tag{32}$$

$$\Phi_i(v) = \Big((\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}}) \widehat{\Psi}(v; 1-\gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i) \widehat{\Psi}(v; -\gamma_A) \Big) s(v)^{\gamma_A}, \tag{33}$$

where $s(v)$ solves equation (14) and $\widehat{\Psi}(v;\theta)$ satisfies a differential-difference equation

$$\begin{aligned}\frac{\widehat{\sigma}_v^2}{2}\widehat{\Psi}''(v;\theta) &+ \Big(\widehat{\mu}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v\Big)\widehat{\Psi}'(v;\theta)\\ &- \left(\lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\widehat{\Psi}(v;\theta)\\ &+ \lambda(1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v}; v + \widehat{J}_v\};\theta\Big) + s(v)^\theta = 0,\end{aligned} \tag{34}$$

subject to the reflecting boundary conditions

$$\widehat{\Psi}'(\underline{v};\theta) = 0, \quad \widehat{\Psi}'(\overline{v};\theta) - \widehat{\Psi}(\overline{v};\theta) = 0, \tag{35}$$

where $\widehat{\mu}_v$, $\widehat{\sigma}_v \geq 0$, and $\widehat{J}_v \leq 0$ are constants given by:

$$\widehat{\mu}_v = (\gamma_A - \gamma_B)\left(\mu_D - \frac{\sigma_D^2}{2}\right) + \lambda - \lambda_B - \frac{\delta^2}{2}, \tag{36}$$

$$\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta, \tag{37}$$

$$\widehat{J}_v = (\gamma_A - \gamma_B)\ln(1 + J_D) + \ln\left(\frac{\lambda_B}{\lambda}\right). \tag{38}$$

The boundaries $\overline{v}$ and $\underline{v}$ solve the following equations:

$$\frac{\widehat{\Psi}(\overline{v}; 1-\gamma_A)}{\widehat{\Psi}(\overline{v}; -\gamma_A)} = l_A, \quad \frac{\widehat{\Psi}(\underline{v}; 1-\gamma_A)}{\widehat{\Psi}(\underline{v}; -\gamma_A)} = 1 - l_B. \tag{39}$$

We observe that equation (34) is linear, in contrast to economies with constraints directly imposed on trading strategies of investors (e.g., Gârleanu and Pedersen, 2012; Chabakauri, 2013, 2015; Rytchkov, 2014). This equation is a differential-difference equation with a "delayed" argument in the fourth term on the left-hand side of the equation because $\widehat{J}_v \leq 0$. This term is further complicated by the fact that the delayed argument is restricted to stay above the lower boundary $\underline{v}$, which gives rise to the dependence of the fourth term on a peculiar argument $\max\{\underline{v}; v + \widehat{J}_v\}$. This term captures investors' decisions in anticipation of hitting their collateral constraint.

Before deriving the equilibrium in the general case, in Corollary 1 below, we provide analytical price-dividend ratios when there is no crisis and investors have log preferences.

Corollary 1 (Analytical asset prices in a special case). *Suppose, investors A and B have logarithmic preferences and there is no production crisis, that is, $\lambda = \lambda_B = 0$. Then, price-dividend ratio $\Psi(v)$ is given by:*

$$\Psi(v) = \frac{1}{\rho} + \frac{C_1 e^{\varphi_+ v} + C_2 e^{\varphi_- v}}{1 + e^v}, \tag{40}$$

where $\varphi_{\pm} = 0.5(1 \pm \sqrt{1 + 8\rho/\delta^2})$*, and constants* C_1 *and* C_2 *are given by equations* (A41) *and* (A42) *in the Appendix, respectively.*

In Section 4 below, we argue that the analytical price-dividend ratio (40) captures some important properties of price-dividend ratio that hold in the general case with arbitrary risk aversions and crises. Hence, this special case can be used as a tractable benchmark in asset pricing research. Nevertheless, we undertake a comprehensive investigation of equilibrium in the general case.

Proposition B.1 in Appendix B presents the closed-form price-dividend ratio for general risk aversions and beliefs. Although the closed-form solution in Proposition B.1 is complex, it helps us build a theory of collateral constraints (10). First, it provides a proof that the boundary-value problem (34)–(35) has a unique solution given the boundaries $\underline{v}$ and $\overline{v}$. Second, using this closed-form solution we prove the existence of boundaries $\underline{v}$ and $\overline{v}$ in Proposition B.2 in Appendix B. Third, it helps avoid numerical methods based on value function iterations (e.g., Krusell and Smith, 1998), widely used in the literature with market frictions, for which the convergence results, in general, are not available. We double-checked the solution reported in Proposition B.1 by solving problem (34)–(35) using the method of finite differences and verifying that it gives the same result.

We call the interval $v \in [\underline{v}, \underline{v} - \widehat{J}_v]$ in the state-space a period of *anxious economy*, similar to Fostel and Geanakoplos (2008).[5] When the economy falls into this state, even a small possibility of a crisis renders the collateral requirement binding and leads to deleveraging in the economy. To explore the economic effects of the anxious economy, we provide closed-form expressions for the interest rates r_t and risk premia in normal times $\mu_t - r_t$, which can be easily obtained using previously derived equations for asset prices and the state price density. Proposition 3 below reports the results.

Proposition 3 (Interest rates and risk premia in the limit). *For a sufficiently small interval* Δt*, the interest rate* r_t *and the risk premium* $\mu_t - r_t$ *in normal times are given by:*

$$r_t = \begin{cases} \tilde{r}_t - \lambda(1 + J_D)^{-\gamma_A} \left(\dfrac{s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)}{s_t} \right)^{-\gamma_A} + O(\Delta t), \ \text{for } \underline{v} < v_t < \overline{v}, \\ \dfrac{(1 - s_t)\Gamma_t\Big(\mathbf{1}_{\{v = \underline{v}\}} - \mathbf{1}_{\{v = \overline{v}\}}\Big) - \gamma_B}{2\gamma_B\sqrt{\Delta t}} \widehat{\sigma}_v + O(1), \ \text{for } v = \underline{v} \text{ or } v = \overline{v}, \end{cases} \tag{41}$$

[5] However, in contrast to Fostel and Geanakoplos (2008), the disagreement about the consumption growth dynamics in our economy does not increase during these periods.

$$
\begin{aligned}
\mu_t - r_t = &\left(\gamma_A \sigma_D - \frac{(1-s_t)\Gamma_t \widehat{\sigma}_v}{\gamma_B} + \frac{(1-s_t)\Gamma_t \widehat{\sigma}_v (\mathbf{1}_{\{v=\underline{v}\}} + \mathbf{1}_{\{v=\overline{v}\}}) - \gamma_B \widehat{\sigma}_v \mathbf{1}_{\{v=\overline{v}\}}}{2\gamma_B}\right)\sigma_t \\
&- \lambda(1+J_D)^{-\gamma_A} J_t \left(\frac{s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)}{s_t}\right)^{-\gamma_A} + O(\sqrt{\Delta t}),
\end{aligned} \tag{42}
$$

where $\tilde{r}_t$ *is the interest rate in the unconstrained economy without crisis risk, given by:*

$$
\begin{aligned}
\tilde{r}_t = &\lambda + \rho + \gamma_A \mu_D - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2 + \left(\frac{\gamma_A \sigma_D \widehat{\sigma}_v - \widehat{\mu}_v}{\gamma_B}\right)(1-s_t)\Gamma_t \\
&- \widehat{\sigma}_v^2 \left(\frac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2 + \frac{1}{2\gamma_A^2\gamma_B^2} s_t (1-s_t)\Gamma_t^3\right),
\end{aligned} \tag{43}
$$

drift $\widehat{\mu}_v$*, volatility* $\widehat{\sigma}_v$*, and* $\widehat{J}_v$ *of the state variable* v *are given by equations* (36)–(38), *volatility* σ_t *and jump size* J_t *are given by equations* (B7)–(B8), *respectively, and* $\Gamma_t \equiv \gamma_A\gamma_B/\Big(\gamma_A(1-s_t) + \gamma_B s_t\Big)$ *is the risk aversion of a representative investor.*

The effects of collateral requirements on interest rates and risk premia arise due to the investors' concern that a potential crisis may render the constraint binding next period when the economy is close to boundary $\underline{v}$. The last term in the first equation in (41) for the interest rate quantifies the impact of collateral requirements on precautionary savings due to a downward jump in the aggregate consumption, which we further discuss in Section 4.

Equations (41) and (42) also feature terms with indicator functions $\mathbf{1}_{\{v=\underline{v}\}}$ and $\mathbf{1}_{\{v=\overline{v}\}}$, which are non-zero only at the boundaries $\underline{v}$ and $\overline{v}$. For the interest rate r_t these terms have the order of magnitude proportional to $1/\sqrt{\Delta t}$, and hence, the interest rate has singularities at the boundaries $\underline{v}$ and $\overline{v}$ when $\Delta t \to 0$. Similar singularities arise in a continuous-time model of Detemple and Serrat (2003). Our discrete-time analysis sheds new light on these singularities by uncovering their order of magnitude $1/\sqrt{\Delta t}$. Consequently, the per-period rate $r_t\Delta t$ is finite and has an order of magnitude $O(\sqrt{\Delta t})$.

The intuition for the singularity is that near the boundaries $\underline{v}$ and $\overline{v}$ even a small shock Δw_t may lead to a default. Consequently, when the collateral requirement of an investor binds at time t, the investor allocates a larger fraction of labor income to bond than in the interior region $\underline{v} < v_t < \overline{v}$ and requires a higher risk premium. Therefore, the interest rate decreases and Sharpe ratio increases at the boundaries.

3.2. Stationary distribution of consumption share

Absent any frictions, state variable v follows an arithmetic Brownian motion with a jump. This process is non-stationary and induces non-stationarity of the unconstrained equilibrium where one of the investor's share of consumption gradually converges to zero. Hence, with the exception of some knife-edge parameter combinations, only one of the investors has a significant impact on asset prices in the frictionless economy in the long run (e.g., Blume and Easley, 2001; Yan, 2008; Chabakauri, 2015).

It is intuitive that imposing collateral requirements (10) helps both investors survive and have an impact on equilibrium in the long-run because these constraints protect investors against losing their shares of aggregate consumption beyond certain limits. Similar intuition for the survival of investors in economies with market imperfections has been discussed in the previous literature (e.g., Blume and Easley, 2001). However, this intuition does not tell anything about the shape of the distribution of consumption share s, whether this distribution is well-defined or degenerate (e.g., fully concentrated at boundaries $\underline{s}$ or $\overline{s}$), and which parameters determine the relative dominance of investors in the economy. Armed with the closed-form dynamics of the state variable v_t in (25), we derive the probability density function (PDF) of consumption share s in closed form, and show that this PDF is stationary and well-defined. The latter result is important because it implies non-trivial time-variation of asset prices in the long run. For simplicity, we assume that there is no crisis risk so that $\lambda = \lambda_B = 0$. Proposition 4 reports the results.

Proposition 4 (Stationary distribution of consumption share). *Suppose,* $\lambda = \lambda_B = 0$. *Then, the PDF* $f(s,\tau;s_t;\tau)$ *of consumption share* s *at time* τ *conditional on observing share* s_t *at time* t *is given in closed form by expression* (A62) *in the Appendix. Furthermore, the stationary PDF of consumption share* s *is given by:*

$$f(s) = \frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(\frac{\gamma_A}{s} + \frac{\gamma_B}{1-s}\right)\frac{\Big((1-s)^{\gamma_B}/s^{\gamma_A}\Big)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}}{\Big((1-\underline{s})^{\gamma_B}/\underline{s}^{\gamma_A}\Big)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2} - \Big((1-\overline{s})^{\gamma_B}/\overline{s}^{\gamma_A}\Big)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}}\mathbf{1}_{\{\underline{s}\le s\le\overline{s}\}}, \tag{44}$$

where $\widehat{\mu}_v = (\gamma_A - \gamma_B)(\mu_D - \sigma_D^2/2) - \delta^2/2$, $\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta$, $\mathbf{1}_{\{\underline{s}\le s\le\overline{s}\}}$ *is an indicator function and* $\underline{s}$ *and* $\overline{s}$ *are the bounds on the consumption share* s*, which solve equation* (24) *for* $\overline{v}$ *and* $\underline{v}$*, respectively.*

Proposition 4 confirms that both investors survive in the long run, and that consumption share s has well-defined stationary distribution. The beliefs enter PDF (44) via the

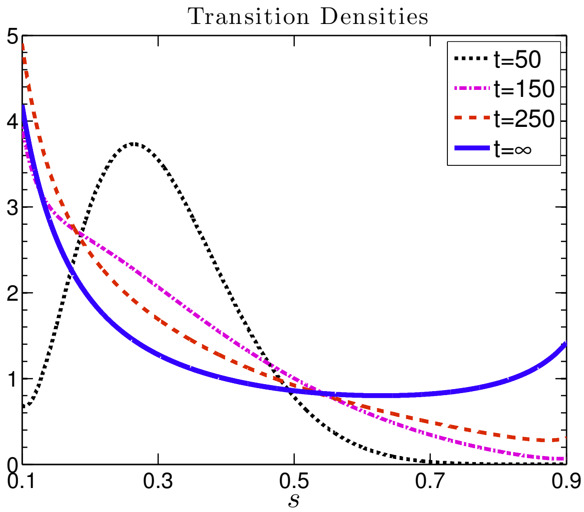

Figure 2
Convergence to stationary distribution of consumption share $s_t = c^*_{A,t}/D_t$
The Figure shows transition densities $f(s,t;s_0,0)$ for the starting point $s_0 = 0.2$ and the stationary distribution $f(s)$ (i.e., density for $t = \infty$). We set $\gamma_A = 2$, $\gamma_B = 1.5$, $\mu_D = 0.018$, $\sigma_D = 0.032$, $\lambda = \lambda_B = 0$, $\rho = 0.02$, $\delta = 0.1125$, $\underline{s} = 0.1$, $\overline{s} = 0.9$, $l_A = 0.123$, and $l_B = 0.14$.

ratio of the drift and variance of process v_t, given by $\widehat{\mu}_v/\widehat{\sigma}^2_v$. This ratio determines the relative dominance of investors in the economy. In particular, for bounds $\underline{s}$ and $\overline{s}$ that are symmetric around 0.5 the PDF is concentrated around $\underline{s}$ if $\widehat{\mu}_v > 0$ and around $\overline{s}$ if $\widehat{\mu}_v < 0$.

Figure 2 plots the stationary PDF (44) and transition densities $f(s,t;s_0,0)$, for parameters described in the legend and explained in Section 4 below. The stationary PDF has a larger mass around $s = 0.1$ because the labor share $l_B = 0.14$ of investor B exceeds the labor share $l_A = 0.123$ of investor A in this example in order to get boundary values $\underline{s} = 0.1$ and $\overline{s} = 0.9$ symmetric around 0.5. From Figure 2 we observe that both rational and irrational investors can occasionally have large consumption shares.

Another notable feature of PDF (44) is that it is bimodal, with a large mass of the distribution concentrated around boundaries $\underline{s}$ and $\overline{s}$. The economic implication of this bimodality is that the periods of binding constraints are likely to be persistent. The closed-form dynamics (25) for the state variable v helps explain the bimodality of the PDF. From this dynamics, we observe that after hitting a boundary the process v_t remains in its vicinity for some time. Hence, because variable v follows an arithmetic Brownian motion in the interval $(\underline{v}, \overline{v})$, the probability of hitting the same boundary again is high.

4. Analysis of Equilibrium

In this section, we demonstrate the economic implications of our model. In Section 4.1, we show that capital requirements amplify the effect of rare crises on generating lower interest rates and higher Sharpe ratios, lead to spikes and crashes of stock prices and stock return volatilities, amplify volatility in good times and decrease it in bad times, and generate volatility clusters. Section 4.2 measures the economic significance of collateralization by quantifying the collateral premium of the stock.

