Illiquidity Spirals in Coupled Over-the-Counter Markets[*]

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Abstract

Traders provide intermediation of two related financial assets—secured debt and the underlying collateral. Each asset is traded in its own over-the-counter trading network. Traders decide whether or not to be active in each network. We give conditions under which the financial connections between the two assets make this a game of strategic complements on two *coupled* networks: incentives to be active in a given network are increasing both in neighbors' activity choices in that network *and* in one's own activity in the other network. Providing the first analysis of such network games, we use the theory of fixed points of monotone functions to characterize the general structure of equilibria in such systems. We focus on illiquidity spirals: following an exogenous shock disabling some intermediaries, a sequence of withdrawals occurs, corresponding to a contagion across the two networks. For a class of market structures associated with random graphs, liquidity changes *discontinuously* in the size of an exogenous shock, in contrast to standard models of network contagion.

Keywords: market liquidity, funding liquidity, over-the-counter markets

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1 Introduction

Many important financial markets are decentralized: financial institutions trade directly with one another rather than through an exchange. Typically, each institution, called a *bank* for short, trades only with a subset of potential counterparties, and that subset is effectively fixed in the short run. Examples of such *over-the-counter* (OTC) markets include the market for repurchase agreements (repo), the interbank market, the market for credit default swaps, and markets for derivatives. The different trading relationships can be viewed as a network. Depending on the nature of the financial instruments traded, these networks are not independent, but instead are *coupled*, in a sense we will make precise. The essence of the coupling is that a bank's circumstances in one market influence its behavior in another.

A leading example of coupling comes from the *repo* market, in which a market for secured debt (repo) is coupled to a market for the collateral. In a repo transaction, a bank sells a security (for example a bond) and simultaneously makes a promise to repurchase that security from the buyer at a given price and a given future time. The prices of the initial sale and the repurchase encode the interest on the loan, as well as the haircut applied to the collateral.[1] Secured loans of this form have become a major source of funding liquidity for a wide array of financial institutions, including banks, money market funds, and security lenders.[2]

They are also fragile in distinctive ways—a fragility the global financial crisis highlighted. The investment bank BNP Paribas, for example, substantiated their decision to close down two of its funds in August 2007 saying *"The complete evaporation of liquidity in certain market segments of the US securitization market has made it impossible to value certain assets fairly regardless of their quality or credit rating. [...] Asset-backed securities, mortgage loans, especially subprime loans don't have any buyers. [...] Traders are reluctant to bid on securities backed by risky mortgages because they are difficult to sell on."*[3] Krishnamurthy (2010) shows that the monthly average of dealer repo activity declined from $450 billion in April 2008 to $250 billion in January 2009. During the summer of 2008, market liquidity for collateral (bonds) deteriorated as well. As a result, collateral haircuts increased: collateral with a higher face value had to be posted to secure a loan of a given amount. Therefore, funding liquidity deteriorated. But borrowers were relying on funding liquidity of the very same sort to purchase the collateral, and thus market liquidity for bonds deteriorated. This leads to a feedback loop: a decline in funding liquidity led to a further decline in market liquidity. This mechanism is at the heart of the destabilizing *illiquidity spirals* described by Brunnermeier and Pedersen (2009).

The basic principles of this price-mediated feedback loop are well-understood for *cen-*

[1] Lenders insure themselves against the risk of a depreciation in the value of the collateral by requiring a *haircut* α, i.e. the size of the loan is equal to $(1-\alpha)$ times the face value of the collateral.

[2] Copeland et al. (2014) estimate the sum of all repo outstanding on a typical day in July and August 2008 to be $6.1 trillion. Less liquid, non-government bond repo outstanding was approximately 500 billion Euros in the EU ($\approx$ 10% of the entire market) and $\approx$ 500 bn USD (mostly agency MBS collateral) in the US; see Baklanova et al. (2015) and ICMA (2016) for detailed discussions.

[3] Source: "BNP Paribas Freezes Funds as Loan Losses Roil Markets" (Bloomberg.com, August 9, 2007). As cited in Acharya et al. (2011).

tralized repo and collateral markets. However both the repo and the collateral market operate in a decentralized way, consistent with the importance of decentralized trade discussed above. Now, there is a new endogenous variable important to every intermediary: whether the particular intermediaries it trades with choose to be active in the market—i.e., available to trade.[4] For example, a lender wants to extend repo loans to its counterparties only if the lender has an active counterparty in the OTC market for the collateral: this is necessary to liquidate collateral in case the borrower defaults, and also to offset collateral purchases and avoid accumulating inventory (an issue we discuss further below). Similarly, an intermediary wants to be active in the collateral market only if it has access, via his OTC relationships, to the funding liquidity provided by the repo market.

Thus, banks' activity decisions depend on the decisions of those they are connected to in both networks. This creates the potential for a network-driven feedback loop, in which the cessation of intermediation by some agents makes others, who are dependent on them for market access, unwilling to lend, and this contagion propagates through the network. One foundation for such behavior comes from liquidity hoarding.[5] We investigate how and to what extent liquidity deteriorates in this way after the exogenous withdrawal of a subset of banks. One of the main take-aways of this paper is that a financial system in which both repo and collateral are traded OTC is significantly less resilient to such an exogenous shock than a system in which collateral (or repo) is traded on an exchange. Both the network structure and the coupling of two OTC networks is critical for this.

To investigate these issues, we develop a model of banks that interact in two OTC markets—following our leading example, we call them the repo and collateral market. Our key reduced-form assumption about bank behavior is that a bank wants to be active in both markets if it has at least one active counterparty in each (these may be different counterparties), and it has not suffered a negative shock that renders it unable to intermediate; otherwise it wants to be active in neither. The justification for the assumed best-responses comes from assuming that banks face two constraints. First, banks face a cash-in-advance constraint: in order to purchase collateral or provide a short-term loan, banks must first obtain funding, requiring access to the repo market. Second, banks face a capital requirement that constrains their inventories: Because they must hold capital proportional to their assets, they are not able to hold large inventories of collateral but instead enter offsetting trades to keep their inventory small. This requires access to the collateral market. The final ingredient for wanting to engage in intermediation is the absence of a shock: as is standard in models of systemic risk, we permit some intermediaries to experience exogenous distress, which makes them withdraw from the market in the short run.

Formally, we represent the OTC repo and collateral markets as two directed networks, which have the same nodes, the banks. A directed link from bank A to bank B reflects that A

[4] A growing empirical literature shows that during times of distress, and even in the absence of overall market freezes, individual banks can loose access to their wholesale funding (for a recent contribution, see e.g. Perignon et al. (2017)).

[5] There is a large literature on liquidity hoarding, and we discuss it, and its relations to our model, in more detail in Section 2.2 below.

extends repo loans to B. Similarly, a directed link from A to B in the collateral market represents that A buys collateral from B. In the short run these relationships, which determine potential trades, are treated as fixed. Abstracting away from prices, quantities and evaluations of specific counterparties' risk, we study a *coupled-network game* in which banks choose to be active—doing business with all their counterparties, if asked—or not, and their decisions are strategic complements according to the best-responses described in the previous paragraph. We study the Nash equilibria of this game. In an equilibrium, our measure of the amount of liquidity is simply the number of banks willing to provide repo and buy collateral. In general, there may be multiple equilibria, which can be ranked by their liquidity. Throughout this paper, we focus on the equilibrium that permits the *maximum* possible liquidity provision.

Our main results implement a sort of "stress test" in a variety of networks, looking at liquidity provision before and after an exogenous shock. The benchmark is one in which no banks experience exogenous distress. We then consider a random set of banks, comprising a certain share of the population, experiencing an exogenous shock, and we consider the post-shock equilibrium. Banks' choices are made following the realizations of shocks—formally, we study the Nash equilibria of the resulting complete-information game.

Broadly speaking, our analysis describes the outcome of this game in terms of the trading network in both markets. First, in Section 3 we characterize the general conditions on the network structure of the OTC markets for the existence of an equilibrium with non-zero liquidity provision. Second, in Section 3.3 we turn to star and core-periphery networks as concrete examples and investigate the effect of the withdrawal of a subset of banks on the equilibrium liquidity measure. Finally, in Section 4 we broaden our scope and consider liquidity in a larger class of random networks in which each bank has a given number of counterparties—a model that can capture important moments of realistic trading networks.

We show that solving the network game can be reduced to finding connected subnetworks comprising sets of banks which have at least one incoming directed link in both repo and collateral networks, after the shock has disabled some banks. We refer to these subnetworks as mutually stable components. By adding up their sizes, we can compute the equilibrium liquidity measure. We provide a simple numerical algorithm for finding the mutually stable components for any network, which is a version of the standard algorithm for finding the greatest fixed point on a lattice. Our numerical implementation provides a practical application for supervisors interested in studying contagion in coupled financial networks, possibly using available supervisory data.[6]

We investigate the effect of the exogenous shock on star and core-periphery networks as well as for important types of random networks. In a star network there are two types of banks: a single hub bank and a set of peripheral banks. Peripheral banks are only connected to the hub. A core-periphery network is a generalization of the star network such that several banks (all interconnected among themselves) comprise the hub. For star and core-periphery networks,

[6] An emerging literature studies contagion in financial networks using supervisory data, see for example Greenwood et al. (2015) and Duarte and Eisenbach (2015). These papers, however, do not consider the case of coupled financial networks.

we exhibit an "illiquidity spiral" such that additional banks withdraw from the collateral and repo markets after an initial exogenous shock. For core periphery networks, the extent of the illiquidity spiral depends the type (core vs periphery) of the shocked bank and the likelihood that two core banks share the same peripheral bank. As this becomes more likely, the illiquidity spiral becomes less severe. Furthermore, we show that the complementarity between the over-the-counter, decentralized structure of repo and collateral markets is essential to this spiral. Importantly, when one of the two markets is replaced by a centralized exchange, post-shock liquidity is always greater than in the pure OTC case.

We then turn to a class of random networks of a fairly general sort, which permit specifying a full joint distribution of in- and out-degrees (i.e., the number of counterparties to whom one provides liquidity). This is of particular practical relevance since strict confidentiality constraints typically prohibit the knowledge of the detailed OTC network structure. Our results allow us to make statements for the more realistic case when at least the degree distribution of the networks are known. We thereby also provide guidance to supervisors about which data to collect in order to study the resilience of coupled financial networks.

In these networks we investigate *market freezes* in the repo and collateral markets as an extreme outcome of an illiquidity spiral. In a market freeze, liquidity in both markets evaporates entirely and abruptly as we vary the size of an external shock. For the case of large random networks we can obtain analytical results that show how and when the repo and collateral markets may freeze given an exogenous shock. In particular, we study the equilibrium liquidity measure as we vary the size of the exogenous shock, increasing it from a low to a high level. As this occurs, the subset of banks withdrawing due to the exogenous shock increases in size. As stated above, our main finding is that, subject to a number of technical conditions, liquidity may entirely evaporate, and this occurs discontinuously in the size of the exogenous shock. That is, starting from a healthy amount of market and funding liquidity, the withdrawal of one additional bank may result in the total freeze of the repo and collateral markets. Again, this stark amplification of an exogenous shock is caused by the coupling of the repo and collateral markets. When one of the two OTC markets is replaced by a centralized exchange, markets may still freeze but liquidity evaporates continuously. That is, starting from any initial condition, the withdrawal of an additional bank may only result in a small decrease in equilibrium liquidity. Furthermore, the number of banks that have to withdraw for equilibrium liquidity to vanish is always larger when at least one market is centralized. These results indicate that the network structure of the repo and collateral markets has an important impact on the resilience of funding and market liquidity. Hence, our results provide a novel perspective on the illiquidity spirals in the markets for short-term collateral debt that were at the heart of the global financial crisis.