We study the equilibrium for calibrated parameters. We set the parameters of the aggregate consumption process to $\mu_D = 0.018$, $\sigma_D = 0.032$, $J_D = -0.2$, and the crisis intensities of investors A and B to $\lambda = 0.017$ and $\lambda_B = 0.01$, respectively.[6] The risk aversions are $\gamma_A = 2$ and $\gamma_B = 1.5$, and the time discount is $\rho = 0.02$. The disagreement parameter is $\delta = 0.1125$, which corresponds to the mean growth rate (6) under investor B's beliefs equal to $1.2\mu_D$. The shares of labor income $l_A = 0.123$ and $l_B = 0.14$ are chosen to generate symmetric bounds on investor A's consumption share: $\underline{s} = 0.1$ and $\overline{s} = 0.9$.[7]

We plot the equilibrium distributions and processes as functions of consumption share $s_t = c^*_{At}/D_t$ because s lies in the interval $[0, 1]$ and is more intuitive than variable v. We observe that consumption share s is countercyclical in the sense that $\mathrm{corr}_t(ds_t, dD_t) < 0$. Intuitively, the aggregate wealth and consumption shift to (away from) investor A following negative (positive) shocks to output because this investor is more risk averse and pessimistic than investor B. We call a process procyclical (countercyclical) if that process is a decreasing (increasing) function of s. We interpret periods of low (high) s_t as good (bad) times in the economy, because during these periods the output D_t is high (low).

4.1. Equilibrium processes

Figure 3 depicts investor B's leverage/market ratio L_t/S_t and stock holdings n_{Bt} in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) demonstrates the cyclicality of leverage. The leverage is lowest when either investor A or investor B bind on their constraints. Intuitively, when $s = \overline{s}$, investor B's financial wealth is zero, and

[6] Drift μ_D and volatility σ_D are within the ranges considered in the literature (e.g., Basak and Cuoco, 1998; Chan and Kogan, 2002; Rytchkov, 2014), intensity $\lambda = 0.017$ is from Barro (2009).

[7] To avoid finding bounds $\underline{s}$ and $\overline{s}$ numerically, we set them exogenously to $\underline{s} = 0.1$ and $\overline{s} = 0.9$ and then recover the shares of labor incomes $l_A = 0.123$ and $l_B = 0.14$ that imply these bounds in equilibrium. First, we find $\underline{v}$ and $\overline{v}$ from equation (14) for v, and then find l_A and l_B from equations (39).

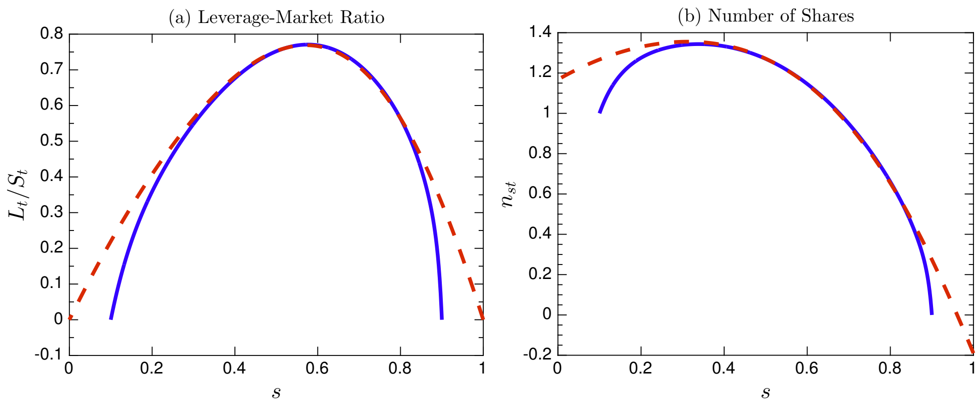

Figure 3
Leverage and stock holdings of optimistic and less risk averse investor B
Panels (a) and (b) depict optimistic and less risk averse investor B's leverage/market price ratio L_t/S_t and the number of shares n_{St}, respectively, as functions of consumption share $s_t = c^*_{At}/D_t$. The solid and dashed lines correspond to constrained and unconstrained economies, respectively.

hence, B lacks collateral and cannot borrow. When $s = \underline{s}$, investor A's financial wealth is zero and the labor income $l_A D_t \Delta t$ is infinitesimally small in the continuous-time limit. The liquidity dries up because investor A cannot supply credit. The leverage cycles are present only in the constrained economy. They do not occur in the unconstrained economy where the state variable s is non-stationary and gradually converges to 0 or 1.

Panel (b) presents the number of stocks held by investor B. Consider first the unconstrained economy where the labor income is pledgeable. From panel (b) we observe that in this economy investor B shorts stocks despite being more optimistic than investor A when consumption share s is close to 1. The intuition is that in bad times, following a sequence of negative shocks to output, investor B shorts stocks to finance consumption and backs short positions by the pledgeable labor income. The stream of labor income $l_B D_t \Delta t$ is equivalent to dividends from holding $\widehat{n}_B = l_B/(1 - l_A - l_B)$ units of non-tradable shares in the Lucas tree. Short-selling allows the investor to circumvent the non-tradability of labor income and freely adjust the effective share $\hat{n}_B + n_{B,St}$ in the Lucas tree. Overcoming the non-tradability of labor incomes makes this economy similar to the non-stationary unconstrained economy where investors can freely trade shares in the Lucas tree. The financial wealth can then become negative. The collateral requirement imposes non-negative wealth constraint, which precludes investor B from shorting. The trading strategy of investor A equals $1 - n^*_{Bt}$ in equilibrium and can be analyzed similarly. Investor A also has an additional motive to short stocks due to being more pessimistic than investor B.

Figure 4 depicts the interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio Ψ, and excess stock return volatility $(\sigma_t - \sigma_D)/\sigma_D$ in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) shows the interest rates r_t.[8] The interest rate declines sharply when the economy enters into an anxious state close to the boundary $\overline{s}$ where even a small possibility of a crisis next period makes the constraint of investor B binding. The intuition is as follows. In the unconstrained economy, a crisis around state $\overline{s}$ generates wealth transfer to the pessimistic and more risk averse investor A and increases her consumption share s above $\overline{s}$. In the constrained economy, consumption share s is capped by $\overline{s}$. Consequently, following a crisis, investor A's marginal utility $(c_A^*)^{-\gamma_A}$ is higher in the constrained than in the unconstrained economy. As a result, investor A is more willing to smooth consumption in the constrained economy, and hence, the interest rate declines due to the precautionary savings motive. In particular, the investor buys more bonds, which drives interest rates down. Panel (b) of Figure 4 shows that the Sharpe ratio increases to compensate investor A for buying risky assets from investor B.

Our results on interest rates and Sharpe ratios indicate that the rare crises and collateral requirements reinforce the effects of each other. In particular, the decreases in interest rates and increases in Sharpe ratios during anxious times arise only when both the crises and the constraints (10) are simultaneously present. Removing the constraint but keeping the crisis risk increases the interest rate and decreases the Sharpe ratio. Equation (41) for the interest rate and equation (42) for the risk premium show that removing the crisis risk (i.e., setting $\lambda = \lambda_B$=0) but keeping the constraint leads to r_t and $\mu_t - r_t$ which are the same as in the frictionless economy when $\underline{v} < v_t < \overline{v}$, consistent with findings in Detemple and Serrat (2003). Absent any crises, the constraints affect r_t and $\mu_t - r_t$ only at the boundaries of the state-space, as shown in Section 3.1.

From panel (c), we observe that the collateral requirements give rise to higher price-dividend ratio Ψ than in the unconstrained economy, $\Psi_t^{constr} - \Psi_t^{unc} > 0$, as proven in Proposition 2. The increases in ratio Ψ are larger around the boundaries $\underline{s}$ and $\overline{s}$, which makes ratio Ψ a U-shaped function of s sensitive to small shocks close to boundaries. The U-shape is a robust phenomenon that does not require rare crises or investors that differ both in risk aversions and beliefs. When both investors have identical risk aversions $\gamma_A = \gamma_B = 1$ but different beliefs and there is no crisis risk (i.e., $\lambda_A = \lambda_B = 0$), the U-shape

[8]We exclude the singularities in the dynamics of r_t and focus on the dynamics in the unconstrained region because the economy spends an infinitesimal amount of time at the boundaries.

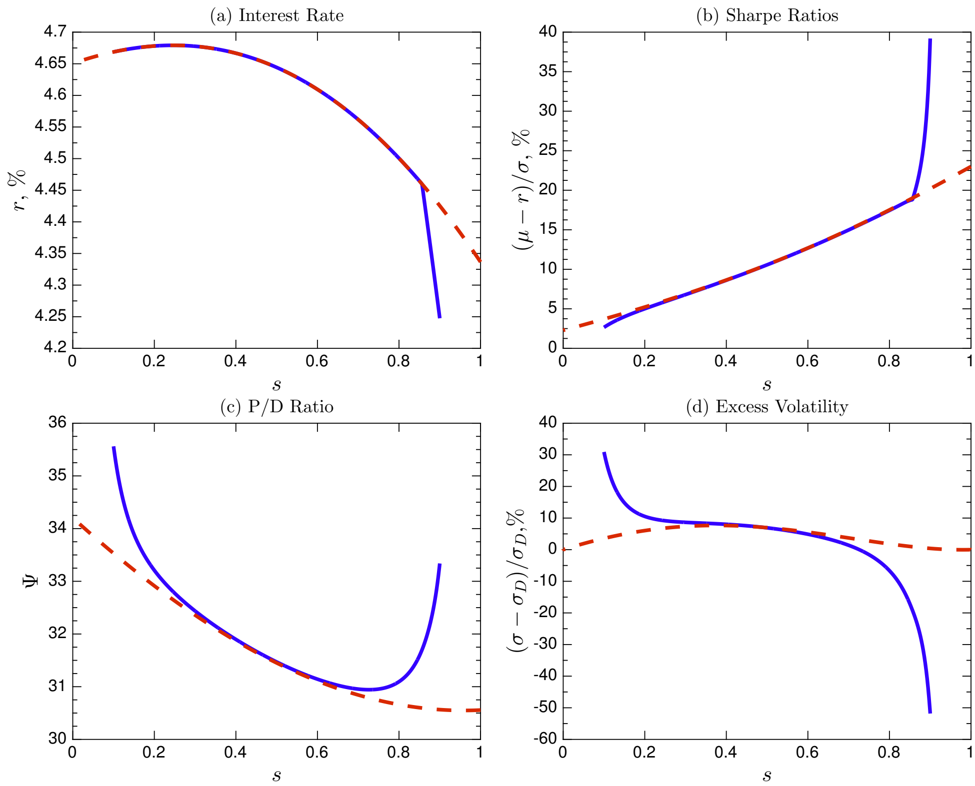

Figure 4
Equilibrium processes
Panels (a)–(d) show interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio Ψ_t, and excess volatility $(\sigma_t - \sigma_D)/\sigma_D$ as functions of $s_t = c^*_{At}/D_t$ for the constrained (solid lines) and unconstrained (dashed lines) economies.

is an analytical result that follows from the closed-form expression (40) for ratio Ψ. This ratio remains U-shaped when investors have different risk aversions but identical beliefs.

The intuition for the U-shape is as follows. Suppose, consumption share s is close to the boundary $\overline{s}$, where investor B's constraint is likely to bind but investor A is unconstrained. Because investor A's constraint is loose the state price density ξ_{At} is proportional to investor A's marginal utility $(c^*_{At})^{-\gamma_A}$. In the constrained economy the consumption share of investor A is capped by $\overline{s} < 1$ whereas in the unconstrained economy it can increase above $\overline{s}$. Therefore, the marginal utility of investor A and, hence, the state price density are expected to be higher in the constrained than in the unconstrained economy. Consequently, stocks are more valuable in the constrained economy around the boundary $\overline{s}$. The intuition around $\underline{s}$ can be analyzed in a similar way. An additional economic force contributing to higher stock price is that the stock can be used as collateral that helps relax the constraint, which gives rise to a premium. This force is explored in Section 4.2.

The results on panel (d) demonstrate that the constraint makes volatility more procyclical, reduces volatility in bad times (around $\overline{s}$) and increases it in good times (around $\underline{s}$). This is because U-shaped price-dividend ratio in the constrained economy is more procyclical in good times (i.e., around $\underline{s}$) and more countercyclical in bad times (i.e., around $\overline{s}$) than in the unconstrained economy. Stock price $S_t = \Psi_t D_t$ is more volatile in good times (around $\underline{s}$) because both Ψ and D_t change in the same direction, and is less volatile in bad times (around $\overline{s}$) because Ψ and D_t change in opposite directions and partially offset the effects of each other. Lower volatility in bad times is in line with the previous literature on the effects of portfolio constraints on asset prices (e.g., Chabakauri, 2013, 2015; Brunnermeier and Sannikov, 2014, among others). The empirical literature finds that the volatility tends to be higher in bad times (e.g., Schwert, 1989). However, high volatility can be explained by high uncertainty about the economic growth and learning effects in bad times (e.g., Veronesi, 1999), which are absent in our model.

Boundary conditions (35) allow us to explore volatility σ_t near the boundaries $\underline{s}$ and $\overline{s}$ using closed form expressions in Corollary 2 below.[9]

Corollary 2 (Stock return volatility at the boundaries). *Stock return volatility in normal times σ_t satisfies the following boundary conditions:*

$$\sigma(\underline{s}) = \sigma_D + \frac{\gamma_B \underline{s} \widehat{\sigma}_v}{\gamma_A(1-\underline{s}) + \gamma_B \underline{s}} > \sigma_D, \quad \sigma(\overline{s}) = \sigma_D - \frac{\gamma_A(1-\overline{s})\widehat{\sigma}_v}{\gamma_A(1-\overline{s}) + \gamma_B \overline{s}} < \sigma_D. \tag{45}$$

By continuity, inequalities (45) also hold in a vicinity of the boundaries. Panel (d) shows that volatility σ_t is very steep at the boundaries: it spikes close to $\underline{s}$ and crashes close to $\overline{s}$, consistent with Corollary 2. It also evolves in three regimes of low, medium, and high volatility, which resembles volatility clustering documented in the empirical literature (e.g., Bollerslev, 1987). The distribution of consumption share s on Figure 2 implies that the economy persists in these clusters for some time.

Figure 5 plots the simulated dynamics of P/D ratio and stock return volatility over a period of 50 years. Consistent with our discussion above, the dynamics of P/D ratio on panel (a) exhibits intervals of booms and busts around the times when the collateral requirements become binding. These intervals resemble periods of inflating and deflating bubbles in the economy. The volatility σ on panel (b) evolves in clusters of high and low

[9] We observe that $\sigma_t(1+r_t\Delta t) \neq 0$, and hence, as shown in Proposition B.1, the matrix of asset payoffs is invertible under our calibration, which is assumed in the beginning of Section 3.

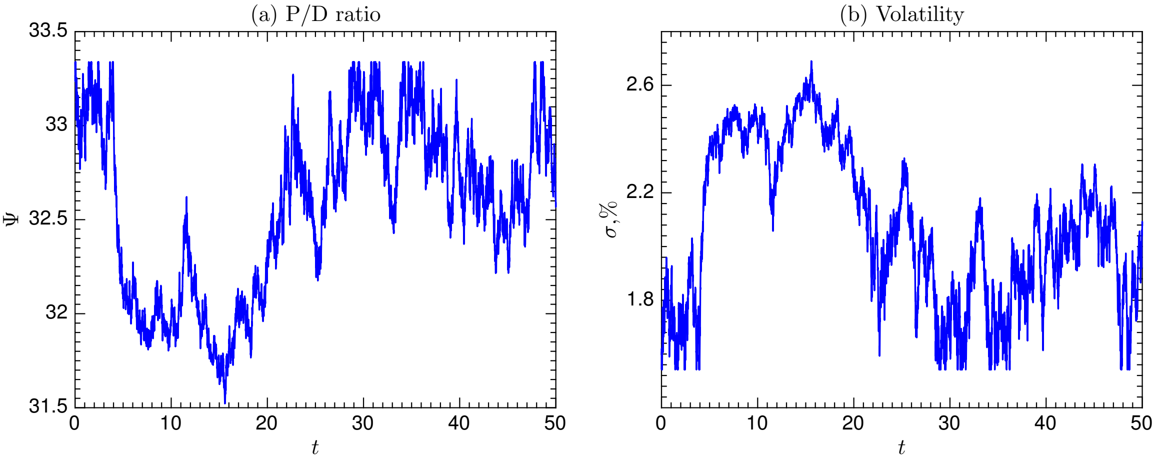

Figure 5
Simulated P/D ratio Ψ and stock return volatility σ over time
Panels (a) and (b) show the spikes and crashes of simulated P/D ratio and volatility σ, and clustering of volatility σ over the period of 50 years.

volatility, as explained above.

4.2. Collateral liquidity premium

In this section, we measure the liquidity premium of stocks over labor income arising because stocks can be used as collateral. We consider a marginal representative investor i that does not affect asset prices and characterize this investor's shadow indifference price $\widehat{S}_{it}$ of labor income. We define $\widehat{S}_{it}$ as the price such that exchanging marginal Δl_i units of labor income for $\widehat{S}_{it}\Delta l_i$ units of wealth leaves the investor's utility unchanged. Consider the investor's value function $V_i(W_{it}, v_t; l_i)$ satisfying the dynamic programming equation (20) subject to constraints (21) and (22). Price $\widehat{S}_{it}$ is the solution of equation $V_i(W^*_{it}, v_t; l_i) = V_i(W^*_{it} + \widehat{S}_{it}\Delta l_i, v_t; l_i - \Delta l_i)$ when $\Delta l_i \to 0$. In the limit, we find:

$$\widehat{S}_{it} = \frac{\partial V_i(W^*_{it}, v_t; l_i)/\partial l_i}{\partial V_i(W^*_{it}, v_t; l_i)/\partial W_{it}}. \tag{46}$$

The definition of shadow indifference price $\widehat{S}_{it}$ comes from the literature on the valuation of derivative securities in incomplete markets (e.g., Davis, 1997).

The labor incomes $l_i D_t \Delta t$ are proportional to dividends $(1 - l_A - l_B)D_t\Delta t$. Therefore, if claims on labor incomes were tradable and pledgeable, shadow price $\widehat{S}_{it}$ would have been equal to $S_t/(1 - l_A - l_B)$. However, labor incomes are non-tradable and non-pledgeable.

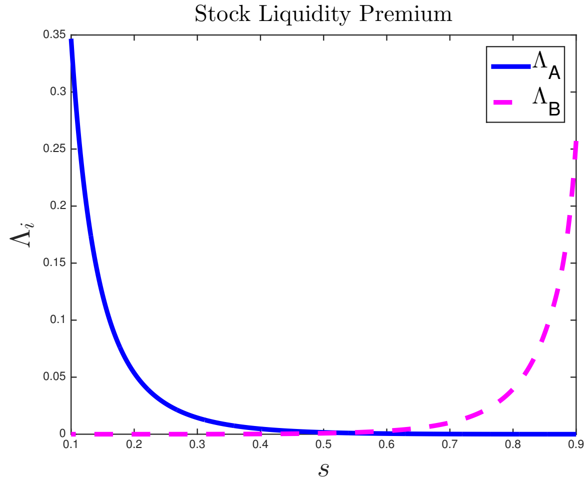

Figure 6
Collateral liquidity premia from the view of investors A and B
The Figure shows the collateral liquidity premia (47) of stocks over non-pledgeable labor incomes from the view of investors A and B.

Hence, from the view of investor i, the stock enjoys liquidity premium, which we define as

$$\Lambda_{it} = \frac{S_t/(1 - l_A - l_B) - \widehat{S}_{it}}{S_t/(1 - l_A - l_B)}. \tag{47}$$

We find derivatives in equation (46) using the envelope theorem. Then, we derive prices $\widehat{S}_{it}$ and show that premia (47) are positive and large. Proposition 5 reports our results.

Proposition 5 (Shadow prices and the liquidity premium). *In the limit $\Delta t \to 0$, investor i's shadow price of a unit of labor income is given by:*

$$\widehat{S}_{it} = \widehat{\Psi}_i(v; -\gamma_A) s(v)^{\gamma_A} D_t, \quad i = A, B, \tag{48}$$

where $\widehat{\Psi}_i(v;\theta)$ satisfies differential-difference equation (34) *subject to the following boundary conditions for investors A and B*

$$\widehat{\Psi}'_A(\underline{v};\theta) = 0, \qquad \widehat{\Psi}'_A(\overline{v};\theta) = 0, \tag{49}$$

$$\widehat{\Psi}'_B(\underline{v};\theta) = \widehat{\Psi}_B(\underline{v};\theta), \quad \widehat{\Psi}'_B(\overline{v};\theta) = \widehat{\Psi}_B(\overline{v};\theta). \tag{50}$$

The investors' liquidity premia for stocks Λ_A and Λ_B are positive, and hence,

$$S_t/(1 - l_A - l_B) > \widehat{S}_{At}, \quad S_t/(1 - l_A - l_B) > \widehat{S}_{Bt}. \tag{51}$$

The premium $\Lambda_{it} > 0$ arises because the stock can be used as a collateral whereas the labor income cannot. The non-tradability of labor income per se does not give rise to the liquidity premium. Intuitively, as discussed in Section 4.1, in an unconstrained economy with fully pledgeable labor income the investors can circumvent the non-tradability of labor income by establishing short positions in the stock which are backed by this pledgeable income. We further remark that the shadow prices and liquidity premia can be found in closed form, similar to stock prices in Section 3, but we do not present them for brevity.

Figure 6 plots the liquidity premia (47) for the same calibrated parameters as in Section 4.1. We observe that investors A and B have different valuations of their labor incomes due to differences in preferences and beliefs. Their premia Λ_i are close to zero when the investors are far away from the boundaries where their respective collateral requirements become binding. The premia increase up to 35% close to the boundaries where the stock is more valuable for the purposes of relaxing the constraints. Large premia Λ_{it} imply the economic significance of stock pledgeability.

5. Conclusion

We develop a parsimonious and tractable theory of asset pricing under collateral requirements. We show that requiring investors to collateralize their trades has significant effects on asset prices and their moments. The constraints decrease interest rates and increase Sharpe ratios when optimistic investors are close to default boundaries. They also increase price-dividend ratios, amplify volatilities in good states and dampen them in bad states. Hence, collateral requirements emerge as viable instruments for stabilizing markets in bad times. The tractability of our model allows us to obtain asset prices and the distributions of consumption shares in closed form.

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Appendix A: Proofs

Lemma A.1 (Change of variable). *Let* $\widehat{n}_i = k_i l_i/(1 - l_A - l_B)$. *Maximization of expected discounted utility* (7) *subject to budget constraints* (8) *and* (9), *and constraint* (11) *is equivalent to maximizing* (7) *with respect to* c_{it}, b_{it} *and* $\widetilde{n}_{it}$ *subject to the following set of constraints:*

$$\widetilde{W}_{it} + l_i D_t \Delta t = c_{it}\Delta t + b_{it} B_t + \widetilde{n}_{it}(S_t, P_t)^\top, \tag{A1}$$

$$\widetilde{W}_{i,t+\Delta t} = b_{it} + \widetilde{n}_{it}\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t,\, \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}}\Big)^\top, \tag{A2}$$

$$\widetilde{W}_{i,t+\Delta t} \geq 0, \tag{A3}$$

where $\widetilde{W}_{it} = W_{it} + \widehat{n}_i S_t$ *and* $\widetilde{W}_{i,t+\Delta t} = W_{i,t+\Delta t} + \widehat{n}_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t})$.

Proof of Lemma A.1. Substituting $n_{it} = \widetilde{n}_{it} - (\widehat{n}_i, 0)$ into (8) and (9), we obtain constraints (A1) and (A2). Rewriting constraint (11) in terms of variable $\widetilde{W}_{i,t+\Delta t}$, we obtain (A3). Finally, we note that $\widetilde{W}_{it} = W_{it} + \widehat{n}_i S_t$ is worth $\widetilde{W}_{i,t+\Delta t}$ next period. Hence, (A1) and (A2) can be seen as self-financing budget constraints. ■

Proof of Lemma 1.

1) We start by demonstrating the equivalence of the dynamic (8)–(9) and static budget constraints (21). Multiplying equation (9) by $\xi_{i,t+\Delta t}/\xi_{it}$, taking expectation operator $\mathbb{E}_t^i[\cdot]$ on both sides, and using equations (16)–(18) for asset prices, we obtain:

$$\mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t}\right] = b_{it} B_t + n_{it}(S_t, P_t)^\top. \tag{A4}$$

From the budget constraint equation (8), we observe that the right-hand side of (A4) equals $W_{it} + l_i D_t \Delta t$, and hence, we obtain the static budget constraint (21). Conversely, if there exists $W_{i,t+\Delta t}$ satisfying constraints (21) and (22) there exist trading strategies b_{it} and n_{it} that replicate $W_{i,t+\Delta t}$ because the underlying market is effectively complete (i.e., the payoff matrix is invertible). Then, rewriting the optimization problem (7) in a recursive form, we obtain the dynamic programming equation (20) for the value function.

2) Consider wealth levels W_{it} and $\widehat{W}_{it}$. Let $\{c_{it}^*, b_{it}^*, n_{it}^*\}$ and $\{\widehat{c}_{it}^*, \widehat{b}_{it}^*, \widehat{n}_{it}^*\}$ be optimal consumptions and portfolios that correspond to W_{it} and $\widehat{W}_{it}$, respectively, and satisfy constraints (8)–(10). For any $\alpha \in [0,1]$, policies $\{\alpha\widehat{c}_{it}^* + (1-\alpha)c_{it}^*, \alpha\widehat{b}_{it}^* + (1-\alpha)b_{it}^*, \alpha\widehat{n}_{it}^* + (1-$

$\alpha)n_{it}^*\}$ are admissible for wealth $\alpha W_{it} + (1-\alpha)\widehat{W}_{it}$. By concavity of CRRA utilities:

$$\begin{aligned} V_i(\alpha W_{it} + (1-\alpha)\widehat{W}_{it}, v_t; l_i) &\geq \sum_{\tau=t}^{\infty} u_i(\alpha \widehat{c}_{it}^* + (1-\alpha)c_{it}^*) \\ &\geq \sum_{\tau=t}^{\infty} (\alpha u_i(\widehat{c}_{it}^*) + (1-\alpha)u_i(c_{it}^*)) \\ &= \alpha V_i(W_{it}, v_t; l_i) + (1-\alpha)V_i(\widehat{W}_{it}, v_t; l_i). \end{aligned} \tag{A5}$$

Therefore, $V_i(W_{it}, v_t; l_i)$ is a concave function of wealth.

3) Consider the following Lagrangian:

$$\begin{aligned} \mathscr{L} &= u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i\Big[V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)\Big] \\ &+ \eta_{it}\left(W_{it} + l_i D_t \Delta t - c_{it}\Delta t - \mathbb{E}_t^i\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t}\Big]\right) + \mathbb{E}_t^i\Big[e^{-\rho\Delta t}\ell_{i,t+\Delta t}W_{i,t+\Delta t}\Big)\Big], \end{aligned} \tag{A6}$$

where multiplier $\ell_{i,t+\Delta t}$ satisfies the complementary slackness condition $\ell_{i,t+\Delta t}W_{i,t+\Delta t} = 0$. Differentiating the Lagrangian (A6) with respect to c_{it} and $W_{i,t+\Delta t}$, we obtain:

$$u_i'(c_{it}^*) = \eta_{it}, \tag{A7}$$

$$e^{-\rho\Delta t}\left(\frac{\partial V_i(W_{t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} + \ell_{i,t+\Delta t}\right) = \eta_{it}\frac{\xi_{i,t+\Delta t}}{\xi_{it}}. \tag{A8}$$

By the envelope theorem (e.g, Back (2010, p.162)):

$$\frac{\partial V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} = u_i'(c_{i,t+\Delta t}^*). \tag{A9}$$

Substituting the partial derivative of the value function (A9) and the marginal utility (A7) into equation (A8), and then dividing both sides of the equation by $u_i'(c_{it}^*)$, we obtain the expression for the SPD (23). ■

Proof of Proposition 1.

Step 1. Consider the case when constraints do not bind, and hence, $\ell_{i,t+\Delta t} = 0$. Then, using equation (13) for the state variable v_t and the first order conditions (23), we obtain:

$$v_{t+\Delta t} - v_t = \ln\left(\frac{(c_{A,t+\Delta t}^*/c_{At}^*)^{-\gamma_A}}{(c_{B,t+\Delta t}^*/c_{Bt}^*)^{-\gamma_B}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) = \ln\left(\frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right).$$

From the above equation and the change of measure equation (19), which relates SPDs $\xi_{A,t+\Delta t}$ and $\xi_{B,t+\Delta t}$, we obtain the dynamics of v_t when constraints do not bind:

$$v_{t+\Delta t} - v_t = \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right). \tag{A10}$$

Let $\overline{v}$ and $\underline{v}$ be the boundaries satisfying Equations (24), at which the constraints of investors A and B bind, respectively. Let investor A's constraint be binding so that $v_{t+\Delta t} = \overline{v}$, and hence, $\ell_{A,t+\Delta t} \geq 0$. Using Equation (13) for v_t, first order conditions (23), and $\ell_{A,t+\Delta t} \geq 0$, we obtain:

$$\begin{aligned}\overline{v} - v_t &\leq \ln\left(\frac{((c^*_{A,t+\Delta t})^{-\gamma_A} + \ell_{A,t+\Delta t})/(c^*_{At})^{-\gamma_A}}{(c^*_{B,t+\Delta t}/c^*_{Bt})^{-\gamma_B}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\ &= \ln\left(\frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) = \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right).\end{aligned} \tag{A11}$$

Similarly, for $v_{t+\Delta t} = \underline{v}$ we obtain that $\underline{v} - v_t \geq \ln\Big(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\Big(D_{t+\Delta t}/D_t\Big)^{\gamma_A-\gamma_B}\Big)$. The latter two inequalities imply that when the constraint binds $v_{t+\Delta t}$ is given by:

$$v_{t+\Delta t} = \max\left\{\underline{v}; \min\left\{\overline{v}; v_t + \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right)\right\}\right\}. \tag{A12}$$

We observe that (A12) is also satisfied in the unconstrained case where $\underline{v} < v_{t+\Delta t} < \overline{v}$. It remains to prove that v_t does not escape $[\underline{v}, \overline{v}]$ interval. Consider a marginal investor of type A. We guess that v_t follows dynamics (A12) and verify that the consumption choice of investor A indeed implies this dynamics. The analysis for investor B is similar.

We have shown above that v_t satisfies inequality (A11) when investor A is constrained. Now, we show the opposite: investor A is constrained when v_t satisfies (A11). Hence, $v_{t+\Delta t}$ cannot exceed $\overline{v}$. Consider v_t such that $v_t + \ln\left(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\,(D_{t+\Delta t}/D_t)^{\gamma_A-\gamma_B}\right) > \overline{v}$ for some $\omega_{t+\Delta t}$ and $v_t \in (\underline{v}, \overline{v})$. Because $\underline{v} < v_t < \overline{v}$, investor A consumes $c^*_{At} = s(v_t)D_t$, as shown above. We show that the constraint of investor A binds and $c^*_{A,t+\Delta t} = s(\overline{v})D_{t+\Delta t}$. This consumption level confirms that $v_{t+\Delta t} = \overline{v}$ is indeed an equilibrium outcome.

Consider the constraint of investor A at date t in the state $\omega_{t+\Delta t}$ where $v_{t+\Delta t} = \overline{v}$:

$$W_{A,t+\Delta t} \geq 0 \equiv \Phi_A(\overline{v})D_{t+\Delta t}, \tag{A13}$$

where the last equality holds by the definition of $\overline{v}$. Using the concavity of the value function, proven in Lemma 1, and condition (A9) from the envelope theorem, we obtain:

$$u'_A(c^*_{A,t+\Delta t}) = \frac{\partial V_A(W_{A,t+\Delta t}, \overline{v}; l_A)}{\partial W} \leq \frac{\partial V_A(\Phi_A(\overline{v})D_{t+\Delta t}, \overline{v}; l_A)}{\partial W} = u'_A(s(\overline{v})D_{t+\Delta t}). \tag{A14}$$

Because $u'_i(c)$ is a decreasing function, we find that $c^*_{A,t+\Delta t}/D_{t+\Delta t} \geq s(\overline{v})$.