While we illustrate our results in the repo market, the applicability of our model is not restricted to this example. We now discuss some other cases in which the same forces apply. As Acharya et al. (2013) point out, banks have been moving an increasing share of their assets off their balance sheets using special purpose vehicles (SPVs), peaking at $1.3 trillion in 2007. These SPVs issued commercial paper that was backed, for example, by mortgages and bought by a variety of financial institutions. Crucially, the SPVs were endowed with explicit and implicit liquidity guarantees, forcing banks to take them back onto their balance sheets in times of crisis. These liquidity guarantees create a coupling between the ABCP market and the market for the collateral. To take another example, In the market for credit default swaps (CDS), banks purchase protection for a bond they provided to a borrower. CDS markets have a non-trivial network structure (Peltonen et al. (2014)) and are coupled to the market for the underlying bonds. Thus, our results have implications for a variety of other markets.

We also offer a methodological contribution by showing how the theory of multilayer networks can be applied to analyze interlinked markets. Though networks have been used extensively to model over-the-counter markets and other trading relationships, to our knowledge our paper is the first to point out the financial implications of the distinctive features of contagion in multilayer networks, where each vertex must remain active in several in networks at once or drop out of all of them.[7] This perspective leads to fundamentally new and counterintuitive phenomena. For example, we show that the amplification of shocks and the extra fragility that is due to the coupling of the two markets strictly declines when the trading networks in the two markets become *more* correlated – that is, when one's trading partners in one network are more likely to be trading partners in the other.

2 Model

2.1 Model Formulation

There is a set $N = \{1, \dots, n\}$ of intermediaries, called *banks* for short. Banks may trade bilaterally with other banks in a *r*epo and a *c*ollateral market, $\mu \in \{R, C\}$. In the repo market, a repo seller provides financing to a repo buyer against a security as collateral. In the collateral market, banks trade the security.

Each bank can trade with only a subset of other banks in either market. The set of trading relationships in market μ is taken as exogenous and described by a directed network $\mathcal{G}_\mu$. A directed network $\mathcal{G}$ is a set of nodes $V(\mathcal{G})$ together with a set $E(\mathcal{G})$ of directed links, i.e., ordered pairs (i, j) with $i, j \in N$, which we often write as $i \to j$. In the repo market, a link $i \to j$ in $\mathcal{G}_R$ means that i provides repo financing (a secured loan) to j, providing cash in exchange for a security and a promise to repurchase it at a later date. We describe such a link by saying that i provides *funding liquidity* to j. In the collateral market, the link $i \to j$ in $\mathcal{G}_C$ means that i purchases collateral from j. Therefore, we say that i provides *market liquidity* to j. We assume there are no self-links $i \to i$. Both markets share the same node set: $V(\mathcal{G}_R) = V(\mathcal{G}_C) = N$.

We focus on repo as the prime example of secured lending, but our model can be translated to any secured lending market. More generally, though we use the repo market terminology throughout, the same primitives can be used to study any two trading networks that are coupled through the behavior of their nodes. See Section 2.2 for more on this.

[7] Trading in single-layer networks is studied in various setups. See, for example, Gai et al. (2011), Duffie et al. (2014), Condorelli et al. (2016).

It will be useful to define the set of a bank's trading partners as its neighborhoods.

Definition 2.1 (In-neighborhood)**.** The in-neighborhood of bank i in market μ, i.e. in network $\mathcal{G}_\mu$ is

$$K^-_{i,\mu} = \{j \mid j \to i \in E(\mathcal{G}_\mu)\}$$

and can be interpreted as the set of banks providing liquidity to i. The in-degree of bank i in market μ is the size of the in-neighborhood $d^-_{i,\mu} = |K^-_{i,\mu}|$.

Analogously, we define the set and number of banks that obtain liquidity from i as the out-neighborhood $K^+_{i,\mu}$ and out-degree $d^+_{i,\mu}$, respectively, by replacing $j \to i$ with $i \to j$ in the above definition.

Banks play a strategic game of complete information. Let $a^R \in \{0,1\}$ and $a^C \in \{0,1\}$ correspond to banks' decisions of whether to be *active* in each market, with the pair (a^R_i, a^C_i) denoting bank i's action. These are the endogenous variables of the model. We take a reduced-form approach to the actual trade: payoffs of our game will represent the consequences of agents' being active; these consequences could be derived from a market of a model (e.g., a trading game), which we do not model. These payoffs depend on the network and on the activity decisions of the others. We denote by $\boldsymbol{a} = \left(a^R_i, a^C_i\right)_{i\in N}$ an action profile.

A bank's best response depends on the actions of its counterparties and the realization of an exogenous shock vector $\boldsymbol{w}$. Let $w_i \in \{0,1\}$ denote the shock realization for bank i. This can be interpreted as an event that determines whether or not a bank operates in the two markets.[8] For $w_i = 0$, bank i's best response is to be be inactive, irrespective of the actions of other banks. For $w_i = 1$, bank i's best response is to be active in both markets, as long as the number of active counterparties in i's in-neighborhood in *each* market μ exceeds a threshold θ_μ, i.e.,

$$\sum_{j\in K^-_{i,\mu}} a^R_j \geq \theta_\mu. \tag{1}$$

It will be useful to define an auxiliary variable $S^\mu_i(\boldsymbol{a}_{-i}) = \sum_{j\in K^-_{i,\mu}} a^\mu_j$, where $S^\mu_i(\boldsymbol{a}_{-i})$ is the number of active counterparties in i's in-neighborhood in market μ. Note that, because a given bank's in-neighborhood does not include itself, this quantity depends only on the actions of the others, $\boldsymbol{a}_{-i} = (a_j)_{j\neq i}$. A bank's best response to its in-neighbors' actions can be summarized as follows:

$$\mathcal{R}_i(\boldsymbol{a}_{-i}, w_i) = \begin{cases} (1,1) & \text{if } S^\mu_i \geq \theta_\mu \text{ for } \mu = R, C \text{ and } w_i = 1 \\ (0,0) & \text{otherwise.} \end{cases} \tag{2}$$

Note that the bank takes the same action in both markets in any best response, and so as a shorthand we will let the variable y_i denote the action of i (in both markets). For simplicity, we will restrict our analysis to the special case where the threshold θ_μ is equal to 1 for $\mu \in \{R, C\}$. In this case, we can express the market-μ component of the best response as a binary-valued

[8]Examples of shocks include news about the security that trigger banks' internal risk limits and force the bank to cease operations, or the bankruptcy of a bank as a consequence of, e.g., fraud.

function,

$$[\mathscr{R}_i(\boldsymbol{a}_{-i}, w_i)]_\mu = w_i \cdot B\left(S_i^R(\boldsymbol{a}_{-i}) \cdot S_i^C(\boldsymbol{a}_{-i})\right), \tag{3}$$

where we define the operator $B(x) = 1$ if $x > 0$ and $B(x) = 0$ otherwise.

Because in any strategy profile where banks are playing best responses, they set their action in both markets to be the same, we will often abuse notation and refer to a bank's action in such strategy profiles as simply 0 or 1.

2.2 Model Interpretation, Discussion of Assumptions, and Relation to the Literature

Market freezes and liquidity hoarding. In the liquidity provision game between banks outlined in the previous section, a bank's choice to be active in either market represents its willingness to provide market and funding liquidity to its trading partners. While we make simple reduced-form assumptions on liquidity provision in order to focus attention on network considerations, our model relates to an active literature on liquidity hoarding as a source for financial market freezes. This literature considers the precise mechanisms of liquidity provision in more detail. Banks in Gale and Yorulmazer (2013) choose to hoard liquidity, even if there is willing borrower in the market, because of a precautionary or a speculative motive. Heider et al. (2015) show that interbank markets can break down and banks start hoarding liquidity if banks have private information about their assets and adverse selection is prevalent. Empirical evidence corroborates these theoretical models. Ashcraft et al. (2011) and Acharya and Merrouche (2013), for example, show that banks in the US and UK indeed were hoarding liquidity during the global financial crisis. Liquidity in an OTC market is, unlike in centralized markets, *local*: due to frictions (e.g. search frictions), a situation can exist in such markets where some market participants have liquidity supply while others have liquidity demand, but no trade ensues. Aggregate liquidity is then the result of the individual decisions by market participants whether or not to be active in the market.

Bank's activity decision and balance sheets. In the repo market, bank i is active ($a_i^R = 1$) and provides *funding liquidity* if it is willing to provide repo financing to all counterparties in its out-neighborhood. Similarly, bank i is active ($a_i^C = 1$) and provides *market liquidity* in the collateral market, if it is willing to purchase collateral from all counterparties in its out-neighborhood. See Section 5 for more discussion of this simplification. The best responses we assume in the liquidity provision game between banks are stylized, but capture basic forces shaping banks' incentives, which we now exposit in discussing a very simple balance sheet. In Section 5 we examine some of these assumptions further.

Our stylized bank i has three assets. It (i) provides repo financing,[9] A_i^R, to its neighbors; (ii) holds an inventory A_i^C of the collateral asset; and (iii) holds an inventory, A_i^O, of other, illiq-

[9]Recall that we think of a repo loan as a short-term loan that is secured by the collateral asset.

uid, assets. The bank funds its assets with three liabilities: (i) L_i^R, short-term reverse repos from its counterparties;[10] (ii) L_i^L, long-term debt; and (iii) E_i, equity. The usual accounting identity requires that $A_i^R + A_i^C + A_i^O = L_i^R + L_i^L + E_i$. Bank i's risk-weighted capital ratio is given by $\lambda_i = E_i/(A_i^C + A_i^O)$. We assign the same risk weight to collateral and other assets and a zero risk weight to repo loans due to their short maturity and secured nature.[11] The bank faces a regulatory capital constraint, the requirement that $\lambda \geq \overline{\lambda}$. It also faces cash-in-advance constraint, which requires that it needs to obtain repo financing from one of its trading partners in order to provide repo financing to another bank.

Assumptions about bank behaviour. We assume that: (i) A bank's capital constraint is tight: $\lambda = \overline{\lambda}$. (ii) A bank can only obtain funding in the OTC repo market. (iii) A bank can only sell collateral in the OTC collateral market. (iv) A bank will only extend repo loans if it can liquidate the collateral in case of default.

In this setup, a bank's feasible intermediation activity in the repo and collateral markets looks as follows. Suppose a bank extends a repo loan of size ΔA_R to one of its counterparties. Increasing repo lending (A_R) leaves the capital ratio unchanged but requires an equal increase in reverse repo borrowing $\Delta L_R = \Delta A_R$ (because of the accounting equality between assets and liabilities, along with the assumption that funding can come only from the OTC repo market). Note that for this transaction the bank requires access to the collateral market where it would liquidate the collateral in case of counterparty default. In other words, extending a repo loan requires access to both funding liquidity and market liquidity.

Now suppose, instead, that a bank purchases an amount ΔA_C of the collateral asset from one of its counterparties. Due to the cash-in-advance constraint, the bank obtains reverse repo to fund the initial purchase $\Delta L_R = \Delta A_C$.[12] This transaction leads to a temporary expansion of the bank's balance sheet. The bank reverses this expansion by selling an amount ΔA_C of the collateral asset to one of its other counterparties and by repaying the reverse repo loan.[13] In summary, purchasing the collateral asset requires access to both funding liquidity and market liquidity.

[10] A reverse repo is a repo agreement where bank i is the repo seller instead of the repo buyer, i.e. where i obtains repo funding. By convention, reverse repos are accounted on the liability side of the balance sheet. A reverse repo obtained by i from j will appear as repo financing on j's asset side.

[11] The repo buyer requires a *haircut* on the collateral, i.e. will provide a fraction $\alpha \in [0,1]$ of the collateral's face value as funding. The haircut insures the repo buyer against the risk of a devaluation of the collateral. The repo buyer will also require interest $r \geq 0$ per unit of repo financing. For simplicity, we assume that the collateral is safe, but possibly illiquid, i.e. that $\alpha = 1$ and $r = 0$.

[12] Note the similarity of this mechanism to the model in Acharya et al. (2011). They consider a model in which debt has a short maturity relative to the collateral and buyers of the collateral require collateralized funding to purchase the asset if it needs to be liquidated. Acharya et al. (2011) show that, in the presence of liquidation costs, the debt-bearing capacity of a collateral asset is determined by the future liquidation value of the collateral, which in turn is determined by its future debt-bearing capacity. Here, we consider a simplified version of this model with a binary debt-bearing capacity depending on the availability of funding and market liquidity in the neighborhood of a given intermediary.