Investor B is unconstrained when $v_{t+\Delta t} = \overline{v}$, and hence, has SPD

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = e^{-\rho\Delta t}\left(\frac{c^*_{B,t+\Delta t}}{c^*_{Bt}}\right)^{-\gamma_B} = e^{-\rho\Delta t}\left(\frac{(1-s(\overline{v}))D_{t+\Delta t}}{(1-s(v_t))D_t}\right)^{-\gamma_B}. \tag{A15}$$

From the change of measure equation (19) and the FOC (23), the SPD of investor A is

$$\begin{aligned}\frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= \frac{\xi_{B,t+\Delta t}}{\xi_{Bt}}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \\ &= e^{-\rho\Delta t}\frac{(c^*_{A,t+\Delta t})^{-\gamma_A}+\ell_{A,t+\Delta t}}{(c^*_{At})^{-\gamma_A}}.\end{aligned} \tag{A16}$$

From (A16) and (A15), we find the Lagrange multiplier:

$$\begin{aligned}\frac{l_{A,t+\Delta t}}{(c^*_{A,t+\Delta t})^{-\gamma_A}} &= \left(\frac{c^*_{A,t+\Delta t}}{c^*_{At}}\right)^{\gamma_A}\left(\frac{(1-s(\overline{v}))D_{t+\Delta t}}{(1-s(v_t))D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}-1 \\ &\geq \left(\frac{s(\overline{v})D_{t+\Delta t}}{s(v_t)D_t}\right)^{\gamma_A}\left(\frac{(1-s(\overline{v}))D_{t+\Delta t}}{(1-s(v_t))D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}-1 \\ &= \left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right)e^{v_t-\overline{v}}-1>0.\end{aligned}$$

The first inequality follows from the fact that $c^*_{A,t+\Delta t}\geq s(\overline{v})D_{t+\Delta t}$ we proved above. The second equality holds by the definition of state variable (13). The second inequality comes from the assumption that $v_t+\ln\left(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\,(D_{t+\Delta t}/D_t)^{\gamma_A-\gamma_B}\right)>\overline{v}$. Hence, the Lagrange multiplier $l_{A,t+\Delta t}$ is strictly positive. From the complementary slackness condition, the constraint (A13) must be binding. Therefore, inequality (A14) becomes an equality, and hence, $c^*_{A,t+\Delta t}=s(\overline{v})D_{t+\Delta t}$.

Step 2. We now look for coefficients μ_v, σ_v and J_v such that:

$$\begin{aligned}\mu_v\Delta t+\sigma_v\Delta w_t+J_v\Delta j_t &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\ &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\right)+(\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t+\sigma_D\Delta w_t+J_v\Delta j_t).\end{aligned} \tag{A17}$$

We write identity (A17) in each of the states $\omega_{t+\Delta t}\in\{\omega_1,\omega_2,\omega_3\}$ and obtain the following system of three linear equations with three unknowns μ_v, σ_v and J_v:

$$\begin{aligned}\mu_v\Delta t+\sigma_v\sqrt{\Delta t} &= \ln\left(\frac{(1-\lambda_B\Delta t)(1+\delta\Delta t)}{1-\lambda\Delta t}\right)+(\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t+\sigma_D\sqrt{\Delta t}), \\ \mu_v\Delta t-\sigma_v\sqrt{\Delta t} &= \ln\left(\frac{(1-\lambda_B\Delta t)(1-\delta\Delta t)}{1-\lambda\Delta t}\right)+(\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t-\sigma_D\sqrt{\Delta t}), \\ \mu_v\Delta t+J_v &= \ln\left(\frac{\lambda_B}{\lambda}\right)+(\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t+J_D).\end{aligned} \tag{A18}$$

Solving the above system, we obtain μ_v, σ_v and J_v reported in Proposition 1.

Step 3. Finally, we show that the boundaries are reflecting for a sufficiently small Δt. Suppose, two conditions are satisfied: $\mu_v \Delta t - \sigma_v \sqrt{\Delta t} < 0$ and $\mu_v \Delta t + \sigma_v \sqrt{\Delta t} > 0$. Then, the boundaries are reflecting: 1) if $v_t = \overline{v}$, then $v_{t+\Delta t} = \overline{v} + \mu_v \Delta t - \sigma_v \sqrt{\Delta t} < \overline{v}$ with positive probability; 2) if $v_t = \underline{v}$, then $v_{t+\Delta t} = \underline{v} + \mu_v \Delta t + \sigma_v \sqrt{\Delta t} > \underline{v}$ with positive probability. It can be easily verified that as $\Delta t \to 0$, $\mu_v \to \widehat{\mu}_v$ and $\sigma_v \to \widehat{\sigma}_v$, where $\widehat{\mu}_v$ and $\widehat{\sigma}_v$ are constants given by equations (36) and (37), respectively. Because $\sigma_v > 0$ and $\sqrt{\Delta t}$-terms dominate Δt-terms for small Δt, we find that $\mu_v \Delta t - \sigma_v \sqrt{\Delta t} < 0$ and $\mu_v \Delta t + \sigma_v \sqrt{\Delta t} > 0$ for all sufficiently small Δt. Hence, the boundaries are reflecting. ■

Proof of Proposition 2. 1) First, we derive the SPD ξ_{At} under the correct beliefs of investor A. When investor A's constraint does not bind, substituting $c^*_{At} = s(v_t) D_t$ into the first order condition (23) we find that

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left(\frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left(\frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A}. \tag{A19}$$

Equation (A19) is consistent with SPD (29) because when the constraint does not bind $v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t < \overline{v}$, and hence the exponential term in (29) vanishes.

When the constraint of investor A binds, the constraint of investor B is loose: the constraints cannot bind simultaneously lest to violate the market clearing conditions. Therefore, the ratio $\xi_{B,t+\Delta t}/\xi_{Bt}$ is given by FOC (23) for investor B with $\ell_B = 0$. Using equation (19), we rewrite the latter SPD under the correct beliefs of investor A:

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left(\frac{1 - s(v_{t+\Delta t})}{1 - s(v_t)} \right)^{-\gamma_B} \left(\frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_B} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}. \tag{A20}$$

Next, from equation (14) for consumption share s we find that $(1 - s_t)^{-\gamma_B} = e^{-v_t} s_t^{-\gamma_A}$. Substituting the latter equality into equation (A20), and also using equation (A17) for the increment $v_{t+\Delta t} - v_t$, we obtain:

$$\begin{aligned} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= e^{-\rho \Delta t} \left(\frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left(\frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} e^{v_t - v_{t+\Delta t}} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \left(\frac{D_{t+\Delta t}}{D_t} \right)^{\gamma_A - \gamma_B} \\ &= e^{-\rho \Delta t} \left(\frac{s(v_{t+\Delta t})}{s(v_t)} \right)^{-\gamma_A} \left(\frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \exp\{v_t - v_{t+\Delta t} + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}. \end{aligned} \tag{A21}$$

The fact that the constraint of investor A is binding means that $v_{t+\Delta t} = \overline{v}$ and $v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \geq \overline{v}$ (because otherwise $v_{t+\Delta t} < \overline{v}$, and hence, the constraint does not bind). Therefore, the exponential term $\exp(v_t - v_{t+\Delta t})$ in equation (A21) can be replaced with

$\exp(\max\{0, v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \overline{v}\})$. When the constraint of investor A does not bind the latter term vanishes and we obtain equation (A19). Therefore, both equations (A19) and (A21) are summarized by equation (29) for $\xi_{A,t+\Delta t}/\xi_{At}$.

2) The equation (30) for the price-dividend ratio can be easily obtained by substituting $S_t = (1 - l_A - l_B)\Psi_t$ into equation (17) for stock prices in terms of SPD and then dividing both sides by D_t. The equation (31) for the wealth-aggregate consumption ratio can be obtained by substituting $W_{it} = D_t \Phi_{it}$ into the equation for the static budget constraint (21) and dividing both sides by D_t.

To derive the matrix of asset returns, we rewrite the stock price dynamics as follows:

$$\frac{\Delta S_t + D_{t+\Delta t}\Delta t}{S_t} = \mu_t \Delta t + \sigma_t \Delta w_t + J_t \Delta j_t.$$

Therefore, the matrix of time-$(t + \Delta t)$ bond, stock and insurance returns is given by:

$$\begin{pmatrix} 1 + r_t\Delta t & 1 + \mu_t \Delta t + \sigma_t\sqrt{\Delta t} & 0 \\ 1 + r_t\Delta t & 1 + \mu_t \Delta t - \sigma_t\sqrt{\Delta t} & 0 \\ 1 + r_t\Delta t & J_t & 1/P_t \end{pmatrix}.$$

It is easy to see that the determinant of the above matrix is given by $-2\sigma_t \Delta t(1+r_t\Delta t)/P_t$. Therefore, the matrix is non-degenerate when $\sigma_t(1 + r_t\Delta t) \neq 0$.

3) In the unconstrained economy, the state variable v_t^{unc} follows dynamics:

$$v_t^{unc} = \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t. \tag{A22}$$

Define processes $U_{t+\Delta t} = U_t + \Delta U_t$ and $V_{t+\Delta t} = V_t + \Delta V_t$, where increments are given by:

$$\Delta U_t = \max\{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \overline{v}\}, \quad \Delta V_t = \max\{0; \underline{v} - v_t - \mu_v \Delta t - \sigma_v \Delta w_t - J_v \Delta j_t\}. \tag{A23}$$

The process for the state variable in the constrained economy can be rewritten as

$$v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t + \Delta V_t - \Delta U_t. \tag{A24}$$

If the state variables have the same value at time 0, i.e., $v_0 = v_0^{unc}$, we obtain:

$$v_t = v_t^{unc} + V_t - U_t \tag{A25}$$

Next, we prove that the SPD is higher in the constrained economy.

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\exp(\Delta U_t), \tag{A26}$$

$$\frac{\xi^{unc}_{A,t+\Delta t}}{\xi^{unc}_{At}} = e^{-\rho\Delta t}\left(\frac{s(v^{unc}_{t+\Delta t})}{s(v^{unc}_t)}\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}. \tag{A27}$$

Iterating the above equations, we obtain:

$$\frac{\xi_{At}}{\xi_{A0}} = e^{-\rho t}\left(\frac{s(v_t)}{s(v_0)}\frac{D_t}{D_0}\right)^{-\gamma_A}\exp(U_t),$$

$$\frac{\xi^{unc}_{At}}{\xi^{unc}_{A0}} = e^{-\rho t}\left(\frac{s(v^{unc}_t)}{s(v_0)}\frac{D_t}{D_0}\right)^{-\gamma_A}.$$

By the definition of $s(v)$ in equation (14), we have $e^v = (1-s(v))^{\gamma_B}\cdot s(v)^{-\gamma_A}$. Hence,

$$\begin{aligned}\frac{\xi_{At}/\xi_{A0}}{\xi^{unc}_{At}/\xi^{unc}_{A0}} &= \left(\frac{s(v_t)}{s(v^{unc}_t)}\right)^{-\gamma_A}\exp(U_t)\\ &= \left(\frac{s(v^{unc}_t+V_t-U_t)}{s(v^{unc}_t)}\right)^{-\gamma_A} e^{v^{unc}_t}e^{-(v^{unc}_t-U_t)}\\ &\ge s(v^{unc}_t-U_t)^{-\gamma_A}e^{-(v^{unc}_t-U_t)}\cdot s(v^{unc}_t)^{\gamma_A}e^{v^{unc}_t}\\ &= (1-s(v^{unc}_t-U_t))^{-\gamma_B}\cdot(1-s(v^{unc}))^{\gamma_B}\ge 1.\end{aligned} \tag{A28}$$

Therefore, we conclude that $\xi_{At}/\xi_{A0} > \xi^{unc}_{At}/\xi^{unc}_{A0}$. The latter inequality and the equation for stock prices (17) imply that $\Psi(v_0) \ge \Psi^{unc}(v_0)$. The proof for the case when time-t variables in the constrained and unconstrained economies coincide is analogous. ■

Proof of Lemma 2. Define the following function in discrete time:

$$\widehat{\Psi}(v_t;\theta) = \mathbb{E}^A_t\left[e^{-\rho\Delta t+\Delta U_t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] + s(v_t)^\theta\Delta t, \tag{A29}$$

where ΔU_t is given by equation

$$\Delta U_t = \max\{0; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\}. \tag{A30}$$

Comparing equation (A29) with equations (30) and (31) for Ψ and Φ_i and using the linearity of equation (A29), it easy to observe that $\Psi(v_t)$ and $\Phi_i(v_t)$ are given by the following equations in terms of $\widehat{\Psi}(v_t;\theta)$:

$$\Psi(v_t) = \widehat{\Psi}(v_t,-\gamma_A)s(v_t)^{\gamma_A} - \Delta t,$$

$$\Phi(v_t) = \Big((\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}})\widehat{\Psi}(v;1-\gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i)\widehat{\Psi}(v;-\gamma_A)\Big)\, s(v)^{\gamma_A}.$$

Taking limit $\Delta t \to 0$, we obtain equations (32) and (33) for $\Psi(v_t)$ and $\Phi_i(v_t)$.

First, we derive the equation for $\widehat{\Psi}(v_t;\theta)$ when v_t belongs to the interior $(\underline{v}, \overline{v})$. For a sufficiently small Δt we have $\Delta U_t = 0$, where ΔU_t is given by (A30). Then, we rewrite the expectation of $(D_{t+\Delta t})/D_t)^{1-\gamma_A}\widehat{\Psi}(v_t;\theta)$ as follows:

$$\begin{aligned}\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] &= (1-\lambda\Delta t)\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\Big|\text{normal}\right] \\ &+\lambda\Delta t\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\Big|\text{crisis}\right].\end{aligned} \tag{A31}$$

Noting that in the crisis $D_{t+\Delta t}/D_t = 1 + \mu_v\Delta t + J_D$ and $v_{t+\Delta t} = \max\{\underline{v}; v_t + \mu_v\Delta t + J_v\}$ and in the normal state $D_{t+\Delta t}/D_t = 1 + \mu_D\Delta t + \sigma_D\Delta w_t$ and $v_{t+\Delta t} = v_t + \mu_v\Delta t + \sigma_v\Delta w_t$, using Taylor expansions for $(D_{t+\Delta t}/D_t)^{1-\gamma_A}$ and $\widehat{\Psi}(v_{t+\Delta t};\theta)$, we find:

$$\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{crisis}\right] = (1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v}; v_t + J_v\};\theta\Big). \tag{A32}$$

$$\begin{aligned}\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{normal}\right] &= \left[1+\left((1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A\sigma_D^2}{2}\right)\Delta t\right]\widehat{\Psi}(v_t;\theta) \\ +\Big(\mu_v + (1-\gamma_A)\sigma_D\sigma_v\Big)\widehat{\Psi}'(v_t;\theta)\Delta t &+ \frac{\sigma_v^2}{2}\widehat{\Psi}''(v_t;\theta)\Delta t + o(\Delta t).\end{aligned} \tag{A33}$$

Substituting (A32)-(A33) into (A29), we obtain:

$$\begin{aligned}\widehat{\Psi}(v_t;\theta) &= \left[1-\left(\lambda+\rho-(1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\Delta t\right]\widehat{\Psi}(v_t;\theta) \\ &+ \Big(\mu_v + (1-\gamma_A)\sigma_D\sigma_v\Big)\widehat{\Psi}'(v;\theta)\Delta t + \frac{\sigma_v^2}{2}\widehat{\Psi}''(v;\theta)\Delta t \\ &+ \lambda(1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v}; v_t + J_v\};\theta\Big)\Delta t + s(v)^\theta\Delta t + o(\Delta t).\end{aligned} \tag{A34}$$

Canceling similar terms, diving by Δt, taking limit $\Delta t \to 0$, and noting that μ_v, σ_v and J_v converge to $\widehat{\mu}_v$, $\widehat{\sigma}_v$ and $\widehat{J}_v$ given by (36)-(38), we obtain equation (34) for $\widehat{\Psi}(v_t;\theta)$.

Next, we derive the boundary conditions for $\widehat{\Psi}(v_t;\theta)$. From equation (25), the state variable dynamics at lower bound is $v_{t+\Delta t} = \underline{v} + \max\{0, \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}$. We use Δv_t to denote the difference of $v_{t+\Delta t}$ and v_t. In this case,

$$\Delta v_t = \max\{0, \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}. \tag{A35}$$

For sufficiently small Δt increment Δv_t is positive only in state ω_1 and is zero otherwise. In state ω_1, $\Delta v_t = \mu_v\Delta t + \sigma_v\sqrt{\Delta t}$. Therefore, the order of $\mathbb{E}_t^A[\Delta v_t]$ is $\sqrt{\Delta t}$, but second order terms involving Δv_t have lower order:

$$\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A\left[\Delta v_t\right]}{\sqrt{\Delta t}}=\frac{\widehat{\sigma}_v}{2},$$
$$\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A\left[(\Delta v_t)^2\right]}{\sqrt{\Delta t}}=\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A\left[\Delta v_t\Delta t\right]}{\sqrt{\Delta t}}=\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A\left[\Delta v_t\Delta w_t\right]}{\sqrt{\Delta t}}=\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A\left[\Delta v_t\Delta j_t\right]}{\sqrt{\Delta t}}=0. \tag{A36}$$

Taylor expansion of $\widehat{\Psi}(v_{t+\Delta t};\theta)$ at $v_t=\underline{v}$ is given by

$$\widehat{\Psi}(v_{t+\Delta t};\theta)=\widehat{\Psi}(\underline{v};\theta)+\widehat{\Psi}'(\underline{v};\theta)\Delta v_t+\frac{1}{2}\widehat{\Psi}''(\underline{v};\theta)\Delta v_t^2+o(\sqrt{\Delta t}). \tag{A37}$$

In subsequent calculations we keep terms with order of $\sqrt{\Delta t}$. Using the above results, we obtain the following expansion:

$$\begin{aligned}&\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right]\\&=\mathbb{E}_t^A\left[(1+\mu_D\Delta t+\sigma_D\Delta w_t+J_v\Delta j_t)^{1-\gamma_A}\left(\widehat{\Psi}(\underline{v};\theta)+\widehat{\Psi}'(\underline{v};\theta)\Delta v_t+\frac{1}{2}\widehat{\Psi}''(\underline{v};\theta)\Delta v_t^2\right)\right]\\&=\widehat{\Psi}(\underline{v};\theta)+\widehat{\Psi}'(\underline{v};\theta)\mathbb{E}_t^A[\Delta v_t]+o(\sqrt{\Delta t}).\end{aligned} \tag{A38}$$

Substituting (A38) into (A29), taking into account that $\Delta U_t=0$ at $v_t=\underline{v}$, and canceling $\widehat{\Psi}(\underline{v};\theta)$ on both sides, we obtain the first boundary condition $\widehat{\Psi}'(\underline{v};\theta)=0$.

At the upper bound $v_t=\overline{v}$ investor A is constrained, and hence, ΔU_t in (A30) is positive. From (25) the state variable at the upper bound is

$$v_{t+\Delta t}=\min\{\overline{v},v_t+\mu_v\Delta t+\sigma_v\Delta w_t+J_v\Delta j_t\}=v_t+\mu_v\Delta t+\sigma_v\Delta w_t+J_v\Delta j_t-\Delta U_t. \tag{A39}$$

The order of $\mathbb{E}_t^A\left[\Delta U_t\right]$ is $\sqrt{\Delta t}$, but second order terms involving ΔU_t have order $o(\sqrt{\Delta t})$. Proceeding in the same way as (A36)-(A38), we arrive at

$$\widehat{\Psi}(\overline{v};\theta)=\widehat{\Psi}(\overline{v};\theta)+\left[\widehat{\Psi}(\overline{v};\theta)-\widehat{\Psi}'(\overline{v};\theta)\right]\mathbb{E}_t^A[\Delta U_t]+o(\sqrt{\Delta t}).$$

Canceling similar terms, taking limit $\Delta t\to 0$, we obtain condition $\widehat{\Psi}(\overline{v};\theta)-\widehat{\Psi}'(\overline{v};\theta)=0$.

Finally, we derive the equations for $\overline{v}$ and $\underline{v}$. Taking limit $\Delta t\to 0$ in equations (24), we find that these equations become: $\Phi_A(\overline{v})=0$, $\Phi_B(\underline{v})=0$. Substituting $\Phi_i(v)$ and $\Psi(v)$ in terms of $\widehat{\Psi}(v;\theta)$ from equations (33) into the latter equations for the boundaries, after some algebra, we obtain equations (39). ■

Proof of Corollary 1. Consider the case $\lambda=\lambda_B=0$ and $\gamma_A=\gamma_B=1$. Then, $s(v)$ solving equation (25) is given by $s(v)=1/(1+e^v)$, $\Psi(v)=\widehat{\Psi}(v)s(v)$, where $\widehat{\Psi}(v)$ solves a special case of equation (34) given by:

$$\frac{\delta^2}{2}\widehat{\Psi}''(v) - \frac{\delta^2}{2}\widehat{\Psi}'(v) - \rho\widehat{\Psi}(v) + 1 + e^v = 0, \tag{A40}$$

subject to boundary conditions (35). It can be easily verified that $\widehat{\Psi}(v) = C_1 e^{\varphi_- v} + C_2 e^{\varphi_+ v} + (1+e^v)/\rho$ satisfies (A40). Substituting $\widehat{\Psi}(v)$ into boundary conditions (35), we obtain the following system for coefficients C_1 and C_2:

$$C_1\varphi_- e^{\varphi_-\underline{v}} + C_2\varphi_+ e^{\varphi_+\underline{v}} + e^{\underline{v}}/\rho = 0; \quad C_1(\varphi_- - 1)e^{\varphi_-\overline{v}} + C_2(\varphi_+ - 1)e^{\varphi_+\overline{v}} - 1/\rho = 0.$$

Solving these equations, we obtain:

$$C_1 = \quad \frac{1}{\rho}\frac{(\varphi_+ - 1)e^{\underline{v}+\varphi_+\overline{v}} + \varphi_+ e^{\varphi_+\underline{v}}}{\varphi_+(\varphi_- - 1)e^{\varphi_-\overline{v}+\varphi_+\underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+\overline{v}+\varphi_-\underline{v}}}, \tag{A41}$$

$$C_2 = -\frac{1}{\rho}\frac{(\varphi_- - 1)e^{\underline{v}+\varphi_-\overline{v}} + \varphi_- e^{\varphi_-\underline{v}}}{\varphi_+(\varphi_- - 1)e^{\varphi_+\underline{v}+\varphi_-\overline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+\overline{v}+\varphi_-\underline{v}}}. \blacksquare \tag{A42}$$

Proof of Proposition 3. From equation (16) for the bond price and the fact that $1 = B_t(1 + r_t\Delta t)$ we find that the riskless interest rate r_t is given by:

$$\begin{aligned} r_t &= \frac{1 - \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t} \\ &= \frac{1 - (1-\lambda\Delta t)\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}] - \lambda\Delta t\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t}, \end{aligned} \tag{A43}$$

where $\xi_{A,t+\Delta t}/\xi_{At}$ is given by equation (29). We separately calculate $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ and $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]$, and then take the limit $\Delta t \to 0$.

We start with the derivation of $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ when $\underline{v} < v_t < \overline{v}$, and hence, by continuity, for a sufficiently small Δt the economy is unconstrained next period, so that $\underline{v} < v_{t+\Delta t} < \overline{v}$. In the unconstrained region $\Delta v_t = \widehat{\mu}_v\Delta t + \widehat{\sigma}_v\Delta w_t$ and the SPD is given by (A19). From the expression for the SPD, using expansions (A52) and (A54), we obtain:

$$\begin{aligned} \mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{normal}\right] &= \mathbb{E}_t\left[\Big((1 + a_t\Delta v_t + b_t(\Delta v_t)^2\Big)\Big(1 - r_{\scriptscriptstyle A}\Delta t - \kappa_{\scriptscriptstyle A}\Delta w_t\Big)|\text{normal}\right] + o(\Delta t) \\ &= \mathbb{E}_t\Big[1 + a_t\Delta v_t + b_t(\Delta v_t)^2 - r_{\scriptscriptstyle A}\Delta t - \kappa_{\scriptscriptstyle A}\Delta w_t - \kappa_{\scriptscriptstyle A} a_t\Delta v_t\Delta w_t\Big|\text{normal}\Big] + o(\Delta t) \\ &= 1 + a_t\widehat{\mu}_v\Delta t + b_t\widehat{\sigma}_v^2\Delta t - r_{\scriptscriptstyle A}\Delta t - \kappa_{\scriptscriptstyle A} a_t\widehat{\sigma}_v\Delta t + o(\Delta t). \end{aligned} \tag{A44}$$

Conditioning on the crisis state, we have:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{crisis}\right] = (1-\rho\Delta t)(1+\mu_D\Delta t + J_D)^{-\gamma_A}\left(\frac{s(\max\{\underline{v}, v_t+\mu_v\Delta t + J_v\})}{s(v_t)}\right)^{-\gamma_A}$$
$$= (1+J_D)^{-\gamma_A}\left(\frac{s(\max\{\underline{v}, v_t+\widehat{J}_v\})}{s(v_t)}\right)^{-\gamma_A} + o(\Delta t). \tag{A45}$$

Substituting a_t and b_t from (A53) into equation (A44), and then substituting (A44) and (A45) into equation (A43), after simple algebra, we obtain r_t in (41) for the case $\underline{v} < v_t < \overline{v}$.

Now, we derive r_t at the boundaries $\underline{v}$ and $\overline{v}$. The SPD is given by (29). Using expansions (A52) and (A54), we obtain the following expansion:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{normal}\right] = \mathbb{E}_t\Big[\Big((1+a_t\Delta v_t + b_t(\Delta v_t)^2\Big)\Big(1 - r_A\Delta t - \kappa_A\Delta w_t\Big)$$
$$\times(1+\Delta U_t + 0.5(\Delta U_t)^2)|\text{normal}\Big] + o(\Delta t)$$
$$= \mathbb{E}_t\Big[1 + a_t\Delta v_t + b_t(\Delta v_t)^2 - r_A\Delta t - \kappa_A\Delta w_t - \kappa_A a_t\Delta v_t\Delta w_t$$
$$+ \Delta U_t - \kappa_A\Delta w_t\Delta U_t + a_t\Delta U_t\Delta v_t + 0.5(\Delta U_t)^2\Big|\text{normal}\Big] + O(\Delta t), \tag{A46}$$

where ΔU_t is given by equation (A30). Using equation (25) for the process v_t and equation (A30) for ΔU_t, for a fixed v_t and sufficiently small Δt, we find that Δv_t and ΔU_t at the boundaries are given by:

$$\Delta v_t = \begin{cases} \min(0, \mu_v\Delta t + \sigma_v\Delta w_t), & \text{if } v_t = \overline{v}, \\ \max(0, \mu_v\Delta t + \sigma_v\Delta w_t), & \text{if } v_t = \underline{v}, \end{cases} \tag{A47}$$

$$\Delta U_t = \begin{cases} 0, & \text{if } v_t < \overline{v}, \\ \max(0, \mu_v\Delta t + \sigma_v\Delta w_t), & \text{if } v_t = \overline{v}, \end{cases} \tag{A48}$$

We note that for a sufficiently small Δt the sign of $\mu_v\Delta t + \sigma_v\Delta w_t$ is solely determined by the second term $\sigma_v\Delta w_t$ because it has the order of magnitude $\sqrt{\Delta t}$. Volatility σ_v is positive because under our assumptions investor A is more risk averse and more pessimistic. Using the latter observation, substituting equations (A47) and (A48) into equation (A46) and computing the expectation, we obtain:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{normal}\right] = 1 + \begin{cases} \left(\dfrac{a_t(\mu_v - \kappa_A\sigma_v)}{2} + \dfrac{b_t\sigma_v^2}{2} + \dfrac{\mu_v + \kappa_A\sigma_v + \sigma_v^2}{2} - r_A\right)\Delta t \\ \qquad + \dfrac{\sigma_v(1-a_t)}{2}\sqrt{\Delta t} + O(\Delta t), \quad \text{if} \quad v_t = \underline{v}, \\ \left(\dfrac{a_t\mu_v - a_t\kappa_A\sigma_v + b_t\sigma_v^2}{2}\right)\Delta t + \dfrac{a_t\sigma_v}{2}\sqrt{\Delta t} + O(\Delta t), \quad \text{if} \quad v_t = \overline{v}. \end{cases} \tag{A49}$$

Substituting (A49) and (A45) into equation (A43) for the interest rate r_t, we obtain r_t in (41) for the case when v_t is at the boundary.

To obtain the risk premium, we first decompose stock returns as follows:

$$\frac{\Delta S_t + D_{t+\Delta t}\Delta t}{S_t} = \mu_t\Delta t + \sigma_t\Delta w_t + J_t\Delta j_t. \tag{A50}$$

Multiplying both sides of (A50) by $\xi_{A,t+\Delta t}/\xi_{At}$ and taking expectations, we obtain:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t + D_{t+\Delta t}\Delta t}{S_t}\right] = \mu_t\Delta t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right] + \sigma_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right] + J_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta j_t\right].$$

On the other hand, from the equation for stock price (17) we find that:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t + D_{t+\Delta t}\Delta t}{S_t}\right] = 1 - \mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right].$$

Combining the last two equations and the equation (A43) for the interest rate, we obtain:

$$\mu_t - r_t = -\left(\sigma_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right] + J_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta j_t\right]\right)\frac{1 + r_t\Delta t}{\Delta t}. \tag{A51}$$

Then, proceeding in the same way as with the calculation of interest rates and using similar expansions, we obtain equation (42) for the risk premium. ■

Lemma A.2 (Useful expansions).

1) *For small increment $\Delta v_t = v_{t+\Delta t} - v_t$ the ratio $\big(s(v_{t+\Delta t})/s(v_t)\big)^{-\gamma_A}$ has expansion:*

$$\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A} = 1 + a_t\Delta v_t + b_t(\Delta v_t)^2 + o(\Delta t), \tag{A52}$$

where coefficients a_t and b_t are given by:

$$a_t = \frac{(1-s_t)\Gamma_t}{\gamma_B}, \quad b_t = \frac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2 + \frac{1}{2\gamma_A^2\gamma_B^2}s_t(1-s_t)\Gamma_t^3, \tag{A53}$$

$\Gamma_t = \gamma_A\gamma_B/(\gamma_A(1-s)+\gamma_B s)$ *is the risk aversion of the representative investor and* s_t *is consumption share of investor A that solves equation* (14).

2) *For the case* $J_D = 0$*, the SPD in a one-investor economy can be expanded as follows:*

$$e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A} = 1 - r_A\Delta t - \kappa_A\Delta w_t + o(\Delta t), \tag{A54}$$

where r_A *and* κ_A *are the riskless rate and the Sharpe ratio in an economy populated only by investor A, given by:*

$$r_A = \rho + \gamma_A\mu_D - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2, \quad \kappa_A = \gamma_A\sigma_D. \tag{A55}$$

Proof of Lemma A.2. 1) We expand the ratio on the left-hand side of (A52) using Taylor's formula, and observe that $a_t = (s(v_t)^{-\gamma_A})'/s(v_t)^{-\gamma_A}$ and $b_t = 0.5(s(v_t)^{-\gamma_A})''/s(v_t)^{-\gamma_A}$. Differentiating, we obtain the following expressions for a_t and b_t:

$$a_t = -\gamma_A\frac{s'(v_t)}{s(v_t)}, \quad b_t = \frac{\gamma_A(1+\gamma_A)}{2}\left(\frac{s'(v_t)}{s(v_t)}\right)^2 - \frac{\gamma_A}{2}\frac{s''(v)}{s(v)}. \tag{A56}$$

To find derivatives $s'(v)$ and $s''(v)$, we differentiate equation (14) twice with respect to v, and obtain two equations for the derivatives:

$$1 = -\left(\frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t}\right)s'(v_t), \tag{A57}$$

$$0 = \left(\frac{\gamma_A}{s_t^2} - \frac{\gamma_B}{(1-s_t)^2}\right)(s'(v_t))^2 - \left(\frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t}\right)s''(v_t). \tag{A58}$$

Finding $s'(v)$ and $s''(v)$ from the system (A57)–(A58) and substituting them into expressions (A56) for coefficients a_t and b_t, after some algebra, we obtain expressions (A53).