[13] It is worth noting that the mechanics of intermediation outlined above are broadly consistent with the practices of intermediaries in financial markets, who target a flat book in the asset they are intermediating.

In this description, banks intermediate liquidity and collateral; there are fundamental traders who provide the basic demand and supply that drives the chains of lending and relending described above. We do not need to explicitly model these for our purpose of analyzing banks' decisions to be active or not. In Appendix A we outline a simple foundation for bank behaviour that illustrates how our model of banks' decision to be active can be understood in a model with microfoundations.

Interpretation of the exogenous shock. In the context we have described above, the exogenous shock vector w is interpreted as the realization of uncertainty in the value of the illiquid asset. A shock realization $w_i = 0$ corresponds to an adverse shock that forces bank i to cease its intermediation activity and become inactive. This, combined with the patterns of intermediation described above, motivate the best response functions of the liquidity provision game: As long as bank i has at least one trading partner active in the repo market (to provide funding liquidity), and one, possibly different, trading partner active in the market for collateral (to provide market liquidity), and as long as it does not experience a shock, the bank is willing to be active in both markets. Note that choosing to be active in a market does not imply that actual transactions occur. Instead the outcome of the liquidity provision game constrains potential transactions in these markets.

The best response functions (2) can, in principle, be obtained from a simple utility function for the bank. A bank is active in a market if it makes profits from intermediating (repo or collateral). These profits are a spread charged from trading the collateral less a cost of doing business. The price of the collateral depends on the demand, i.e. on the number of trading partners that are active. Our assumption that a bank is active as long as it has one active trading partner abstracts from prices. In a more realistic scenario, where the price declines in the demand (i.e. the number of active buyers in the bank's neighborhood) declines, a bank becomes inactive even without being cut off the market entirely. In this sense our model is conservative about banks' withdrawal.

Illiquidity spirals in the literature. In the literature on secured lending, the theoretical papers most closely related to ours are Brunnermeier and Pedersen (2009) and Acharya et al. (2011). Our model of illiquidity spirals in repo and collateral markets explicitly takes into account the OTC network structure of these markets. This sets us apart from Brunnermeier and Pedersen (2009) who study the feedback between market and funding liquidity in centralized markets. Our model shows that the network structure alone, abstracting from haircut and pricing feedback, can lead to a significant amplification of exogenous shocks in collateral and repo markets. Acharya et al. (2011) show how a bank's ability to obtain secured funding depends on the risk and liquidation value of the collateral and how this dependency leads to a feedback between collateral and debt markets mediated by the debt capacity (essentially, quantity) offered. In contrast to these works, we show that, given the complementarity of collateral and secured debt markets and while abstracting away from prices, the over-the-counter nature of these markets is sufficient to generate a feedback between market and funding liquidity that amplifies exogenous shocks.

Our paper is also related to Martin et al. (2014) who model repo runs arising from pure coordination failure in a dynamic model. They show that repo markets in which haircuts cannot be adjusted, such as a centralized or tri-party repo market, can be more fragile than bilateral repo markets in which haircuts can be adjusted. We contribute to this debate by showing that if at least one OTC market is replaced by a centralized exchange, the resilience of liquidity improves. This suggests at least two opposing effects that need to be taken into account when judging the merits of centralized exchanges: the flexibility of haircuts and adverse network effects.

Empirical evidence of financial market freezes. A number of authors empirically study the fragility of repo markets during the 2007/2008 financial crisis. Gorton and Metrick (2012) argue that a central aspect of the crisis was a system-wide run on short-term collateralized debt, and in particular on certain non-government bond repo markets. Krishnamurthy et al. (2014) show that, while the trip-party repo market has been more stable during the crisis than bilateral repo, markets for asset-backed commercial paper (ABCP) experienced a significant contraction.[14] This finding is mirrored by Copeland et al. (2014), who document substantial heterogeneity in access even to tri-party repo funding in late 2008.

Financial over-the-counter networks. The structure of the over-the-counter markets is at the heart of our model. Empirical studies of financial networks often find them to have a core-periphery structure. Studying the inter-dealer corporate bond market, Maggio et al. (2017) show that this market has a persistent core-periphery structure. Similarly, Li and Schürhoff (2014) show that the market for municipal bonds has a core-periphery structure. Anecdotal evidence suggests that most interbank lending nowadays is secured. Hence, we can hope to infer from the structure of the interbank market about the structure of other over-the-counter markets as well. Craig and Von Peter (2014) show, for example, that the German interbank market follows a core-periphery structure fairly closely. In the international context, Gabrieli and Georg (2014) show that the Euroarea interbank market follows a core-periphery structure less closely, with large international banks connecting the different national core-periphery networks.

A large literature studies contagion in financial networks that ensues when the default of one financial institution causes the subsequent default of other financial institutions (see, for example, Acemoglu et al. (2015), Elliott et al. (2014), and Zawadowski (2013), as well as Glassermann and Young (2016) for an extensive overview. See also Burkholz et al. (2016) for a contagion analysis in a multiplex network). While our model could be interpreted as a contagion model, contagion occurs via the banks' decision variables to withdraw from markets rather than via actual defaults. This is in line with the empirical evidence from the 2008 financial crisis: even the default of a large bank, such as Lehman Brothers, did not trigger many subsequent defaults,[15]

[14] Covitz et al. (2012) study the fragility of asset-backed commercial paper markets. Our model naturally extends to ABCP markets.

[15] An exception is the Reserve Primary Fund, who, due to a large exposure to Lehman Brothers, filed for

while it is likely to have led to a freeze in markets for short-term collateralized debt. In reality, financial networks are endogenous, and this should be part of our ultimate model of intermediation; for one recent contribution and a survey of related work, see Farboodi (2016).

Games on networks. In a broader sense, our paper is related to the networks literature in economic theory, especially contagion and games on networks. Papers such as Blume (1993) and Ellison (1993) first suggested that local interaction, modeled via a network structure, can be used to study the likelihood that various equilibria would be played and how an economy may reach an equilibrium. Whereas these early papers focused on noisy heuristic adjustment procedures, Morris (1997, 2000) studied games with standard (no-noise) solution concepts and related networks to games of incomplete information. The latter paper's results, applied to a network game, show when a network can support heterogeneous actions, and what conditions result in equilibria such as (in our context) "everyone withdraws." Jackson and Yariv (2007) and Galeotti et al. (2009) developed this sort of model to accommodate random networks described by a degree distribution. Our approach has much in common with this theoretical literature on games in networks. We also use the structure of supermodular games (as in Milgrom and Roberts (1990)) to identify benchmark equilibria, and look at their structure for large random networks. The main innovation relative to these papers on games in networks is that we study multilayer networks, and analyze how the multilayer aspect of their structure affects the best-response structure of the game, especially when the underlying networks are random. Equilibria depend more sharply on the parameters on the network than has been reported previously, due to the discontinuities discussed above. Thus, our paper relates to the network games literature broadly, and offers new game-theoretic implications arising from multilayer network structures.

3 Equilibrium for general networks

In the following we will consider arbitrary directed networks of trading relationships, $\mathcal{G}_R$ and $\mathcal{G}_C$, for the repo and collateral markets.

3.1 Definition and existence of equilibrium

Banks play a game of complete information given a realization of the exogenous shock. An *equilibrium* in this game is a fixed point of the best-response correspondence $\mathscr{R}$, whose components are the functions $\mathscr{R}_i$ defined in (3).

To study these, we first introduce some notation and terminology. First, for any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ define the pointwise ordering $\boldsymbol{x} \leq \boldsymbol{y} \iff x_i \leq y_i$ for all $i \in N$. With this ordering, we have a game of strategic complements: the best response of each agent is monotone in its

bankruptcy the day after Lehman Brothers filed for bankruptcy.

argument (holding the shock $\boldsymbol{w}$ fixed). Thus, a standard application of Tarski's fixed-point theorem shows that equilibria exist and are nicely ordered. Indeed, they form a complete lattice, containing a greatest element (a maximal equilibrium that pointwise dominates every other) and a least element. This lattice structure permits a rich theory of comparative statics: see Milgrom and Roberts (1990) and Zhou (1994).[16]

We focus on the maximum equilibrium, following the rationale in Elliott et al. (2014), and call that equilibrium $\boldsymbol{y}^*$. A natural equilibrium liquidity measure is then

$$\mathscr{L}(\boldsymbol{y}^*) = \sum_i y_i^*, \tag{4}$$

which simply counts the number of banks willing to provide liquidity.

3.2 Equilibrium, network structure and shocks

3.2.1 Simple examples

Let us briefly consider conditions under which the maximum equilibrium will look very simple: $\boldsymbol{y}^* = \mathbf{0}$ or $\boldsymbol{y}^* = \mathbf{1}$. If the shock vector takes the form $\boldsymbol{w} = \mathbf{0}$, the equilibrium will be $\boldsymbol{y}^* = \mathbf{0}$. Equally, for an arbitrary shock vector but in the absence of links between the banks, the equilibrium will be $\boldsymbol{y}^* = \mathbf{0}$. If the shock vector takes the form $\boldsymbol{w} = \mathbf{1}$ and all banks have at least one incoming link in both $\mathscr{G}_C$ and $\mathscr{G}_R$, in the maximum equilibrium, all banks will choose to be active and $\boldsymbol{y}^* = \mathbf{1}$.

3.2.2 Characterization and algorithm

As the simple examples above suggest, the equilibrium of the liquidity provision game depends on two factors: the network structure of the over-the-counter markets and the realization of the exogenous shock. We will examine this dependence.

Suppose the network structure satisfies the following assumption.

Assumption 3.1. All banks have at least one incoming edge in $\mathscr{G}_R$ and $\mathscr{G}_C$.

In this case, for the shock realization $\boldsymbol{w} = \mathbf{1}$, the maximal equilibrium is $\boldsymbol{y}^* = \mathbf{1}$. We can think of this as a *pre-shock* situation, in which no assets have lost any value; our assumption guarantees that no banks withdraw. Starting from such a baseline, we can consider equilibrium outcomes for more interesting realizations of the shock, i.e. for shock vectors with some $w_i = 0$. We refer to this regime as the *post-shock* regime. In the following we will characterize how the post-shock equilibrium depends on the structure of the networks $\mathscr{G}_C$ and $\mathscr{G}_R$.

Let W denote the set of banks for which $w_i = 0$. Let $\mathscr{G}_C(W)$ and $\mathscr{G}_R(W)$ denote the networks after the nodes and edges corresponding to the banks in W have been removed. $\mathscr{G}_C(W)$

[16] For similar arguments in other financial network applications, see Eisenberg and Noe (2001) or Elliott et al. (2014).

and $\mathscr{G}_R(W)$ are simply the subgraphs implied by conditioning on $w_i = 1$. It is useful to consider these subgraphs since the remaining banks' actions will be determined exclusively by the network structure. To link the equilibrium outcome to the network structure first define a stable component in a network $\mathscr{G}$.

Recall that a strongly connected component of a network $\mathscr{G}$ is a maximal set of nodes S so that for every $i, j \in S$, there is a path from i to j consisting of edges in $\mathscr{G}$.

Definition 3.1 (Stable component)**.** A *stable component* of a network $\mathscr{G}$ is a nonsingleton strongly connected component.

This definition rules out trivial strongly connected components, and in fact selects those strongly connected components for which each node has at least one incoming edge. It also, equivalently, selects the components that have no *exiting* edges.

This concept of stable components can be generalized to the coupled networks $\mathscr{G}_R$ and $\mathscr{G}_C$.

Definition 3.2 (Mutually stable component)**.** Let $\mathscr{G}_R$ and $\mathscr{G}_C$ be directed graphs. Let $\mathscr{C}_R$ (respectively, $\mathscr{C}_C$) denote a stable component in $\mathscr{G}_R$ (respectively, $\mathscr{G}_C$). Then a mutually stable component is $MSC(\mathscr{G}_R, \mathscr{G}_C) = \mathscr{C}_R \cap \mathscr{C}_C$.