2) Substituting $D_{t+\Delta t}/D_t$ from (1) into equation (A54), after some algebra, we obtain:

$$\begin{aligned} e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right) &= e^{-\rho\Delta t}\left(1+\mu_D\Delta t + \sigma_D\Delta w_t\right)^{-\gamma_A} \\ &= (1-\rho\Delta t)\left(1 - \left(\gamma_A\mu_D - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2\right)\Delta t - \gamma_A\sigma_D\right) + o(\Delta t) \\ &= 1 - r_A\Delta t - \kappa_A\Delta w_t + o(\Delta t). \quad \blacksquare \end{aligned} \tag{A59}$$

Proof of Proposition 4. Consider a reflected arithmetic Brownian motion with boundaries $\underline{v}$ and $\overline{v}$ and dynamics $dv_t = \widehat{\mu}_v dt + \widehat{\sigma}_v dw_t$ when $\underline{v} < v_t < \overline{v}$, where w_t is a Brownian motion. The transition density for this process is given by (see Veestraeten, 2004):

$$f_v(v,\tau;v_t,t)=\frac{1}{\sqrt{2\pi\widehat{\sigma}_v^2(\tau-t)}}\sum_{n=-\infty}^{+\infty}\left[\exp\left(-\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}n(\overline{v}-\underline{v})-\frac{\big(v-v_t-\widehat{\mu}_v(\tau-t)+2n(\overline{v}-\underline{v})\big)^2}{2\widehat{\sigma}_v^2(\tau-t)}\right)\right.$$

$$\left.+\exp\left(-\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\big(v_t-\underline{v}+n(\overline{v}-\underline{v})\big)-\frac{\big(v-v_t-\widehat{\mu}_v(\tau-t)+2(v_t-\underline{v}+n[\overline{v}-\underline{v}])\big)^2}{2\widehat{\sigma}_v^2(\tau-t)}\right)\right]$$

$$+\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\sum_{n=0}^{+\infty}\left[\exp\left(-\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\big(\overline{v}-v+n[\overline{v}-\underline{v}]\big)\right)\mathcal{N}\left(\frac{v_t+\widehat{\mu}_v(\tau-t)-v-2(\overline{v}-v+n[\overline{v}-\underline{v}])}{\widehat{\sigma}_v\sqrt{\tau-t}}\right)\right.$$

$$\left.-\exp\left(\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\big(v-\underline{v}+n[\overline{v}-\underline{v}]\big)\right)\left(1-\mathcal{N}\left(\frac{v_t+\widehat{\mu}_v(\tau-t)-v+2(v-\underline{v}+n[\overline{v}-\underline{v}])}{\widehat{\sigma}_v\sqrt{\tau-t}}\right)\right)\right],\tag{A60}$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution of a standard normal distribution. By $F_v(v,\tau;v_t,t)$ $=\mathrm{Prob}\{v_\tau\leq v|v_t\}$ we denote the corresponding cumulative distribution function of v conditional on observing v_t at time t. We observe that $s_t=s(v_t)$ is a decreasing function of v_t implicitly defined by equation (14). From the latter equation we also find that $s^{-1}(x)=\gamma_B\ln(1-s)-\gamma_A\ln(s)$. The cumulative distribution function of consumption share s_τ at time τ conditional on observing s_t at time t can then be found as follows:

$$\begin{aligned}F(x,\tau;s_t,t)&=\mathrm{Prob}\{s_\tau\leq x|s_t\}\equiv\mathrm{Prob}\{s(v_\tau)\leq x|s_t\}\\&=1-\mathrm{Prob}\{v_\tau\leq s^{-1}(x)|v_t\}\\&=1-\mathrm{Prob}\{v_\tau\leq\gamma_B\ln(1-x)-\gamma_A\ln(x)|v_t\}\\&=1-F_v(\gamma_B\ln(1-x)-\gamma_A\ln(x),\tau;v_t,t).\end{aligned}\tag{A61}$$

Substituting $v_t=\gamma_B\ln(1-s_t)-\gamma_A\ln(s_t)$ into (A61), differentiating CDF $F(x,\tau;s_t,t)$ with respect to x and setting $x=s$, we find that the transition PDF for s is given by:

$$f(s,\tau;s_t,t)=\left(\frac{\gamma_A}{s}+\frac{\gamma_B}{1-s}\right)f_v\Big(\gamma_B\ln(1-s)-\gamma_A\ln(s),\tau;\gamma_B\ln(1-s_t)-\gamma_A\ln(s_t),t\Big),\tag{A62}$$

where transition density $f_v(v,\tau;v_t,t)$ is given by equation (A60).

The stationary distribution of variable v, calculated in Veestraeten (2004), is given by:

$$f_v(v)=\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\frac{\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)v\Big)}{\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)\overline{v}\Big)-\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)\underline{v}\Big)}.\tag{A63}$$

Proceeding in the same way as for the derivation of transition PDF (A62), we obtain stationary PDF (44) for consumption share s. ■

Proof of Corollary 2. The proof easily follows by substituting boundary conditions (35) into the equation (B7) for volatility σ_t at the boundary values $\underline{v}$ and $\overline{v}$. ■

Proof of Proposition 5. Consider Lagrangian (A6) for the dynamic optimization of investor i. Differentiating this Lagrangian with respect to l_i and c_{it}, we obtain:

$$\frac{\partial V_i(W^*_{it}, v_t; l_i)}{\partial l_i} = \eta_{it} D_t \Delta t + e^{-\rho \Delta t} \mathbb{E}^i_t \left[\frac{\partial V_i(W^*_{i,t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial l_i} \right], \tag{A64}$$

$$u'(c^*_{it}) = \eta_{it}. \tag{A65}$$

By the envelope theorem (e.g., Back (2010, p.162)):

$$\frac{\partial V_i(W_{it}, v_t; l_i)}{\partial W} = u'_i(c^*_{it}), \tag{A66}$$

$$\frac{\partial V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} = u'_i(c^*_{i,t+\Delta t}). \tag{A67}$$

Substituting (46), (A65), (A66), and (A67) into equation (A64), and simplifying, we find:

$$\widehat{S}_{it} = D_t \Delta t + \mathbb{E}^i_t \left[e^{-\rho \Delta t} \frac{u'_i(c^*_{i,t+\Delta t})}{u'_i(c^*_{it})} \widehat{S}_{i,t+\Delta t} \right]. \tag{A68}$$

From equation (29), we recall that the SPD of investor A is given by

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t + \Delta U_t} \frac{(c^*_{A,t+\Delta t})^{-\gamma_A}}{(c^*_{At})^{-\gamma_A}} \frac{D_{t+\Delta t}}{D_t}, \tag{A69}$$

where $\Delta U_t = \max\{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \overline{v}\}$. Rewriting equation (A68) for investor A in terms of SPD (A69), we obtain:

$$\widehat{S}_{At} = D_t \Delta t + \mathbb{E}^A_t \left[e^{-\Delta U_t} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \widehat{S}_{A,t+\Delta t} \right]. \tag{A70}$$

Following the same steps as in the proof of Lemma 2, we find that $\widehat{S}_{At} = \widehat{\Psi}_i(v_t; -\gamma_A) s(v_t)^{\gamma_A} D_t$, where $\widehat{\Psi}_i(v; \theta)$ satisfies differential-difference equation (34) with boundary conditions (49).

Iterating equation (17) for stock and equation (A70) for shadow prices, we obtain:

$$S_t + (1 - l_A - l_B) D_t \Delta t = \frac{1}{\xi_t} \mathbb{E}^A_t \left[\sum_{\tau=t}^{\infty} \xi_\tau (1 - l_A - l_B) D_\tau \Delta t \right], \tag{A71}$$

$$\widehat{S}_{At} = \frac{1}{\xi_t} \mathbb{E}^A_t \left[\sum_{\tau=t}^{\infty} e^{-(U_\tau - U_t)} \xi_\tau D_\tau \Delta t \right]. \tag{A72}$$

Inequality $(S_t + (1 - l_A - l_B) D_t \Delta t)/(1 - l_A - l_B) > \widehat{S}_{At}$ follows from the fact that $U_t = \sum_{\tau=0}^{t} \Delta U_\tau$ is a non-decreasing processes. In the continuous-time limit, we obtain that $S_t/(1 - l_A - l_B) > \widehat{S}_{At}$. Hence, the liquidity premium Λ_{At} is positive. The derivation of the shadow price of investor B is analogous and available upon request.

Appendix B: Existence of boundaries $\underline{v}$ and $\overline{v}$.

Proposition B.1 (Closed-form solutions).

1) *In the limit* $\Delta t \to 0$ *the price-dividend ratio* Ψ *and wealth-consumption ratios* Φ_i *are given by equations* (32) *and* (33)*, where function* $\widehat{\Psi}(v;\theta)$ *is given by:*

$$\widehat{\Psi}(v;\theta) = \int_{\underline{v}}^{v} s(y)^{\theta}\widehat{\psi}(v-y)dy + \frac{\int_{\underline{v}}^{\overline{v}} s(y)^{\theta}\left[\widehat{\psi}'(\overline{v}-y) - \widehat{\psi}(\overline{v}-y)\right]dy}{1 + H\left(\widehat{\psi}(\overline{v}-\underline{v}) - \int_0^{\overline{v}-\underline{v}}\widehat{\psi}(y)dy\right)}\left(1 - H\int_0^{v-\underline{v}}\widehat{\psi}(y)dy\right), \tag{B1}$$

where $s(y)$ *solves equation*[10] (14), and $\widehat{\psi}(x)$*,* H *and some auxiliary variables are given by:*

$$\widehat{\psi}(x) = \frac{2}{\widehat{\sigma}_v^2}\sum_{n=0}^{\infty}\Bigg[\left(\frac{2\lambda(1+J_D)^{1-\gamma_A}}{\widehat{\sigma}_v^2}\right)^n \frac{\exp\left((\zeta_+ + \zeta_-)(x + n\widehat{J}_v)/2\right)}{(\zeta_+ - \zeta_-)^{2n+1}n!} \tag{B2}$$

$$\times\, Q_n\left(\frac{(\zeta_+ - \zeta_-)(x + n\widehat{J}_v)}{2}\right)\mathbf{1}_{\{x+n\widehat{J}_v\geq 0\}}\Bigg], \tag{B3}$$

$$Q_n(x) = \exp(-x)\sum_{m=0}^{n}(2x)^{n-m}\frac{(n+m)!}{m!(n-m)!} - \exp(x)\sum_{m=0}^{n}(-2x)^{n-m}\frac{(n+m)!}{m!(n-m)!}, \tag{B4}$$

$$H = \lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2 - \lambda(1+J_D)^{1-\gamma_A}, \tag{B5}$$

$$\zeta_\pm = -\frac{\widehat{\mu}_v + (1-\gamma_A)\widehat{\sigma}_v\sigma_D \mp \sqrt{(\widehat{\mu}_v + (1-\gamma_A)\widehat{\sigma}_v\sigma_D)^2 + 2\widehat{\sigma}_v^2\left(\lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)}}{\widehat{\sigma}_v^2}. \tag{B6}$$

2) *Stock return volatility in normal times and the jump size* J_t *are given by:*

$$\sigma_t = \sigma_D + \left(\frac{\widehat{\Psi}'(v_t;-\gamma_A)}{\widehat{\Psi}(v_t;-\gamma_A)} - \frac{\gamma_A(1-s(v_t))}{\gamma_A(1-s(v_t)) + \gamma_B s(v_t)}\right)\widehat{\sigma}_v, \tag{B7}$$

$$J_t = \frac{(1+J_D)\widehat{\Psi}\left(\max\{\underline{v}; v_t + \widehat{J}_v\}; -\gamma_A\right)s\left(\max\{\underline{v}; v_t + \widehat{J}_v\}\right)^{\gamma_A}}{\widehat{\Psi}(v_t;-\gamma_A)s(v_t)^{\gamma_A}} - 1. \tag{B8}$$

Numbers of shares $n^*_{i,st}$ *and leverage* $L_{it} = -b_{it}B_{it}$ *to market price* S_t *ratio are given by:*

$$n^*_{i,st} = \frac{\Phi_i(v_t)\sigma_D + \Phi_i'(v_t)\widehat{\sigma}_v}{\Psi(v_t)\sigma_t}, \qquad \frac{L_{it}}{S_t} = n_{i,st} - \frac{\Phi_i(v_t)}{\Psi(v_t)(1-l_A-l_B)}. \tag{B9}$$

[10] Although $s(y)$ is not in closed form, we observe from equation (14) that its inverse is given by $s^{-1}(x) = \gamma_B\ln(x) - \gamma_A\ln(1-x)$. The change of variable $x = s(y)$ eliminates implicit functions, similar to Chabakauri (2015). We keep all integrals in terms of $s(y)$ because $s(y)$ is intuitive and easily computable from (14).

Proof of Proposition B.1. 1) First, we solve the differential-difference equation in Lemma 2. We denote $g(x) = \widehat{\Psi}(x + \underline{v}; \theta)$ and apply the following changes of variables:

$$\begin{aligned} &x = v - \underline{v}, \quad \tilde{\sigma} = \widehat{\sigma}_v, \quad \tilde{\mu} = \widehat{\mu}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v, \quad \tilde{J} = -\widehat{J}_v, \quad \tilde{\lambda} = \lambda(1+J_D)^{1-\gamma_A}, \\ &\tilde{\rho} = \lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2. \end{aligned} \tag{B10}$$

Equations (34) and (35) with new variables now become:

$$\frac{\tilde{\sigma}^2}{2} g''(x) + \tilde{\mu} g'(x) - \tilde{\rho} g(x) + \tilde{\lambda} g(\max\{x - \tilde{J}, 0\}) + s(x + \underline{v})^\theta = 0, \tag{B11}$$

$$g'(0) = 0, \quad g(\overline{v} - \underline{v}) - g'(\overline{v} - \underline{v}) = 0. \tag{B12}$$

Let $\mathcal{L}[g(x)] = \int_0^\infty e^{-zx} g(x)dx$ be the Laplace transform of $g(x)$, and similarly for other functions. The Laplace transforms of $g'(x)$, $g''(x)$ and $g(\max\{x - \tilde{J}, 0\})$ are given by:

$$\begin{aligned} \mathcal{L}[g'(x)] &= z\mathcal{L}[g(x)] - g(0), \\ \mathcal{L}[g''(x)] &= z^2\mathcal{L}[g(x)] - zg(0) - g'(0), \\ \mathcal{L}\left[g(\max\{x - \tilde{J}, 0\})\right] &= \int_0^\infty e^{-zx} g(\max\{x - \tilde{J}, 0\})dx \\ &= \int_0^{\tilde{J}} e^{-zx} g(0)dx + \int_{\tilde{J}}^\infty e^{-zx} g(x - \tilde{J})dx \\ &= \frac{1}{z}(1 - e^{-\tilde{J}z})g(0) + e^{-\tilde{J}z}\mathcal{L}[g(x)]. \end{aligned} \tag{B13}$$

Applying the transform to equation (B11), we arrive at the following equation:

$$\begin{aligned} &\frac{\tilde{\sigma}^2}{2}\left(z^2\mathcal{L}[g(x)] - zg(0) - g'(0)\right) + \tilde{\mu}\left(z\mathcal{L}[g(x)] - g(0)\right) - \tilde{\rho}\mathcal{L}[g(x)] \\ &\qquad + \tilde{\lambda}\left(e^{-\tilde{J}z}\mathcal{L}[g(x)] + \frac{1}{z}(1 - e^{-\tilde{J}z})g(0)\right) + \mathcal{L}\left[s(x + \underline{v})^\theta\right] = 0. \end{aligned} \tag{B14}$$

Applying boundary condition $g'(0) = 0$ and solving for $\mathcal{L}[g(x)]$, we obtain:

$$\mathcal{L}[g(x)] = \frac{\mathcal{L}\left[s(x + \underline{v})^\theta\right]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} + g(0)\left(\frac{1}{z} - \frac{\tilde{\rho} - \tilde{\lambda}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z}\right). \tag{B15}$$

Now define a new function $\widehat{\psi}(x)$ through inverse Laplace transform

$$\widehat{\psi}(x) = \mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}}\right]. \tag{B16}$$

Next, we apply inverse transform to each term in (B15). Noting that $\mathcal{L}^{-1}[1/z] = 1$ and using the theorem which states that Laplace transform of a convolution is the product of Laplace transforms, we derive the following inverse transforms:

$$\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[s(x+\underline{v})^{\theta}\right]}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}}\right] = \int_0^x s(y+\underline{v})^{\theta}\cdot\widehat{\psi}(x-y)dy,$$
$$\mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}}\cdot\frac{1}{z}\right] = \int_0^x \mathbf{1}_{\{y\geq 0\}}\cdot\widehat{\psi}(x-y)dy = \int_0^x \widehat{\psi}(y)dy. \tag{B17}$$

The linearity of the Laplace transform gives the following equation:

$$g(x) = \mathcal{L}^{-1}\left[\mathcal{L}\left[g(x)\right]\right] = \int_0^x s(y+\underline{v})^{\theta}\cdot\widehat{\psi}(x-y)dy + g(0)\left[1-\left(\tilde{\rho}-\tilde{\lambda}\right)\int_0^x \widehat{\psi}(y)dy\right]. \tag{B18}$$

We calculate $g(0)$ below, and then after changing the variable back from x to $v = x+\underline{v}$, substituting in expressions for $\tilde{\rho}$ and $\tilde{\lambda}$ from (B10), we obtain (B1).

Next, we solve for $\widehat{\psi}(x)$ in closed form. We expand $\mathcal{L}\left[\widehat{\psi}(x)\right]$ as series, and sum up the inverse transforms of each term in the summation to get $\widehat{\psi}(x)$.

$$\begin{aligned}\mathcal{L}\left[\widehat{\psi}(x)\right] &= \frac{1}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}} \\ &= (\tilde{\rho}-\tilde{\mu}z-\tfrac{\tilde{\sigma}^2}{2}z^2)^{-1}\cdot(1-\frac{\tilde{\lambda}e^{-\tilde{J}z}}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2})^{-1} \\ &= \sum_{n=0}^{\infty}\frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}.\end{aligned} \tag{B19}$$

The above series converges for z such that $|\tilde{\rho}-\tilde{\mu}z-(\tilde{\sigma}^2/2)z^2| > |\tilde{\lambda}\exp(-\tilde{J}z)|$. This holds if the real part of z is sufficiently large, e.g., $\Re(z) > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho}+\tilde{\lambda}}$. The inverse Laplace transform can then be calculated along the line $(\overline{z}-i\infty, \overline{z}+i\infty)$ in the complex domain where $\overline{z} > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho}+\tilde{\lambda}}$, and hence, the inequality for $\Re(z)$ is satisfied.