A mutually stable component is therefore a subset of nodes which form part of a stable component in both networks. The existence and size of a mutually stable component is closely related to the maximal equilibrium.

Proposition 1 (Maximal equilibrium and mutually stable components)**.** *In the maximal equilibrium $\boldsymbol{y}^*$, the set of active banks ($y_i^* = 1$) equals the set of banks in the union of all mutually stable components of $\mathscr{G}_R$ and $\mathscr{G}_C$.*

In other words, the active banks in the maximal equilibrium are exactly those belonging to any mutually stable component of $\mathscr{G}_R$ and $\mathscr{G}_C$. That this is an equilibrium follows from the form of $\mathscr{R}$: All banks in the mutually stable component have, by definition, at least one incoming link in both networks from other banks in the mutually stable component. To complete the proof of Proposition 1, it suffices to show that no equilibrium expands the set of active banks relative to $\boldsymbol{y}^*$. To this end, observe that $\mathscr{R}(\boldsymbol{y}) = \boldsymbol{y}$ implies that every active bank under $\boldsymbol{y}$ is in a stable component in each network, so the set of banks that are set to be active by the algorithm must be a subset of $MSC(\mathscr{G}_R, \mathscr{G}_C)$.

By the supermodular structure of the game, the maximum lattice point can be found by starting from the maximum feasible actions $\boldsymbol{y} = (1, \dots, 1)$, and repeatedly applying the best response function, $\mathscr{R}$ (Milgrom and Roberts, 1990). Algorithm 1 below makes this precise.

(a) Star network

(b) Core-periphery network.

Figure 1: Example networks

Algorithm 1 Algorithm to compute the equilibrium (greatest fixed point of $\mathscr{R}$).

$\boldsymbol{y} \leftarrow \mathbf{1}$
$\boldsymbol{x} \leftarrow \mathbf{0}$
while $\boldsymbol{x} \neq \boldsymbol{y}$ **do**
  $\boldsymbol{x} \leftarrow \boldsymbol{y}$
  $\boldsymbol{y} \leftarrow \mathscr{R}(\boldsymbol{y}, \boldsymbol{w})$
end while
return $\boldsymbol{y}$

3.3 Star and core-periphery networks

We now consider two important examples to illustrate the interplay between network structure and the realization of the exogenous shock. We will first consider a star network structure for both repo and collateral markets. In a second example we will generalize the star network to a core-periphery network, which is a stylized representation of patterns often seen in empirical studies of over-the-counter markets: see Abad et al. (2016) and Craig and Von Peter (2014).

3.3.1 Star network

Suppose both the repo network $\mathscr{G}_R$ and the collateral network $\mathscr{G}_C$ can be represented by star networks. While this configuration is very simple, it already leads to non-trivial results. We illustrate that an adverse shock in one market can spill over into the other, causing additional banks to withdraw from the market. This contagion between the repo and collateral networks amplifies the initial adverse shock and leads to a reduced equilibrium liquidity measure.

Fig. 1 illustrates a star network. The network consists of two types of nodes: a single hub node (blue rectangle) and a set of peripheral nodes (yellow circles). The hub node has bi-directional links to all peripheral nodes but peripheral nodes are not connected to each other. Thus, the OTC markets $\mu \in \{R, C\}$ are characterized by a partition of the set of banks N into a single hub bank $B_{H,\mu}$ and set of peripheral banks $B_{P,\mu} = N \setminus B_{H,\mu}$.

Let us now consider a shock profile $\boldsymbol{w}^j$ in which bank j, chosen uniformly at random, receives an adverse shock, i.e.

$$w_i^j = \begin{cases} 0 \text{ if } i = j, \\ 1 \text{ otherwise.} \end{cases}$$

The probability that bank j receives an adverse shock is then $P(\boldsymbol{w}^j) = 1/n$. The following results for the post-shock liquidity measure will be computed by averaging over all shock profiles of this form.

We also consider the realization of the network labels—that is, the identity of the hub in each network—to be random. Thus, we have a (simple) random multilayer network, and we will condition on realizations of the identities of the hub and the periphery as random variables.

To compute the post-shock liquidity measure it is sufficient to consider two cases. In the first case the hub banks in the repo and collateral markets are different, i.e. $B_{H,C} \neq B_{H,R}$. In the second case the hub banks are the same, i.e. $B_{H,C} = B_{H,R}$.

Proposition 2 (Coupled star networks – post-shock liquidity measure)**.** *Consider two star networks $\mathscr{G}_C$ and $\mathscr{G}_R$. Let $E[\cdot \mid \cdot]$ denote the conditional expectation operator. The conditional expected post-shock liquidity measures are*

$$E[\mathscr{L} \mid B_{H,C} \neq B_{H,R}] = \frac{(n-2)(n-1)}{n},$$
$$E[\mathscr{L} \mid B_{H,C} = B_{H,R}] = \frac{(n-1)^2}{n}.$$

Given a belief $q = P(B_{H,C} = B_{H,R})$ the post-shock liquidity measure for the coupled star networks is

$$\hat{\mathscr{L}}^{s-s} = E[\mathscr{L}] = qE[\mathscr{L} \mid B_{H,C} = B_{H,R}] + (1-q)E[\mathscr{L} \mid B_{H,C} \neq B_{H,R}]$$
$$= q\frac{(n-2)(n-1)}{n} + (1-q)\frac{(n-1)^2}{n}.$$

For the proof see Appendix B.1.

First, note that the conditional post-shock equilibrium liquidity is always less than $n-1$, the liquidity in case of no additional withdrawals after the adverse shock. This corresponds to an effect we call an *illiquidity spiral.* Second, note that the equilibrium liquidity is smaller when the repo and collateral markets do not share the same hub node. This suggests that OTC markets that are less similar are less resilient to exogenous shocks. In later sections we will extend this observation.

To put the above result in context it is useful to compare it to the equilibrium liquidity obtained when one of the two OTC markets is replaced by a centralized exchange. In a centralized exchange all banks can trade with all other banks. Therefore, we model a centralized exchange as a fully connected (complete) network. Suppose that the collateral market is replaced by a complete network. There can never be contagion "through" the complete network, because no node is critical to connectivity within it. In other words, the mutually stable components of the pair of networks $(\mathscr{G}_R, \mathscr{G}_C)$ are simply the stable components of $\mathscr{G}_R$. Therefore, the expected post-shock liquidity in the star-complete configuration $\hat{\mathscr{L}}^{s-c}$ is the same as in the star-star configuration conditional on the two networks sharing the same hub bank, i.e. $B_{H,C} = B_{H,R}$.

Proposition 3 (Coupled star and complete networks – post-shock liquidity measure)**.** *Consider a star network $\mathscr{G}_C$ and a complete network $\mathscr{G}_R$. The post-shock liquidity measure is*

$$\hat{\mathscr{L}}^{s-c} = E[\mathscr{L}] = \frac{(n-1)^2}{n}.$$

Furthermore, the post-shock liquidity in the star-complete configuration always exceeds the post-shock liquidity in the star-star configuration if $q > 0$:

$$\hat{\mathscr{L}}^{s-c} > \hat{\mathscr{L}}^{s-s}.$$

The latter assertion follows immediately from the fact that $E[\mathscr{L} \mid B_{H,C} \neq B_{H,R}] < E[\mathscr{L} \mid B_{H,C} = B_{H,R}] = \hat{\mathscr{L}}^{s-c}$ and the fact that $\hat{\mathscr{L}}^{s-s}$ is a convex combination of these two quantities. Hence, the coupling of the OTC repo and collateral markets leads to a reduction of the expected equilibrium liquidity relative to the benchmark case of a centralized collateral market.

3.3.2 Core-periphery network

Core-periphery networks, see Fig. 1 for a stylized example, are often used in models of over-the-counter markets since they capture the segmented dealer-client structure of many OTC markets. In this section we will generalize the results from the previous section and consider illiquidity spirals when $\mathscr{G}_R$ and $\mathscr{G}_C$ are modeled as stylized core-periphery networks.

Nodes in a core-periphery network are partitioned into two sets: a set of core nodes and a set of peripheral nodes. A core node is connected to all other core nodes and a subset of peripheral nodes via bi-directional links. A peripheral node is connected only to a single core node via a bi-directional link (see Fig. 1). Let $\Omega_\mu : N \to \{c, p\}$ denote the map that assigns each node in network $\mathscr{G}_\mu$ to the core or periphery. We will consider a random set of coupled core-periphery networks $\mathscr{G}_C$ and $\mathscr{G}_R$ that are jointly parameterized by the number of banks in the core in both networks, the number of banks in the core of $\mathscr{G}_C$ and the periphery of $\mathscr{G}_R$, etc. Let n_{cc} denote the number of banks that are core nodes in both $\mathscr{G}_R$ and $\mathscr{G}_C$, we write $n_{cc} = \#\{i \mid \Omega_R(i) = c \wedge \Omega_C(i) = c\}$. Similarly define $n_{cp} = \#\{i \mid \Omega_R(i) = c \wedge \Omega_C(i) = p\}$, $n_{pc} = \#\{i \mid \Omega_R(i) = p \wedge \Omega_C(i) = c\}$ and $n_{pp} = \#\{i \mid \Omega_R(i) = p \wedge \Omega_C(i) = p\}$. Note that of course, $n_{cc} + n_{cp} + n_{pc} + n_{pp} = n$.

The parameter vector $\boldsymbol{n} = (n_{cc}, n_{cp}, n_{pc}, n_{pp})$ fully determines the set of random coupled core-periphery networks we will study in this section. A random network is constructed by the following mechanism. Banks are randomly assigned to one of the four types (cc, cp, pc, pp) without replacement. All core banks in a given network are connected via bi-directional links. For each peripheral bank in a given network we pick one core bank uniformly at random (with replacement) and establish a bi-directional link.

We parameterize a shock profile $\boldsymbol{w}$ by the number of banks of a particular type with $w_i = 1$ $\boldsymbol{m} = (m_{cc}, m_{cp}, m_{pc}, m_{pp})$. We denote the complement, i.e. banks with $w_i = 0$, by $\overline{\boldsymbol{m}} = (\overline{m}_{cc}, \overline{m}_{cp}, \overline{m}_{pc}, \overline{m}_{pp})$. The shock size is given by $\overline{m}_W = \sum_{tp} \overline{m}_{tp}$. Let the random variable k_R (k_C) denote the number of peripheral neighbors of a core bank in $\mathscr{G}_R$ ($\mathscr{G}_C$) with $w_i = 1$. Given the network parameters, the network construction mechanism and a parameterization of the shock profile, the expected number of peripheral neighbors (with $w_i = 1$) of a core bank in $\mathscr{G}_R$ and $\mathscr{G}_C$ will be, respectively,

$$\overline{k}_R(\boldsymbol{m}) = E[k_R \mid m_{pc}, m_{pp}] = \frac{m_{pc} + m_{pp}}{n_{cc} + n_{cp}},$$
$$\overline{k}_C(\boldsymbol{m}) = E[k_C \mid m_{cp}, m_{pp}] = \frac{m_{cp} + m_{pp}}{n_{cc} + n_{pc}}.$$

We can derive an expression for the post-shock liquidity measure by averaging over all parameterizations of the shock profile. For this we first compute an expression for the expected number of additional banks that will withdraw conditional on the withdrawal of a bank of a particular type, e.g. n_{cc}.

Lemma 1 (Type conditional spill-over). Let $z(\boldsymbol{m}) \in [0, 1]$ denote the expected fraction of peripheral banks in $\mathscr{G}_C$ that are not connected to a core bank in $\mathscr{G}_R$ with $w_i = 0$. Then the expected number of additional banks that will withdraw conditional on the withdrawal of a bank of a particular type is given by

- cc - (core-core): $a_{cc} = \overline{k}_R + \overline{k}_C z$,
- cp - (core-periphery): $a_{cp} = \overline{k}_R$,
- pc - (periphery-core): $a_{pc} = \overline{k}_C z$,
- pp - (periphery-periphery): $a_{pp} = 0$.