Let $\zeta_- < \zeta_+$ be roots of $\tilde{\rho}-\tilde{\mu}z-\tilde{\sigma}^2z^2/2 = 0$, given by (B6). We use the following inversion formula for $1/[(z-\zeta_+)(z-\zeta_-)]^{n+1}$ from Gradshteyn and Ryzhik (2007, p. 1117):

$$\mathcal{L}^{-1}\left[\frac{1}{[(z-\zeta_+)(z-\zeta_-)]^{n+1}}\right] = \frac{\sqrt{\pi}}{\Gamma(n+1)}\frac{x^{n+\frac{1}{2}}}{(\zeta_+-\zeta_-)^{n+\frac{1}{2}}}e^{\frac{\zeta_++\zeta_-}{2}x}I_{n+\frac{1}{2}}\left(\frac{\zeta_+-\zeta_-}{2}x\right). \tag{B20}$$

Function $e^{-n\tilde{J}z}$ in the complex domain corresponds to a shift from x to $x-n\tilde{J}$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}\right]=\tilde{\lambda}^n\left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1}\mathbf{1}_{x\geq n\tilde{J}}$$
$$\times\frac{\sqrt{\pi}}{\Gamma(n+1)}\frac{(x-n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+-\zeta_-)^{n+\frac{1}{2}}}e^{\frac{\zeta_++\zeta_-}{2}(x-n\tilde{J})}I_{n+\frac{1}{2}}\left(\frac{(\zeta_+-\zeta_-)(x-n\tilde{J})}{2}\right). \tag{B21}$$

Consequently, the explicit expression for $\widehat{\psi}(x)$ is given by:

$$\widehat{\psi}(x)=\sum_{n=0}^{\infty}\tilde{\lambda}^n\left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1}\frac{\mathbf{1}_{\{x\geq n\tilde{J}\}}\sqrt{\pi}}{\Gamma(n+1)}\frac{(x-n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+-\zeta_-)^{n+\frac{1}{2}}}e^{\frac{\zeta_++\zeta_-}{2}(x-n\tilde{J})}I_{n+\frac{1}{2}}\left(\frac{(\zeta_+-\zeta_-)(x-n\tilde{J})}{2}\right), \tag{B22}$$

where function $I_{n+\frac{1}{2}}(\cdot)$ is a modified Bessel function of the first kind, $\zeta_-<\zeta_+$ are given by (B6) and $\tilde{\rho}$, $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{\lambda}$, and $\tilde{J}$ are defined in (B10). Bessel function $I_{n+\frac{1}{2}}(\cdot)$ is given by (see equation 8.467 in Gradshteyn and Ryzhik (2007)):

$$I_{n+\frac{1}{2}}(z)=\frac{1}{\sqrt{2\pi z}}\left[e^z\sum_{m=0}^{n}\frac{(-1)^m(n+m)!}{m!(n-m)!(2z)^m}+(-1)^{n+1}e^{-z}\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!(2z)^m}\right]. \tag{B23}$$

Substituting (B23) into (B22), after minor algebra, we obtain expression (B3) for $\widehat{\psi}(x)$. The infinite series (B22) has only finite number of non-zero terms because for a fixed x indicators $1_{\{x\geq n\tilde{J}\}}$ vanish for sufficiently large n, and hence, (B22) is well-defined.

To find $g(0)$ in equation (B18), we first evaluate $\widehat{\psi}(0)$. From the above formula (B22), because $\mathbf{1}_{\{0\geq n\tilde{J}\}}=0$ for all $n>0$, we obtain

$$\widehat{\psi}(0)=-\frac{2}{\tilde{\sigma}^2}\cdot\frac{e^{\zeta_+\cdot 0}-e^{\zeta_-\cdot 0}}{\zeta_+-\zeta_-}=0. \tag{B24}$$

Differentiating (B18) and using $\widehat{\psi}(0)=0$, we find:

$$g'(x)=\int_0^x s(y+\underline{v})^\theta\cdot\widehat{\psi}'(x-y)dy-g(0)\cdot\left(\tilde{\rho}-\tilde{\lambda}\right)\widehat{\psi}(x), \tag{B25}$$

We solve for $g(0)$ from the boundary condition $g(\overline{v}-\underline{v})-g'(\overline{v}-\underline{v})=0$ and obtain:

$$g(0)=\frac{\int_0^{\overline{v}-\underline{v}}s(y+\underline{v})^\theta\cdot\left[\widehat{\psi}'(\overline{v}-\underline{v}-y)-\widehat{\psi}(\overline{v}-\underline{v}-y)\right]dy}{1-\left(\tilde{\rho}-\tilde{\lambda}\right)\int_0^{\overline{v}-\underline{v}}\widehat{\psi}(y)dy+\left(\tilde{\rho}-\tilde{\lambda}\right)\widehat{\psi}(\overline{v}-\underline{v})}. \tag{B26}$$

Substituting (B26) into (B18), we derive equation (B1) for $\widehat{\Psi}(v;\theta)$.

2) Next we solve for stock volatility and jump size. In the unconstrained region $\underline{v} < v_t < \overline{v}$, stock price S_t, dividend D_t and state variable v_t follow processes:

$$\begin{aligned} dS_t &= S_t[\mu_t d_t + \sigma_t dw_t + J_t dj_t], \\ dD_t &= D_t[\mu_D d_t + \sigma_D dw_t + J_D dj_t], \\ dv_t &= \widehat{\mu}_v dt + \widehat{\sigma}_v dw_t + \Big(\max\{\underline{v}; v_t + \widehat{J}_v\} - v_t\Big)\, dj_t. \end{aligned} \tag{B27}$$

Applying Ito's lemma to $S_t = (1 - l_A - l_B)\widehat{\Psi}(v_t; -\gamma_A)s(v_t)^{\gamma_A} D_t$, and matching dw_t and dj_t terms, after some algebra, we obtain σ_t and J_t in Proposition B.1.

Equation equation (9) for $W_{i,t+\Delta t}$, implies the following expressions for $n^*_{i,st}$ and b^*_{it}:

$$\begin{aligned} n^*_{i,st} &= \sqrt{\frac{\mathrm{var}_t[W_{i,t+\Delta t} - W_{it}|\text{normal}]}{\mathrm{var}_t[\Delta S_t + (1 - l_A - l_B)D_t \Delta t|\text{normal}]}}, \\ b^*_{it} &= \mathbb{E}_t[W_{i,t+\Delta t|}\text{normal}] - n_{it}\mathbb{E}_t[S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t|\text{normal}]. \end{aligned}$$

Taking limit $\Delta t \to 0$ in the above expressions and using expansions similar to those in the proof of Lemma 2, we obtain the number of stocks and the leverage per the market value of stocks in equation (B9).

Proposition B.2. *Let Ψ_i be the price-dividend ratio in the economy populated only by investor $i = A, B$. If ratios Ψ_i, given by equations* (B37)-(B38) *in the Appendix, are positive and finite, then there exist boundaries $\underline{v}$ and $\overline{v}$ that satisfy equations* (39).

Proof of Proposition B.2. It is easy to observe that because $1 - l_B > l_A$ the following inequality is satisfied: $\gamma_B \ln(l_B) - \gamma_A \ln(1 - l_B) < \gamma_B \ln(1 - l_A) - \gamma_A \ln(l_A)$. The boundaries $\underline{v}$ and $\overline{v}$ solve equations (39). Define

$$L_B(\underline{v}, \overline{v}) = \frac{\widehat{\Psi}(\underline{v}; 1 - \gamma_A)}{\widehat{\Psi}(\underline{v}; -\gamma_A)}. \tag{B28}$$

Substituting $\widehat{\Psi}(v; \theta)$ from (B1) into equation (B28), after some algebra, we obtain:

$$L_B(\underline{v}, \overline{v}) = \frac{\displaystyle\int_{\underline{v}}^{\overline{v}} \Big[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\Big]\, s(y)^{-\gamma_A} \cdot s(y) dy}{\displaystyle\int_{\underline{v}}^{\overline{v}} \Big[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\Big]\, s(y)^{-\gamma_A} dy}. \tag{B29}$$

$L_B(\underline{v}, \overline{v})$ is a weighted average of a decreasing function $s(y)$ from $\underline{v}$ to $\overline{v}$. By (B42) in Lemma B.1 below, function $\Big[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\Big]\, s(y)^{-\gamma_A}$ is positive. Consequently $L_B(\underline{v}, \overline{v}) <$

$s(\underline{v})$ and the function is decreasing in its first argument because

$$\frac{\partial}{\partial \underline{v}} L_B(\underline{v}, \overline{v}) = \frac{\left[\widehat{\psi}(\overline{v} - \underline{v}) - \widehat{\psi}'(\overline{v} - \underline{v})\right] s(\underline{v})^{-\gamma_A}}{\int_{\underline{v}}^{\overline{v}} \left[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\right] s(y)^{-\gamma_A} dy} \left[L_B(\underline{v}, \overline{v}) - s(\underline{v})\right] < 0. \tag{B30}$$

Consequently, for any $\overline{v} \geq \gamma_B \ln(1 - l_A) - \gamma_A \ln(l_A)$,

$$L_B(\gamma_B \ln(l_B) - \gamma_A \ln(1 - l_B), \overline{v}) < s(\gamma_B \ln(l_B) - \gamma_A \ln(1 - l_B)) = 1 - l_B. \tag{B31}$$

Below, we prove that there exists a $\underline{V} < 0$ such that $L_B(\underline{V}, \overline{v}) > 1 - l_B$ for any $\overline{v} \geq \gamma_B \ln(1 - l_A) - \gamma_A \ln(l_A)$. Then, by the intermediate value theorem, equation (B28) has a solution $\underline{v}$ for any fixed $\overline{v}$.

Using inequalities (B43) and (B44) from Lemma B.1 and inequality (B58) from Lemma B.2 below, we derive the following inequality:

$$\begin{aligned}
1 - L_B(\underline{V}, \overline{v}) &= \frac{\int_{\underline{V}}^{\overline{v}} \left[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\right] s(y)^{-\gamma_A} (1 - s(y)) dy}{\int_{\underline{V}}^{\overline{v}} \left[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\right] s(y)^{-\gamma_A} dy} \\
&< \frac{\int_{\underline{V}}^{\overline{v}} \left[-e^{z^+(\overline{v} - y)} \widehat{\psi}'(0)\right] (2^{\gamma_B + 1} e^{y} + 2^{\gamma_A} e^{\frac{1}{\gamma_B} y}) dy}{\int_{\underline{V}}^{\overline{v} - 1} \left[-e^{z^+(\overline{v} - y - 1)} (z^+ - 1) \widehat{\psi}(1)\right] s(\overline{v})^{-\gamma_A} dy} \\
&= \frac{\widehat{\psi}'(0) e^{z^+} s(\overline{v})^{\gamma_A}}{(z^+ - 1)\widehat{\psi}(1)} \cdot \frac{\int_{\underline{V}}^{\overline{v}} 2^{\gamma_B + 1} e^{(1 - z^+)y} + 2^{\gamma_A} e^{(\frac{1}{\gamma_B} - z^+)y} dy}{\int_{\underline{V}}^{\overline{v} - 1} e^{-z^+ y} dy} \\
&< \frac{\widehat{\psi}'(0) e^{z^+} s(\gamma_B \ln(1 - l_A) - \gamma_A \ln(l_A))^{\gamma_A}}{(z^+ - 1)\widehat{\psi}(1)} \cdot \frac{\int_{\underline{V}}^{\infty} 2^{\gamma_B + 1} e^{(1 - z^+)y} + 2^{\gamma_A} e^{(\frac{1}{\gamma_B} - z^+)y} dy}{\int_{\underline{V}}^{\gamma_B \ln(1 - l_A) - \gamma_A \ln(l_A) - 1} e^{-z^+ y} dy}.
\end{aligned} \tag{B32}$$

As y decreases, the denominator term $e^{-z^+ y}$ increases exponentially faster than any term on the numerator. Consequently, the right-hand side of the above inequality converges to 0 as $\underline{V} \to -\infty$, which can be formally verified by L'Hôpital's rule. Therefore, there exists a $\underline{V} < 0$ not dependent on $\overline{v}$ such that $1 - L_B(\underline{V}, \overline{v}) < l_B$, or, equivalently,

$$L_B(\underline{V}, \overline{v}) > 1 - l_B. \tag{B33}$$

For a given $\overline{v}$, $L_B(\underline{v},\overline{v})$ is an continuously decreasing function of $\underline{v}$ that takes different signs at the endpoints of the interval $[\underline{V}, \gamma_B \ln(l_B) - \gamma_A \ln(1-l_B)]$. Therefore, by the intermediate value theorem, there exists unique $\underline{v} \in [\underline{V}, \gamma_B \ln(l_B) - \gamma_A \ln(1-l_B)]$ such that $L_B(\underline{v},\overline{v}) = 1 - l_B$, and this defines a mapping $\underline{v} = m_B(\overline{v})$. Since L_B has non-zero partial derivative with respect to $\underline{v}$, $m_B(\cdot)$ is continuous by the implicit function theorem.

Similar to (B28), we define

$$L_A(\underline{v},\overline{v}) = \frac{\widehat{\Psi}(\overline{v}; 1-\gamma_A)}{\widehat{\Psi}(\overline{v}; -\gamma_A)}. \tag{B34}$$

Substituting $\widehat{\Phi}(v,\theta)$ from (B1) into (B34), after some algebra, we obtain:

$$L_A(\underline{v},\overline{v}) = \frac{\displaystyle\int_{\underline{v}}^{\overline{v}} \left[q'(\overline{v}-\underline{v})\widehat{\psi}(\overline{v}-y) - q(\overline{v}-\underline{v})\widehat{\psi}'(\overline{v}-y)\right] s(y)^{-\gamma_A} \cdot s(y) dy}{\displaystyle\int_{\underline{v}}^{\overline{v}} \left[q'(\overline{v}-\underline{v})\widehat{\psi}(\overline{v}-y) - q(\overline{v}-\underline{v})\widehat{\psi}'(\overline{v}-y)\right] s(y)^{-\gamma_A} dy}. \tag{B35}$$

Proceeding the same way as above, for any $\underline{v}$ less than or equal to $\gamma_B \ln(l_B) - \gamma_A \ln(1-l_B)$, there exists a $\overline{v} \in \left[\gamma_B \ln(1-l_A) - \gamma_A \ln(l_A), \overline{V}\right)$ that satisfies $L_A(\underline{v},\overline{v}) = l_A$, where $\overline{V}$ does not depend on $\underline{v}$ .This defines a continuous mapping $\overline{v} = m_A(\underline{v})$.

Consider the following system of two equations with two unknowns:

$$\overline{v} = m_A(\underline{v}), \quad \underline{v} = m_B(\overline{v}), \tag{B36}$$

where $m_A(\cdot)$ maps $\underline{v} \in (-\infty, \gamma_B \ln(l_B) - \gamma_A \ln(1-l_B)]$ to $\overline{v} \in \left[\gamma_B \ln(1-l_A) - \gamma_A \ln(l_A), \overline{V}\right]$, and $m_B(\cdot)$ maps $\overline{v} \in [\gamma_B \ln(1-l_A) - \gamma_A \ln(l_A), \infty)$ to $\underline{v} \in [\underline{V}, \gamma_B \ln(l_B) - \gamma_A \ln(1-l_B)]$. Consider now a composition function $m(v) \equiv m_A(m_B(v))$. Function $m(\cdot)$ maps $\overline{v} \in \left[\gamma_B \ln(1-l_A) - \gamma_A \ln(l_A), \overline{V}\right]$ into itself. Because $m(v)$ is continuous, it has a fixed point $\overline{v}$ by the intermediate value theorem. Then, $\overline{v}$ and $\underline{v} \equiv m_B(\overline{v})$ satisfy equations (B36).