To develop an intuition for this result, let us consider the effects of the withdrawal of a bank given its type (cc, cp, pc, pp). If a core bank withdraws in a given network, only its peripheral neighbors will withdraw. If a peripheral bank withdraws, no additional bank will withdraw. Thus if a cc bank withdraws its peripheral neighbors in both networks will withdraw. If a cp (pc) bank withdraws only peripheral neighbors in $\mathscr{G}_R$ ($\mathscr{G}_C$) will withdraw. Finally if a pp bank withdraws no additional bank withdraws.

This would be easy to compute if core banks in $\mathscr{G}_C$ and $\mathscr{G}_R$ did not share peripheral banks. However, if $\sum_i w_i$ becomes large relative to the network size n, some withdrawing core banks are quite likely to share periphery banks. Fig. 2 illustrates this situation. Here the withdrawing core banks labeled 1 and 2 share a peripheral neighbor labeled 5. Treating the amplification effect of 1 and 2 as independent would result in the double counting of bank 5. We can correct for this double counting by appropriately scaling the expected number of peripheral neighbors of a core node in one of the two networks. In Lemma 1 we scale $\overline{k}_C$ by the fraction (z) of peripheral

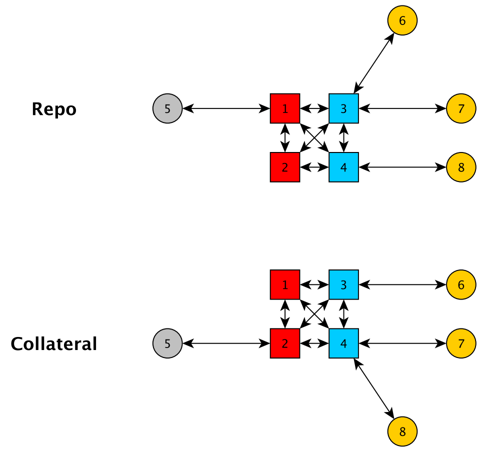

Figure 2: Illustrative example: double counting of withdrawing peripheral nodes. The top network represents $\mathcal{G}_R$ while the bottom network represents $\mathcal{G}_C$. In both networks the core is comprised of nodes $\{1,2,3,4\}$ (squares) while the periphery is comprised of nodes $\{5,6,7,8\}$ (circles). We have $w_i = 0$ for $i = 1,2$ (red) and $w_i = 1$ otherwise. Conditional on the realization of the shock bank 5 is the only additional bank that chooses to withdraw (grey).

banks in $\mathcal{G}_C$ that are not connected to a core bank in $\mathcal{G}_R$ with $w_i = 0$. While it is possible to derive an exact expression for $z(\boldsymbol{m})$, one can use an approximation $z(\boldsymbol{m}) \approx 1-(m_{cc}+m_{cp})/(n_{cc}+n_{cp})$ for simplicity.[17] This approximation performs well in numerical experiments.

In the final step we aggregate the effect quantified in Lemma 1 and average over all parameterizations of the shock profile.

Proposition 4 (Core-periphery - post shock liquidity measure)**.** *The expected number of withdrawing banks given $\overline{\boldsymbol{m}}$ and $\boldsymbol{n}$ is given by*

$$A(\overline{\boldsymbol{m}}) = (1+a_{cc})\overline{m}_{cc} + (1+a_{cp})\overline{m}_{cp} + (1+a_{pc})\overline{m}_{pc} + (1+a_{pp})\overline{m}_{pp}.$$

The expected number of banks providing liquidity conditional on shock size $\overline{m}_W$ is given by

$$\hat{\mathcal{L}}^{cp-cp}(\overline{m}_W) = E[\mathcal{L} \mid \overline{m}_W] = n - \sum_{\overline{\boldsymbol{m}}} A(\overline{\boldsymbol{m}})P(\overline{\boldsymbol{m}},\boldsymbol{n})$$

where $P(\cdot,\cdot)$ is the multivariate hypergeometric distribution with parameters $\boldsymbol{n}$ and $\overline{\boldsymbol{m}}$.

Finally, suppose that the collateral market is replaced by a complete network, i.e. a centralized exchange. This special case is nested in our parameterization of coupled core periphery networks and corresponds to the case where $\boldsymbol{n}' = (n'_{cc}, 0, n'_{pc}, 0)$. Since the number of core and periphery nodes in the repo market have to be preserved we require that $n'_{cc} = n_{cc} + n_{cp}$ and

[17] Computing the exact expression involves keeping track of all network configurations and their probabilities and is complicated by the dependencies introduced by sampling without replacement. At the same time computing this exact expression for z does not add any substantial insights.

$n'_{pc} = n_{pc} + n_{pp}$, where $\boldsymbol{n}$ parameterizes the core-periphery network before the collateral market is centralized.

Proposition 5 (Core-periphery and centralized market - post shock liquidity measure)**.** *Let $\mathcal{G}_R$ be a core periphery network and $\mathcal{G}_C$ be a complete network jointly parameterized by $\boldsymbol{n}'$. Given $\overline{\boldsymbol{m}}$, the expected number of withdrawing banks is*

$$A(\overline{\boldsymbol{m}}) = (1 + \overline{k}_R)\overline{m}_{cc} + \overline{m}_{pc}.$$

The expected number of banks providing liquidity conditional on shock size $\overline{m}_W$ is

$$\hat{\mathcal{L}}^{cp-c}(\overline{m}_W) = E[\mathcal{L} \mid \overline{m}_W] = n - \sum_{\overline{\boldsymbol{m}}} A(\overline{\boldsymbol{m}}) P(\overline{\boldsymbol{m}}, \boldsymbol{n}')$$

where P is defined as above. For a fixed shock size $\overline{m}_W$ and $n_{cp} + n_{pp} > 0$, the post-shock liquidity in case of a centralized collateral market is always greater than post-shock liquidity in the pure core-periphery case

$$\hat{\mathcal{L}}^{cp-c}(\overline{m}_W) > \hat{\mathcal{L}}^{cp-cp}(\overline{m}_W).$$

In other words, when a collateral market with some peripheral nodes is replaced by a centralized exchange, post shock liquidity is always improved. Note that the above result does not rely on the exact functional form of $z(\boldsymbol{m})$. We only require that $z(\boldsymbol{m}) > 0$ for some network configurations when both markets are core-periphery. This is the case for the scenarios we are considering here.[18]

To illustrate the size of this effect and the comparative statics of the liquidity measure for different shock sizes, we numerically evaluate the post shock liquidity measure in Fig. 4 for an example with $(n_{cc} = 0, n_{cp} = 2, n_{pc} = 2, n_{pp} = 50)$ using the approximation $z(\boldsymbol{m}) \approx 1 - (m_{cc} + m_{cp})/(n_{cc} + n_{cp})$.[19] In the supplementary information, Appendix F, we study the effect of the network size on post-shock liquidity. While the risk of a catastrophic (i.e. discontinuous in the shock size) cascade becomes less likely for smaller networks, the extent of the cascade depends on the network characteristics of the exogenously failing nodes.

4 Equilibrium for random networks

We now turn our attention to liquidity in random network models. Intuitively, the set of networks we consider can be obtained through the rewiring of links in a given network while holding the in- and out-degrees of the nodes fixed. Random networks are tractable, allowing us to study the effect of an exogenous shock for network structures beyond the concrete examples given in the previous section.

[18]The case when there are just two nodes is an exception. However, in this case the notion of a core and a periphery are not well defined. Therefore, we exclude this corner case from our analysis.

[19]To speed up the calculation, rather than summing over the entire probability space, $E[\mathcal{L} \mid \overline{m}_W]$ is approximated by its Monte Carlo average.

4.1 Random network models

Let $\mathbf{d}_\mu^+ = (d_{i,\mu}^+)_{i=1}^n$ and $\mathbf{d}_\mu^- = (d_{i,\mu}^-)_{i=1}^n$ be sequences of non-negative integers representing the out-degree and in-degree, respectively, of a bank $i \in N$ in market $\mu \in \{R, C\}$, where as before $n = |N|$. We require that feasible degree sequences satisfy a number of technical assumptions summarized in Appendix B.2.[20] Let $G_\mu(n, \mathbf{d}_\mu^+, \mathbf{d}_\mu^-)$ be the set of graphs with feasible degree sequences $\mathbf{d}_\mu^+$ and $\mathbf{d}_\mu^-$. A random network $\mathcal{G}_\mu$ is then a draw from $G_\mu(n, \mathbf{d}_\mu^+, \mathbf{d}_\mu^-)$ uniformly at random. To obtain our analytical results, we study the limit of large networks, $n \to \infty$. In this limit, we define the joint degree distribution $p_{jk,\mu}$ as the fraction of banks with in-degree j and out-degree k.

In the following, we take the random networks $\mathcal{G}_R$ and $\mathcal{G}_C$ to be *independent* draws from G_R and G_C, respectively. This implies that, for example, bank i's out-degree in $\mathcal{G}_R$ is independent from its out-degree in $\mathcal{G}_C$. In Section 4.5 we relax this assumption.

4.2 Stress in over-the-counter markets

Let the repo (collateral) market correspond to a random network $\mathcal{G}_R$ ($\mathcal{G}_C$) as defined above. There is a pre- and a post-shock state. In the pre-shock state both repo and collateral markets are liquid, that is $\mathcal{L}(\boldsymbol{y}^*) > 0$, conditional on a shock profile $\boldsymbol{w} = \mathbf{1}$. In the post-shock state, a fraction $1-x$ of banks chosen uniformly at random receive an adverse shock $w_i = 0$. These banks withdraw from both markets. We call $1-x$ the size of the exogenous shock. Let $\mathcal{L}^*(x)$ be the expected liquidity of the maximal equilibrium averaged over all shock profiles of size $1-x$. How does $\mathcal{L}^*(x)$ vary as function of the shock size?

Proposition 6. *Given random networks $\mathcal{G}_R$ and $\mathcal{G}_C$, there exists a critical shock size $1 - x_c(\mathcal{G}_R, \mathcal{G}_C)$ at which the expected liquidity of the maximal equilibrium in the OTC repo and collateral market vanishes abruptly:*

1. *Frozen market regime: $\mathcal{L}^*(x) = 0$ for all $x \in [0, x_c)$,*
2. *Liquid market regime: $\mathcal{L}^*(x) > 0$ for all $x \in [x_c, 1]$,*
3. *Discontinuous transition: $\mathcal{L}^*(x_c^-) = 0 \neq \mathcal{L}^*(x_c) > 0$.*

Proposition 6 states that there are two liquidity regimes which materialize depending on the size of the exogenous shock. If the shock is sufficiently small, i.e. less than $1-x_c$, both repo and collateral markets are liquid. However, if the shock increases beyond its critical value by a marginal amount, liquidity in both markets vanishes. This is the market freeze regime. The fact that the transition between the liquid and the frozen market regime is *discontinuous* means that starting from a liquid market, the withdrawal of a single additional bank can be amplified through the coupled structure of the repo and collateral markets to the extent that liquidity freezes.

[20] These assumptions ensure, for example, that the first and second moment of the degree distribution remain bounded in the limit $n \to \infty$ or that the total number of out-degrees matches the total number of in-degrees.

The intuition for Proposition 6 is as follows. By assumption, for shock size $1 - x = 0$ we are in the liquid market regime. As the shock size is increased, more banks withdraw with some of their counterparties withdrawing as a result. For large values of $1 - x$, it becomes unlikely that a bank receives liquidity both in the repo and collateral markets as the corresponding networks have fragmented. Thus post-shock liquidity vanishes (frozen market regime) for sufficiently large shocks. Then there will be some intermediate shock size at which the market switches from the liquid to the frozen regime. The transition from the liquid to the frozen regime is discontinuous because the complementary nature of the repo and collateral markets produces "fragile connectors" (see Fig. 3 for an illustrative example). A bank with few counterparties in the repo market is fragile since it can easily lose access to liquidity. If the same bank is an important intermediary of liquidity in the collateral market, it will be a fragile connector. The withdrawal of such a bank becomes more likely as the shock size is increased and once it occurs, can have devastating consequences for liquidity in both markets. Note that this result holds for any random network which satisfies the assumptions made in Appendix B.2 – it does not hold for arbitrary networks, however. One counterexample are the stylized core-periphery networks discussed in Section 3.3. For these networks, a shock can never spread through the core and hence fragile connectors do not exist.