As demonstrated in Barro (2009), the price-dividend ratios in homogeneous-investor economies populated by investors A and B, respectively, are given by:

$$\Psi_A = \frac{1}{\rho + (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2 - \lambda(1+J_D)^{1-\gamma_A}}, \tag{B37}$$

$$\Psi_B = \frac{1}{\rho + (1-\gamma_B)(\mu_D + \sigma_D\delta) + \frac{(1-\gamma_B)\gamma_B}{2}\sigma_D^2 - \lambda_B(1+J_D)^{1-\gamma_B}}. \tag{B38}$$

After simple algebra, it can be shown that $\Psi_A = 1/(\tilde{\rho}-\tilde{\lambda})$ and $\Psi_A = 1/(\tilde{\rho}-\tilde{\mu}-0.5\tilde{\sigma}^2-\tilde{\lambda}e^{-\tilde{J}})$. Therefore, assumption (B40) in Lemma B.1 is equivalent to conditions $\Psi_A > 0$ and $\Psi_B > 0$. The latter conditions also follow from condition (15) in Section 2 when time is continuous.

The investor's value functions are bounded because

$$\begin{aligned}
&\left|\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(\tau-t)}\frac{(c^*_{i\tau})^{1-\gamma_i}}{1-\gamma_i}d\tau\right]\right| = \left|\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(\tau-t)}\frac{s(v_\tau)^{1-\gamma_i}D_\tau^{1-\gamma_i}}{1-\gamma_i}d\tau\right]\right| \\
&\leq \frac{\max\left\{s(\underline{v})^{1-\gamma_i}, s(\overline{v})^{1-\gamma_i}\right\}}{|1-\gamma_i|}\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(\tau-t)}D_\tau^{1-\gamma_i}d\tau\right] \\
&= \max\left\{s(\underline{v})^{1-\gamma_i}, s(\overline{v})^{1-\gamma_i}\right\}\frac{\Psi_i D_t^{1-\gamma_i}}{|1-\gamma_i|} < +\infty. \quad \blacksquare
\end{aligned} \tag{B39}$$

Lemma B.1 (Inequalities for $\hat{\psi}(x)$ and $\hat{\psi}'(x)$). *Suppose, the model parameters are such that the following two inequalities are satisfied:*

$$\tilde{\rho}-\tilde{\lambda} > 0, \quad \tilde{\rho}-\tilde{\mu}-\frac{\tilde{\sigma}^2}{2}-\tilde{\lambda}e^{-\tilde{J}} > 0, \tag{B40}$$

where $\tilde{\rho}$, $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{J}$ are given by equations (B10). *Let function $q(x)$ be given by*

$$q(x) = 1-(\tilde{\rho}-\tilde{\lambda})\int_0^x \widehat{\psi}(y)dy. \tag{B41}$$

Then, for all $x > 0$ and $\overline{v} > \underline{v}$ the following inequalities are satisfied:

$$\begin{gathered}
\widehat{\psi}(x) < 0, \quad \widehat{\psi}'(x) < 0, \quad \widehat{\psi}(x)-\widehat{\psi}'(x) > 0, \\
q'(\overline{v}-\underline{v})\widehat{\psi}(x) - q(\overline{v}-\underline{v})\widehat{\psi}'(x) > 0.
\end{gathered} \tag{B42}$$

Furthermore, there exists $z^+ > 1$ such that

$$\widehat{\psi}(x)-\widehat{\psi}'(x) > -e^{z^+(x-1)}(z^+-1)\widehat{\psi}(1), \quad \text{for} \ \ x \geq 1, \tag{B43}$$

$$\widehat{\psi}(x)-\widehat{\psi}'(x) < -e^{z^+x}\widehat{\psi}'(0), \quad \text{for} \ \ x > 0. \tag{B44}$$

Proof of Lemma B.1. From definition (B16), $\widehat{\psi}(x)$ satisfies equation:

$$\left[\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}\right]\mathcal{L}\left[\widehat{\psi}(x)\right] = 1. \tag{B45}$$

Dividing the above equation by z, applying inverse Laplace transform and using the fact that $\widehat{\psi}(0) = 0$, we find that $\widehat{\psi}(x)$ satisfies the following integro-differential equation:

$$\frac{\tilde{\sigma}^2}{2}\widehat{\psi}'(x) = -1-\tilde{\mu}\widehat{\psi}(x)+(\tilde{\rho}-\tilde{\lambda})\int_0^x\widehat{\psi}(y)dy+\tilde{\lambda}\int_{\max\{x-\tilde{J},0\}}^x\widehat{\psi}(y)dy. \tag{B46}$$

Letting $x = 0$ in equation (B46), we obtain $\widehat{\psi}'(0) < 0$. Therefore, because $\widehat{\psi}(0) = 0$, $\widehat{\psi}(x) < 0$ in some neighborhood of 0. We first prove that $\widehat{\psi}(x) < 0$ for all $x > 0$. Suppose, on the contrary, that there exists $x > 0$ such that $\widehat{\psi}(x) \geq 0$. Let $\underline{x} = \inf\{x \in R^+ : \widehat{\psi}(x) \geq 0\}$. By the continuity of $\widehat{\psi}(x)$, we have $\widehat{\psi}(\underline{x}) = 0$ and $\widehat{\psi}(x) < 0$ for $x \in (0, \underline{x})$. Evaluating equation (B46) at $\underline{x}$, we obtain:

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\widehat{\psi}'(\underline{x}) &= -1 - \tilde{\mu}\widehat{\psi}(\underline{x}) + (\tilde{\rho} - \tilde{\lambda})\int_0^{\underline{x}} \widehat{\psi}(y)dy + \tilde{\lambda}\int_{\max\{\underline{x}-\tilde{J},0\}}^{\underline{x}} \widehat{\psi}(y)dy \\ &< -1 - \tilde{\mu}\cdot 0 + (\tilde{\rho} - \tilde{\lambda})\int_0^{\underline{x}} 0\cdot dy + \tilde{\lambda}\int_{\max\{\underline{x}-\tilde{J},0\}}^{\underline{x}} 0\cdot dy = -1.\end{aligned} \tag{B47}$$

The inequality (B47) is satisfied because $\tilde{\rho} - \tilde{\lambda} > 0$ by assumption (B40). However, $\widehat{\psi}'(\underline{x}) < 0$ is inconsistent with $\underline{x}$ being the smallest positive number such that $\widehat{\psi}(\underline{x}) = 0$ because $\widehat{\psi}(x)$ cannot be a decreasing function at $\underline{x}$. Therefore, we arrive at a contradiction, and hence, $\widehat{\psi}(x) < 0$ for all $x > 0$.

Consider function $h(z) \equiv \tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}$. By assumption (B40), $h(0) > 0$ and $h(1) > 0$. It can be easily observed that $h(-\infty) = h(+\infty) = -\infty$. Therefore, by the intermediate value theorem there exist two real roots $z^- < 0$ and $z^+ > 1$ that satisfy equation $h(z) = 0$. Furthermore, function $h(z)$ is concave because $h''(z) < 0$. The concavity of $h(z)$ implies that $h(z) \geq 0$ for all $z \in [z^-, z^+]$.

Let $\widehat{z}$ be any number such that $\widehat{z} \in [z^-, z^+]$, and let $\widehat{\alpha}(x) \equiv e^{-\widehat{z}x}\widehat{\psi}(x)$. Next, we establish that $\widehat{\alpha}'(x) < 0$ for all $x \geq 0$. Differentiating equation (B46), we obtain:

$$\frac{\tilde{\sigma}^2}{2}\widehat{\psi}''(x) = -\tilde{\mu}\widehat{\psi}'(x) + \tilde{\rho}\widehat{\psi}(x) - \tilde{\lambda}\widehat{\psi}(x - \tilde{J})1_{x\geq\tilde{J}}. \tag{B48}$$

Substituting $\widehat{\psi}(x) = e^{\widehat{z}x}\widehat{\alpha}(x)$ into equation (B48), after some algebra, we find:

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\widehat{\alpha}''(x) &= -(\tilde{\mu} + \tilde{\sigma}^2\widehat{z})\widehat{\alpha}'(x) + (\tilde{\rho} - \tilde{\mu}\widehat{z} - \tfrac{\tilde{\sigma}^2}{2}\widehat{z}^2 - \tilde{\lambda}e^{-\tilde{J}\widehat{z}})\widehat{\alpha}(x) + \tilde{\lambda}e^{-\tilde{J}\widehat{z}}\left[\widehat{\alpha}(x) - \widehat{\alpha}(x-\tilde{J})1_{x\geq\tilde{J}}\right] \\ &= -(\tilde{\mu} + \tilde{\sigma}^2\widehat{z})\widehat{\alpha}'(x) + (\tilde{\rho} - \tilde{\mu}\widehat{z} - \tfrac{\tilde{\sigma}^2}{2}\widehat{z}^2 - \tilde{\lambda}e^{-\tilde{J}\widehat{z}})\int_0^x \widehat{\alpha}'(y)dy + \tilde{\lambda}e^{-\tilde{J}\widehat{z}}\int_{\max\{x-\tilde{J},0\}}^x \widehat{\alpha}'(y)dy.\end{aligned} \tag{B49}$$

We observe that $\widehat{\alpha}(0) = \widehat{\psi}(0) = 0$, $\widehat{\alpha}'(0) = -\widehat{z}\widehat{\psi}(0) + \widehat{\psi}'(0) < 0$ because $\widehat{\psi}(0) = 0$ and $\widehat{\psi}'(0) < 0$. The rest of the proof for $\widehat{\alpha}'(x) < 0$ is similar to that of $\widehat{\psi}(x) < 0$. Consequently, differentiating $\widehat{\alpha}(x)$ and dividing $\widehat{\alpha}'(x) < 0$ by $e^{-\widehat{z}x}$, we obtain:

$$\widehat{z}\widehat{\psi}(x) - \widehat{\psi}'(x) > 0, \text{ for any } \widehat{z} \in [z^-, z^+]. \tag{B50}$$

In particular for $\widehat{z} = 0$ we find $\widehat{\psi}'(x) < 0$, and for $\widehat{z} = 1$ we find $\widehat{\psi}(x) - \widehat{\psi}'(x) > 0$. Therefore, we have proven the first three inequalities in (B42).

Next, we prove (B43) and (B44). For $x > 1$, using inequality (B50) and the fact that $\widehat{\alpha}(x) = e^{-\widehat{z}x}\widehat{\psi}(x)$ is a decreasing function, we establish inequality (B43) as follows:

$$\begin{aligned}\widehat{\psi}(x) - \widehat{\psi}'(x) = (1 - z^+)\widehat{\psi}(x) + (z^+\widehat{\psi}(x) - \widehat{\psi}'(x)) &> -e^{z^+x}(z^+ - 1)(e^{-z^+x}\widehat{\psi}(x)) \\ &> -e^{z^+x}(z^+ - 1)(e^{-z^+1}\widehat{\psi}(1)).\end{aligned} \tag{B51}$$

To prove (B44), let $\widetilde{\alpha}(x) = -e^{-z^+x}\widehat{\psi}'(x)$. Differentiating equation (B48) and rewriting it in terms of $\widetilde{\alpha}(x)$, we derive the following equation:

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\widetilde{\alpha}''(x) &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\widetilde{\alpha}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\widetilde{\alpha}(x) - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\widetilde{\alpha}(\max\{x - \tilde{J}, 0\}) \\ &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\widetilde{\alpha}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\int_{\max\{x-\tilde{J},0\}}^{x}\widetilde{\alpha}'(y)dy + \left[\tilde{\lambda}e^{-\tilde{J}z^+} - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\right]\widetilde{\alpha}(0).\end{aligned} \tag{B52}$$

Letting $x = 0$ in (B48), we find that $\widehat{\psi}''(0) = -2(\tilde{\mu}/\tilde{\sigma}^2)\widehat{\psi}'(0)$. Consequently,

$$\widetilde{\alpha}(0) = -\widehat{\psi}'(0) > 0 \quad \widetilde{\alpha}'(0) = -\widehat{\psi}''(0) + z^+\widehat{\psi}'(0) = \frac{2}{\tilde{\sigma}^2}(\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}z^+)\widehat{\psi}'(0) < 0, \tag{B53}$$

where the last inequality hold because $z^+ > 1$ and $z^+(\tilde{\mu} + 0.5\tilde{\sigma}^2 z^+) = \tilde{\rho} - \tilde{\lambda}e^{-\tilde{J}z^+} > \tilde{\rho} - \tilde{\lambda} > 0$. Similar to the above, we show that $\widetilde{\alpha}'(x) < 0$. Hence, we derive (B44) as follows:

$$\widehat{\psi}(x) - \widehat{\psi}'(x) < -\widehat{\psi}'(x) = e^{z^+x}\widetilde{\alpha}(x) < e^{z^+x}\widetilde{\alpha}(0) = -e^{z^+x}\widehat{\psi}'(0). \tag{B54}$$

Finally, we prove the last inequality in (B42). We define $\widehat{\beta}(x) = e^{-z^+x}q(x)$ and next prove that $\widehat{\beta}'(x) < 0$. Proceeding in the same way as above, we express equation (B46) first in terms of $q(x)$ and then in terms of $\widehat{\beta}(x)$:

$$\frac{\tilde{\sigma}^2}{2}q''(x) = -\tilde{\mu}q'(x) + \tilde{\rho}q(x) - \tilde{\lambda}q(\max\{x - \tilde{J}, 0\}), \tag{B55}$$

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\widehat{\beta}''(x) &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\widehat{\beta}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\widehat{\beta}(x) - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\widehat{\beta}(\max\{x - \tilde{J}, 0\}) \\ &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\widehat{\beta}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\int_{\max\{x-\tilde{J},0\}}^{x}\widehat{\beta}'(y)dy + \left[\tilde{\lambda}e^{-\tilde{J}z^+} - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\right]\widehat{\beta}(0).\end{aligned} \tag{B56}$$

For $x = 0$ we observe that $\widehat{\beta}(0) = q(0) = 1$, $\widehat{\beta}'(0) = -z^+q(0) + q'(0) = -z^+q(0) - (\tilde{\rho} - \tilde{\lambda})\widehat{\psi}(0) = -z^+ < 0$. Moreover, it is easy to observe that $\left[\tilde{\lambda}e^{-\tilde{J}z^+} - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\right]\widehat{\beta}(0) \leq 0$ for all x. Proceeding as above, we find that $\widehat{\beta}'(x) < 0$, and hence, $q'(x) < z^+q(x)$. Using the latter inequality and $\widehat{\psi}(x) < 0$, we prove the last inequality in (B42):

$$\begin{aligned}q'(\overline{v} - \underline{v})\widehat{\psi}(x) - q(\overline{v} - \underline{v})\widehat{\psi}'(x) &\geq z^+q(\overline{v} - \underline{v})\widehat{\psi}(x) - q(\overline{v} - \underline{v})\widehat{\psi}'(x) \\ &= q(\overline{v} - \underline{v})\left[z^+\widehat{\psi}(x) - \widehat{\psi}'(x)\right] > 0. \quad \blacksquare\end{aligned} \tag{B57}$$

Lemma B.2 (Inequality for consumption shares). *Let $s(v_t)$ denote the consumption share of investor A. Then, for all $v \in \mathbb{R}$ the following inequality is satisfied:*

$$s(v)^{-\gamma_A}(1 - s(v)) \leq 2^{\gamma_B+1}e^v + 2^{\gamma_A}e^{v/\gamma_B}. \tag{B58}$$

Proof of Lemma B.2. First, we rewrite equation (14) in the following equivalent form:

$$s(v)^{-\gamma_A}(1 - s(v))^{\gamma_B} = e^v. \tag{B59}$$

When $\gamma_B \leq 1$, from the above equation we obtain the following inequality:

$$s(v)^{-\gamma_A}(1 - s(v)) \leq s(v)^{-\gamma_A}(1 - s(v))^{\gamma_B} = e^v. \tag{B60}$$

For $\gamma_B > 1$ and $1 - s(v) \geq 1/2$, we find that:

$$s(v)^{-\gamma_A}(1 - s(v)) \leq 2^{\gamma_B-1}s(v)^{-\gamma_A}(1 - s(v))^{\gamma_B} = 2^{\gamma_B-1}e^v. \tag{B61}$$

Finally, for $\gamma_B > 1$ and $s(y) \geq 1/2$ we have the following inequality:

$$s(y)^{-\gamma_A}(1 - s(y)) \leq 2^{\gamma_A-\gamma_A/\gamma_B}s(y)^{-\gamma_A/\gamma_B}(1 - s(y)) < 2^{\gamma_A}e^{v/\gamma_B}. \tag{B62}$$

Combining all the inequalities (B60)-(B62), we obtain inequality (B58). ■