For the proof of Proposition 6 we need to show how the mutually stable component of the repo and collateral networks depends on the size of the exogenous shock. It can be shown that, in the limit $n \to \infty$, random networks posses two regimes. In one regime, there exists a unique giant component which scales with the size of the network, see Cooper and Frieze (2004). In the other regime the network is fragmented into many small components whose share of the network vanishes as $n \to \infty$. The stable component in a network $\mathcal{G}_\mu$, as defined in Section 3, is its giant component. The task is then to find how the mutual giant component, i.e. the set of banks which are in the giant component in both $\mathcal{G}_R$ and $\mathcal{G}_C$ varies with x. For this, we can make use of a branching process approximation[21] since we assume that $\mathcal{G}_R$ and $\mathcal{G}_C$ are independent. We define two branching processes, one per market, which explore the banks in each network that are active, i.e. those which form part of the giant component in $\mathcal{G}_R$ and $\mathcal{G}_C$. The degree sequences $\mathbf{d}_\mu^+$ and $\mathbf{d}_\mu^-$ and the size of the exogenous shock $1 - x$ implicitly determine the branching probabilities. If, starting from a randomly chosen bank, the branching process goes extinct, we know that this bank cannot form part of the giant component. This reasoning allows us to specify as system of coupled equations whose greatest solution yields $\mathcal{L}^*(x)$. For the full proof of Proposition 6 see Appendix B.3.

4.3 Resilience through centralized markets

It is again instructive to compare Proposition 6 to the benchmark of a centralized collateral market. As before for star and core-periphery networks, a centralized collateral market can be

[21] Amini et al. (2013) and Elliott et al. (2014) use a similar approach in their analysis of contagion on financial networks. Buldyrev et al. (2010) also rely on a branching process approximation in deriving similar results to Proposition 6 for percolation processes on coupled networks.

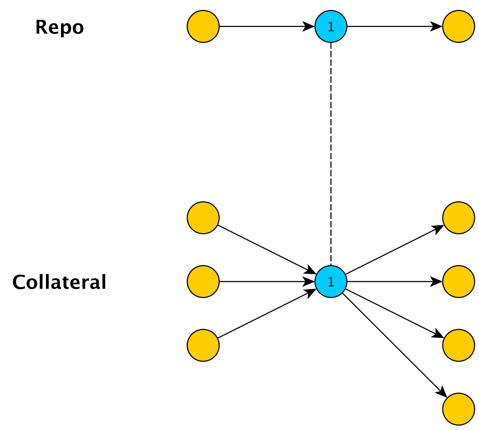

Figure 3: Illustrative example of a fragile connector: Bank 1 is a stylized example of a fragile connector. It receives liquidity in the repo market from only a single bank and is therefore susceptible to the withdrawal of this critical bank. At the same time bank 1 is the sole provider of liquidity to a number of banks in the collateral market – it acts as a connector in the market for collateral. Thus if bank 1 were to withdraw it would lead to a loss of access to liquidity for a large number of banks in the collateral market. The existence of such fragile connectors is crucial in the mechanism underlying Proposition 6.

easily nested into the random network model by replacing the over-the-counter market with a complete network. Let $\bar{\mathscr{G}}_C$ denote the complete network representing the centralized collateral market.

How does $\mathscr{L}^*(x)$ vary as function of the shock size and how does its behavior differ from the pure over-the-counter case discussed in the previous section?

Proposition 7. *Let $\mathscr{G}_R, \mathscr{G}_C$ be random networks and let $\bar{\mathscr{G}}_C$ be a complete network.*

(A) There exists a critical shock size $1 - r_c(\mathscr{G}_R, \bar{\mathscr{G}}_C)$ at which the expected liquidity of the maximal equilibrium in the OTC repo and centralized collateral market vanishes smoothly:

(1) Frozen market regime: $\mathscr{L}^*(x) = g_R(f_R, x) = 0$ for all $x \in (0, r_c]$,

(2) Liquid market regime: $\mathscr{L}^*(x) = g_R(f_R, x) \geq 0$ for all $x \in [r_c, 1]$,

(3) Smooth transition: $\mathscr{L}^*(r_c^-) = \mathscr{L}^*(r_c)$.

(B) The critical shock size for the case of a centralized collateral market is always greater then the critical shock size for the case of an OTC collateral market: $1 - x_c(\mathscr{G}_R, \mathscr{G}_C) < 1 - r_c(\mathscr{G}_R, \bar{\mathscr{G}}_C)$.

Proposition 7 shows that, when one of the two markets is replaced by a centralized exchange, the transition from the liquid to the frozen market regime is no longer abrupt but *smooth*. In addition, the transition always occurs at a larger shock size in the presence of a centralized exchange. Here, liquidity is less sensitive to the withdrawal of a single bank and can only vary smoothly with the size of the exogenous shock: a sudden market freeze is not possible. This result emphasizes the stabilizing effect of a centralized exchange on liquidity in the presence of exogenous shocks.

The intuition for Proposition 7 is similar to the intuition for Proposition 6. The main difference between the pure over-the-counter case and the centralized collateral case is the absence of fragile connectors. Since in the complete network all banks receive and provide liquidity to each other, there can be no contagion through the complete network. While a bank may be fragile in the repo market, its withdrawal cannot lead to further withdrawals in the collateral market. The absence of fragile connectors therefore removes the amplification effect that results from the complementarity of the repo and collateral markets. This leads to a smooth transition and an increased critical shock size.

For the proof of Proposition 7 we first show that finding the maximal equilibrium in the joint system $(\mathcal{G}_R, \bar{\mathcal{G}}_C)$ reduces to finding the giant component of $\mathcal{G}_R$ only. This is because, as mentioned above, the complete $\bar{\mathcal{G}}_C$ cannot fragment and any node in the giant component of $\mathcal{G}_R$ is by construction in the giant component of $\bar{\mathcal{G}}_C$. Finding the giant component of $\mathcal{G}_R$ is a straightforward problem in the literature, see Newman (2002) and Cooper and Frieze (2004), and immediately yields the smooth transition of liquidity regimes in Proposition 7. A consequence of the "no-contagion" property of the complete network is that all other choices of $\mathcal{G}_C$ must lead to a smaller critical shock size.[22] Since the complete network is excluded by assumption from our random network model, we obtain the strict relationship of the critical shock sizes in Proposition 7. For the full proof of Proposition 7 see appendix B.3.

4.4 Liquidity for binomial and power law degree distributions

We illustrate the results in Propositions 6 and 7 by considering networks with binomial (Erdős-Rényi) and scale free (power law) degree distribution.

Erdős-Rényi: Let q denote the probability that a randomly chosen bank is connected to another bank by an outgoing or incoming link. Here, due to the independence of in- and out-degrees the joint degree distribution factorizes into $p_{jk} = p_j p_k$ with $p_j = p_k$ and

$$p_k = \binom{n-1}{k} q^k (1-q)^{n-k-1}.$$

We hold the average in- and out-degree $\lambda = nq$ fixed as $n \to \infty$.

Scale free: As for the Erdős-Rényi networks we assume that the in- and out-degrees are independent, such that the joint degree distribution factorizes into $p_{jk} = p_j p_k$. We take $p_j = p_k$

[22] An exception to this is the special case when $\mathcal{G}_C$ is a copy of $\mathcal{G}_R$, as we will discuss in Section 4.5. However, given the independence of $\mathcal{G}_R$ and $\mathcal{G}_C$ this event has vanishing probability as $n \to \infty$.

and $p_k = C_\mu k^{-\alpha}$ for $\alpha \in (2,3]$ and $k > 1$. The constant that normalizes the degree distribution is $C = 1/(\zeta(\alpha) - 1)$, where $\zeta(\cdot)$ is the Riemann zeta function. The exponent α determines how dispersed the degree distribution is; for $\alpha < 2$, the variance of the degree distribution diverges.

We solve for the liquidity measure of the maximal equilibrium $\mathscr{L}^*(x)$ numerically; the detailed calculations can be found in Appendix C. For each degree distribution, we also compute $\mathscr{L}^*(x)$ when the collateral market is replaced by a complete network. In Fig. 7 and 8 we present the results for the binomial and scale free degree distributions, respectively. The findings of Propositions 6 and 7 are apparent. First, in the case of two coupled OTC markets $(\mathscr{G}_R, \mathscr{G}_C)$ there is a discontinuous transition from the liquid to the frozen market regime. Second, when the collateral market is replaced by a complete network $(\mathscr{G}_R, \bar{\mathscr{G}}_C)$, the transition is smooth and occurs at a greater shock size. Our results are robust to the choice of parameters as long as the degree distributions satisfy the requirements laid out at the beginning of this section.

4.5 Correlations between repo and collateral networks

In our definition of the random network model, we have assumed that $\mathscr{G}_R$ and $\mathscr{G}_C$ are independent draws from their respective distributions. Under this assumption a bank's counterparties in the repo market are different from its counterparties in the collateral market. In many cases however, trading relationships persist across markets.

Let $\mathscr{G}_R$ and $\mathscr{G}_C$ be random networks with the same degree distribution. Then, as $n \to \infty$ $|E(\mathscr{G}_C)| = |E(\mathscr{G}_R)| = M$. Define the overlap measure of network similarity as:

$$\omega = \frac{\#\{i \to j \mid i \to j \in E(\mathscr{G}_C) \wedge i \to j \in E(\mathscr{G}_R)\}}{M}.$$

If $\mathscr{G}_R$ and $\mathscr{G}_C$ are independent, no edges overlap and $\omega = 0$. If $\mathscr{G}_R$ is a copy of $\mathscr{G}_C$, all edges overlap and $\omega = 1$.

Let $\mathscr{G}_C$ and $\mathscr{G}_R$ be two Erdős-Rényi random networks with overlap ω. How are the results in Proposition 6 affected by different levels of overlap? We find that there exists a critical overlap ω_c at which the discontinuous transition from liquid to frozen markets becomes continuous. That is, as the networks become more similar, we move from the pure over-the-counter market case to the case where one market is replaced by a centralized exchange. This transition occurs at an overlap of approximately $\omega_c = 2/3$ if $\mathscr{G}_C$ and $\mathscr{G}_R$ have the same degree distribution.

Thus, as the two over-the-counter markets become more similar, i.e. as banks share more counterparties across markets, liquidity becomes more resilient to exogenous shocks. This may appear counterintuitive at first, since it seemingly contradicts notion that a diversified set of counterparties protects against random shocks to one's counterparties. However, the complementary nature of repo and collateral markets that removes the gain from diversification and leads instead to an amplification of exogenous shocks through the over-the-counter markets.

The intuition behind our finding is as follows. Note that the maximal equilibrium when $\omega = 1$ is the same as the maximal equilibrium when the collateral network is a complete network. This is because, as in the complete network case, any bank that is in the giant component of $\mathcal{G}_R$ is by construction also in the giant component of $\mathcal{G}_C$. Thus for $\omega = 1$, the results in Proposition 7 apply while for $\omega = 0$, the results in Proposition 6 apply. The overlap parameter ω then interpolates between these two extremes and there must be an intermediate overlap at which the discontinuous transition of liquidity becomes continuous. Our calculations are based on a heuristic approximation of the effect of ω on the maximal equilibrium liquidity measure, see Appendix D for details.

We also study $\mathcal{L}^*(x)$ explicitly for two levels of overlap $\omega = \{0.2, 0.8\}$, see Fig. 9. As expected for $\omega = 0.2$, we observe a discontinuous transition from the liquid to the frozen regime, while for $\omega = 0.8$, we observe a continuous transition.[23]

5 Discussion

We interpret the model laid out in Section 2.1 as a liquidity provision game between capital- and cash-constrained intermediaries. Importantly, rather than modeling trades in the repo and collateral markets in detail, we model in reduced-form a single decision about whether to provide liquidity in each market. The graph formed by active banks in the maximal equilibrium can be interpreted as the set of *potential* trades. We abstract away from volume and frequency of trade along a given link.

We assume that banks' best responses are binary. That is, a bank either decides to provide liquidity to all of its counterparties or none. In particular, as long as there is at least one trading partner active in the repo market, and one, possibly different, trading partner active in the market for collateral, bank i is willing to be active in both markets. In a more detailed model that endogenized decisions over which links to keep "open" or "active," there would be two new forces relative to our model. On the one hand, a bank may sometimes choose to stay partly operational in one market if it has that option, while it would have shut down if given a binary option between staying active and shutting down. In this sense our best-response function is a conservative approximation of the best-response function in a richer model, which resolves ambiguity in favor of shutting down. The second force goes in the opposite direction: other banks' possibility of remaining partly operational may make a given bank more likely to remain (partly) operational after a given shock. Thus, there is no easy comparison between our model and one with richer, link-by-link decisions about the willingness to trade. Because the strategic dynamics in this model are already intricate, we view the current model as a useful simplification.

However, the basic structure of a supermodular game would carry over to a suitable version of the more general link-by-link game. Let $a^{\mu}_{ij} \in \{0,1\}$ denote the action of bank i vis-a-vis bank $j \in K^{+}_{i,\mu}$. Then one could impose the following constraint: $\sum_{j\in K^{+}_{i,\mu}} a^{\mu}_{ij} \le c_{i,\mu} + \sum_{j\in K^{-}_{i,\mu}} a^{\mu}_{ji}$.

[23] We compute $\mathcal{L}^*(x)$ both via our approximate method and numerically via algorithm 1. Note that the heuristic solution approximates the numerical solution quite well, though there are clear finite size effects for the numerical solution (we used $n = 2000$).

That is, the liquidity that bank i provides to its counterparties is constrained, up to a constant $c_{i,\mu}$, by the liquidity it receives from its counterparties. Now the best-responses—how a bank withdraws liquidity—would be more complicated, reflecting, for example, bank level heterogeneity such as counterparty risk. The game can still be defined to be supermodular, and so one can again define a maximal equilibrium. The propagation of shocks, however, can be quite different due to the two forces we have sketched. On the one hand, cascades can start from smaller shocks, because banks can react in a less extreme way. On the other hand, sensitivity to parameters will also be less extreme, because liquidity will be withdrawn gradually and a shock that might have triggered full shutdown of a set of banks before will only trigger a milder contagion in a subset of those banks. Such extensions would be interesting to explore in future work.

We motivate the banks' best responses by assuming tight capital and cash-in-advance constraints. One motivation for tight constraints is simply bank profit maximization. A bank increases leverage and reduces cash reserves to boost returns until regulatory constraints become tight. This will be optimal if the bank cannot anticipate the adverse shock. One way to relax the assumption of tight constraints is to lower the threshold $\theta_\mu = 1$ in Eq. (2) to 0 for a subset of banks. These banks can be thought of having slack constraints and would provide liquidity in both markets irrespective of their neighbors' actions. This will naturally increase the liquidity measure of the maximal equilibrium.

6 Conclusion

We develop a model of intermediary liquidity provision in over-the-counter repo and collateral markets and study the maximal equilibria of the resulting complete-information game in a multilayer network. The complementary nature of repo and collateral couples these markets and motivates the banks' best responses. In particular, as long as there is at least one trading partner active in the repo market, and one, possibly different, trading partner active in the market for collateral, a bank is willing to be active in both markets.

We find three main results. First, the presence of *fragile connectors*—banks that are on the brink of isolation in one market and critical intermediaries in the other—makes equilibrium liquidity fragile. In particular, the withdrawal of such a fragile connector can lead to a sudden market freeze. Second, even in the absence of fragile connectors, the complementary nature of repo and collateral markets can amplify exogenous shocks and lead to illiquidity spirals. Third, replacing at least one OTC market by a centralized exchange reduces the extent of illiquidity spirals and improves the resilience of liquidity.

Moving forward, a natural next step would be to extend the model to account for incomplete information. In this setting, the realization of the shock profile or the parameters entering banks' best response functions may only be partially known to the agents. This would, for example, allow us to study how liquidity is affected by changes in banks' beliefs about the distribution of the exogenous shock. In turn this would permit a study of the realistic phenomenon that liquidity may evaporate when bad news is published, if that news coordinates beliefs in a suitable way (Angeletos and Werning, 2006; Morris and Yildiz, 2017).

Our model raises a number of issues for policymakers. First, we illustrate the potential fragility of liquidity in over-the-counter markets and show how it may be reduced by moving towards centralized exchanges. Second, our results highlight the importance of better measurement and empirical and study of the structure of these markets, in particular with respect to fragile connectors.

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A A simple trading model with intermediation and complementary assets

In Section 2 we introduce a model of intermediaries' willingness to trade in two coupled over-the-counter markets. We intentionally abstract from the end customers of repo and collateral assets (i.e. investors and entrepreneurs) and focus instead on the actions of the intermediaries. We can then interpret the activity of intermediaries as a proxy for liquidity in these two markets. Here, we show how our framework can be embedded into a simple trading model with intermediation. This will allow us to illustrate why our proxy for liquidity can indeed be related to the liquidity accessible to end customers.

A.1 A single asset market

As before, consider an OTC market for some asset with a set of intermediaries (which we call banks for short) N. In addition to the trading relationships that banks maintain among each other, each bank $i \in N$ has relationships with a set of end customers E_i. For simplicity, assume that the sets of banks' end customers are non-overlapping and of equal and fixed size. Two customers, s and b, are selected uniformly at random from $\bigcup_i E_i$. Focus on the case where the two customers do not share the same bank, i.e. $b \in E_i$ and $s \in E_j$ with $i \neq j$. As $|N|$ grows and $|E_i|$ remains fixed, this is by far the most likely case. Now suppose that s receives a supply shock of the asset of a single unit. Further, suppose that s has some valuation v_s for the asset that is lower than b's valuation v_b for the asset. Hence b is a natural buyer and s is a natural seller of the asset. However, we assume the existence of a friction that prevents buyer and seller to trade directly with one another. Hence, the asset is traded in an OTC market and the only way to realize the potential surplus $v_b - v_s$ is to trade via their intermediaries i and j.

Provided there exists a path between i and j in the network of trading relationships $\mathcal{G}$, b and s will be able to trade and the total surplus of $v_b - v_s$ will be realized. The distribution of this surplus among buyer, seller and intermediaries will depend on the bargaining protocol as well as the network topology. Whether or not an intermediation path from i to j (and hence from s to b) exists depends on how many banks are available to intermediate. However, whether or not to be active and available to intermediate is a choice for the bank. This links liquidity available to customers to the number of intermediaries active in the OTC market.

A.2 Two complementary asset markets

So far we have considered the market for a single asset, e.g. asset C in isolation. In a number of practically relevant cases, though, banks require access to a complementary asset R in order to provide intermediation services for C. Formally, we consider asset R a complement to asset C if a bank i is available for trade in asset C if and only if bank i can buy or sell asset R if it receives a corresponding valuation shock. In Section 2.2, we provide an argument for why repo and collateral are complements: a repo loan can only be obtained if a borrower can pledge eligible collateral. However, the lender is only willing to accept this collateral if the collateral can be liquidated should the borrower default. But the willingness of other intermediaries to buy collateral depends on whether they have funding available, i.e. whether they can obtain repo funding. How does the introduction of a complementary asset alter the market mechanics of the single-asset case?

Suppose that the markets for assets C and R function as outlined above: randomly selected buyers and sellers trade via an intermediation chain of banks which operate in both markets simultaneously. Again, the customers b_C and s_C in the market for asset C will trade if there exists a path of active intermediaries between their corresponding intermediary banks i and j. For each intermediary i_C along such a path in $\mathcal{G}_C$ there must exist a path from the intermediary in $\mathcal{G}_R$ to a buyer and a seller. This requirement is a direct result of the complementarity condition. The probability that the complementarity condition is satisfied is a non-decreasing function of the number of intermediaries active in the market for the complementary asset R. This links the liquidity available to customers in the market for asset C to the activity of intermediaries in both the market for asset C and asset R.

Since C and R are mutual complements we obtain the situation outlined in Section 2.2. Note that the condition that there exists a path between two randomly selected banks i and j in both $\mathcal{G}_C$ and $\mathcal{G}_R$ is precisely the condition on membership of a mutually stable component introduced in Section 3. Thus, in this simple model of intermediated trading between randomly selected customers, the liquidity available to customers in both asset markets depends directly on the equilibrium liquidity measure introduced in Section 3.

A.3 Complementarity in repo and collateral markets

Next, we show how the banks' best response functions outlined in Section 2 can be obtained from a highly stylized bank profit function under the assumption of asset complementarity outlined above. Let s_C, b_C be the seller and buyer of collateral, respectively. Let s_R and b_R be the lender and borrower of repo, respectively. To ease notation, define the following random variables:

$$\begin{aligned} X_i^C &= \mathbb{1}\{i \text{ in a trading chain from } s_C \text{ to } b_C \text{ in } \mathcal{G}_C\}, \\ X_i^R &= \mathbb{1}\{\text{trading chain from } i \text{ to } s_R \text{ in } \mathcal{G}_R\}, \\ D_i &= \mathbb{1}\{\text{repo loan by } i \text{ has failed}\}. \end{aligned}$$

All variables in the following are bank dependent, but the index has been suppressed for clarity. The profit of a bank from being active in the collateral market can be written as:

$$\Pi^C = \left[\left(\upsilon X^C - r\right) X^R a^R - c\right] a^C,$$

where c is a small fixed cost of being active. The bank earns revenue (captures a share of the total surplus) from collateral intermediation whenever it is on an intermediation path. This yields υX^C, where υ is the collateral intermediation spread and X^c is the binary variable that indicates whether or not a path exists from the intermediary to the collateral buyer. The bank's decision to be active in the collateral network is denoted $a^C \in \{0,1\}$. Since it faces a cash in advance constraint, the bank has to be active in the repo market and borrow at rate r if it intermediates the collateral asset. To be able to borrow on the repo market, however, there has to be a path from the bank to the ultimate lender.

This simple profit function has a number of desirable properties and limits. First, if the bank is not active in the collateral market ($a^C = 0$), the profit is zero. Second, if the bank is not active in the repo market ($a^R = 0$), the profit is $-ca^C$. Third, the profit from intermediating the collateral asset υX^C must be larger than the cost of repo financing r. And fourth, the multiplicative term $X^C X^R$ captures the fact that the assets are complements.

Similarly, the profit of a bank from being active in the repo market is

$$\Pi^R = \left[wX^R - c_D X^R D\left(1 - X^C a^C\right) - c\right] a^R,$$

where w is the repo intermediation spread, c is a small cost for being active in the repo market, and c_D is a default cost. The bank earns revenue from intermediation wX^R if it is on a repo intermediation chain. If a repo loan it provided defaults and if it cannot liquidate the collateral in the collateral market, the bank incurs a default cost c_D. To liquidate collateral, the bank has to be active in the collateral market and in a trading chain to a collateral buyer. Now assume that $\mathbb{E}[X^C X^R] = \mathbb{E}[X^C]\,\mathbb{E}[X^R]$ and $\mathbb{E}[X^C D] = \mathbb{E}[X^C]\,\mathbb{E}[D]$ and $\mathbb{E}[DX^R] = \mathbb{E}[D]\,\mathbb{E}[X^R]$. Let $\mathbb{E}[X^C] = \overline{X}^C$, $\mathbb{E}[X^R] = \overline{X}^R$ and $\mathbb{E}[D] = \overline{D}$. Note that $\overline{X}^C = 0$ and $\overline{X}^R = 0$ if the bank has no incoming links in the respective markets. Then, the expected profit in both markets is:

$$\begin{aligned}
\mathbb{E}[\Pi^C] &= \left[\left(\upsilon\overline{X}^C - r\right)\overline{X}^R a^R - c\right] a^C, \\
\mathbb{E}[\Pi^R] &= \left[w\overline{X}^R - c_D\overline{X}^R\overline{D}\left(1 - \overline{X}^C a^C\right) - c\right] a^C.
\end{aligned}$$

Further, suppose that if the bank has incoming links in both markets, $\overline{X}^C > 0$ and $\overline{X}^R > 0$, it holds that:

$$\begin{aligned}
\left(\upsilon\overline{X}^C - r\right)\overline{X}^R a^R - c &> 0, \\
w\overline{X}^R - c_D\overline{X}^R\overline{D}\left(1 - \overline{X}^C a^C\right) - c &> 0
\end{aligned}$$

and furthermore assume that $w\overline{X}^R - c_D\overline{X}^R\overline{D} - c < 0$ holds. Then the best responses are as outlined in Section 2. The first condition is a condition on a^R and says that the bank is active in the repo market if the revenue from collateral intermediation minus the cost of repo financing is larger than the fixed cost of being active. The second condition is a condition on a^C says that the bank is active in the collateral market if the revenue from being an intermediary in the repo market minus the expected cost of being unable to liquidate the collateral in case the borrower defaults is larger than the fixed cost. These conditions reduce to the condition that the bank has to be active and on the intermediation path in both markets, and that the revenue is large enough to cover the fixed costs.

B Proofs

B.1 Star network

Proof. Proposition 2. First consider the case where $B_{H,C} \neq B_{H,R}$. Suppose that the hub bank in either market is hit by the adverse shock. The withdrawal of the hub bank forces all peripheral banks to stop providing liquidity as they are fully dependent on the hub bank. Thus, in equilibrium $y_i^* = 0$ for all i and $\mathscr{L} = 0$ Now suppose a peripheral bank is hit by the adverse shock. Its withdrawal will not affect any other banks since their provider of liquidity (the hub bank) has not been affected. Thus, in equilibrium $y_i^* = 1$ for all banks that did not receive the adverse shock and $\mathscr{L} = n-1$.

The probability that the adverse shock hits the hub bank in either market is $P(w^j \wedge j = B_{H,\mu}) = 2/n$. Conversely, the probability that the adverse shock does not hit the hub bank is $P(w^j \wedge j \neq B_{H,\mu}) = (n-2)/n$. Combing this with the above, the expected equilibrium liquidity measure is

$$\begin{aligned} E[\mathscr{L} \mid B_{H,C} \neq B_{H,R}] &= P(w = B_{H,\mu})\mathscr{L}(w = B_{H,\mu}) + P(w \neq B_{H,\mu})\mathscr{L}(w \neq B_{H,\mu}) \\ &= \frac{2}{n}0 + \frac{n-2}{n}(n-1) = \frac{(n-2)(n-1)}{n}. \end{aligned}$$

Now consider the case where $B_{H,C} = B_{H,R}$. Clearly, if the hub bank is hit by the adverse shock, in equilibrium $\mathscr{L} = 0$. This occurs with probability $1/n$. Conversely, if a peripheral bank is hit by the adverse shock, in equilibrium $\mathscr{L} = n-1$. The expected equilibrium measure is then given by

$$E[\mathscr{L} \mid B_{H,C} = B_{H,R}] = \frac{1}{n}0 + \frac{n-1}{n}(n-1) = \frac{(n-1)^2}{n}.$$

□

B.2 Preliminaries: theory of random networks

B.2.1 The configuration model

In the following we introduce random networks generated from the configuration model. All of the concepts introduced below apply equally to both markets $\mu \in \{R, C\}$. To avoid notional clutter, we drop the subscript μ for now.

Let $\mathbf{d}_n^+ = (d_{i,n}^+)_{i=1}^n$ and $\mathbf{d}_n^- = (d_{i,n}^-)_{i=1}^n$ be a sequence of non-negative integers representing the out-degree and in-degree respectively of banks $i \in N$, where as before $n = |N|$. Note that all out-edges must have a corresponding in-edge, therefore $\sum_i^n d_{i,n}^+ = \sum_i^n d_{i,n}^-$. Denote the empirical distribution of degrees by

$$p_{jk,n} := \frac{1}{n}\#\{i \in N \mid d_{i,n}^+ = j, d_{i,n}^- = k\}.$$

In the configuration model a random graph is generated by randomly matching the in- and out-degrees of the banks.

Definition B.1 (Configuration model, Amini et al. (2013))**.** Consider a set of nodes $N = \{1, ..., n\}$ and a degree sequences $\mathbf{d}_n^+ = (d_{i,n}^+)_{i=1}^n$ and $\mathbf{d}_n^- = (d_{i,n}^-)_{i=1}^n$. This defines for each node i a set of incoming and outgoing half edges H_i^- and H_i^+ respectively. The set of all incoming and outgoing half edges is denoted by H^- and H^+ respectively. An edge is formed by the matching of two half edges drawn at random without replacement from H^- and H^+. A random directed multigraph $\tilde{\mathcal{G}}(n, \mathbf{d}_n^+, \mathbf{d}_n^-)$ drawn from the configuration model is then given by a particular matching of all incoming and outgoing half edges.

The random directed multigraph as generated by the above procedure may include self-edges and multiple edges between two nodes. A graph without self-edges or multiple edges is a simple graph. We are only interested in simple graphs and therefore need to impose conditions on the degree sequences to ensure that the above procedure only generates such graphs. We follow Amini et al. (2013) and Britton et al. (2007) and impose the following standard conditions.

Assumption B.1 (Conditions for degree sequences). For each n, $\mathbf{d}_n^+$ and $\mathbf{d}_n^-$ are sequences of non-negative integers such that $\sum_i^n d_{i,n}^+ = \sum_i^n d_{i,n}^-$ and for some probability distribution over in- and out-degrees $(p_{jk})_{j,k\geq 0}$,

1. $p_{jk,n} \to p_{jk}$ for every $j, k \geq 0$ as $n \to \infty$,
2. $\lambda := \sum_{j,k} p_{jk} j = \sum_{j,k} p_{jk} k \in (0, \infty)$,
3. $\sum_{i=1}^n (d_{i,n}^+)^2 + (d_{i,n}^-)^2 = O(n)$.

Note that conditions (2) and (3) imply that the average degree of the banks and the second moment of the degree sequence cannot diverge as the network becomes large. We follow Cooper and Frieze (2004) and further require that the in- and out-degree sequences are *proper*.[24]

Assumption B.2 (Proper degree sequences, Cooper and Frieze (2004)). Let Δ_n denote the maximum degree. Then

1. Let $\rho_n = \max(\sum_{j,k} \frac{j^2 k p_{jk,n}}{\lambda_n}, \sum_{j,k} \frac{k^2 j p_{jk,n}}{\lambda_n})$ If $\Delta_n \to \infty$ with n then $\rho_n = o(\Delta_n)$.
2. $\Delta_n \leq \frac{n^{1/12}}{\log n}$.

We assume that degree sequences are such, given an exogenous shock, the size of the largest component of a random network $\mathcal{G}_\mu$ decreases faster as the size of the shock increases.

[24]These technical assumptions require that a weighted measure of the degree sequence's second moment must grow much slower with the network size than the maximum degree of the sequence. This ensures that, while the maximum degree may go to infinity, the degree sequence does not become too dispersed.

Assumption B.3 (Concavity of size of giant component). Let the giant component of network $\mathcal{G}_\mu$ be given by Definition B.2. Let $1-x$ denote the fraction of banks that receive an adverse shock $w_i = 0$. Let $g_\mu(x)$ denote the size of the giant component of network $\mathcal{G}_\mu$ given a shock of size $1-x$. Then we assume that as $n \to \infty$ the degree sequences $\mathbf{d}^+_{n,\mu}$ and $\mathbf{d}^-_{n,\mu}$ are such that $g_\mu(x)$ is concave in x for $\mu \in \{R, C\}$.

We call a degree sequence that satisfies Assumptions B.1, B.2 and B.3 *feasible*. Let $G(n, \mathbf{d}^+_n, \mathbf{d}^-_n)$ be the set graphs with feasible degree sequences $\mathbf{d}^+_n, \mathbf{d}^-_n$. The random networks $\mathcal{G}_R$ and $\mathcal{G}_C$ are then drawn uniformly at random from G_R and G_C, respectively.

Finally, we assume that:

Assumption B.4 (Network independence). The networks $\mathcal{G}_R$ and $\mathcal{G}_C$ are independent draws from G_R and G_C, respectively. The following pairs of random variables are independent: $d^+_{iC,n}, d^+_{iR,n}$ and $d^+_{iC,n}, d^-_{iR,n}$ and $d^-_{iC,n}, d^+_{iR,n}$ and $d^-_{iC,n}, d^-_{iR,n}$ for all $i \in N$.

In other words, the in (out) degree of a bank in the repo market does not predict its in (out) degree in the collateral market.

B.2.2 Equilibrium and the mutual giant component

We can now establish a connection between the equilibrium defined in section 3 and the asymptotic properties of the random graphs defined above. First, let us define the giant out-component.

Definition B.2 (Giant out-component)**.** Let $\mathcal{G}$ be a directed graph as defined above. Let S denote the largest connected component such that for all $i \in S$ we have $d^-_{i,n} > 0$. Then S is a giant out-component if

$$\lim_{n\to\infty} \frac{1}{n}|S| \to c > 0.$$

We denote the giant out-component by $GC_o(\mathcal{G})$.

In other words, the giant out-component is the set of all connected nodes with at least one incoming edge that scales with the size of the graph. It can be shown that, under the technical assumptions made above, the giant out-component is unique if it exists, see Cooper and Frieze (2004). It is obvious that the giant out-component is also a stable component. Now let us define the mutual giant out-component.

Definition B.3 (Mutual giant out-component)**.** Let $\mathcal{G}_R$ and $\mathcal{G}_C$ be directed graphs as defined above. Then the mutual giant out-component is

$$MGC_o(\mathcal{G}_R, \mathcal{G}_C) = \{i \mid i \in GC_o(\mathcal{G}_C) \wedge i \in GC_o(\mathcal{G}_R)\}.$$

Again it is obvious that a mutual giant out-component is also a mutually stable component. The size of the mutual giant out-component and the equilibrium liquidity measure are then related as follows.

Lemma 2. Let $\boldsymbol{y}^*$ be an equilibrium for $\mathscr{G}_R$, $\mathscr{G}_C$ and a shock profile $\boldsymbol{w}$ as given in section 3. Then

$$\mathscr{L}(\boldsymbol{y}^*) = \sum_i y_i^* \geq |MGC_o(\mathscr{G}_R(W), \mathscr{G}_C(W))|.$$

In the limit of large networks we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_i y_i^* \to \frac{1}{n} |MGC_o(\mathscr{G}_R(W), \mathscr{G}_C(W))|,$$

The size of the mutual giant out-component is a lower bound on the number of active banks in equilibrium. It is a lower bound since there may exist small, mutually stable components outside the mutual giant out-component. However, as the network becomes large, the relative size of these small mutually stable components vanishes. Therefore, in the limit of large networks the size of the mutual giant out-component is sufficient to compute the equilibrium liquidity. In the following we will discuss how the mutual giant out-component can be found.

B.2.3 A branching process approximation of equilibrium liquidity

In this section we will invoke results from the theory of branching processes and probability generating functions to compute the size of the giant mutual out-component (see Cooper and Frieze (2004) and Buldyrev et al. (2010)). We will first characterize the giant out-component in an isolated network and will then proceed to derive the mutual giant out-component.

Computing the giant out-component Let us first introduce the idea of a probability generating function.

Definition B.4 (Probability generating function.)**.** Let $X \in \mathbb{N}^0$ be a discrete random variable in the set of non-negative integers. Let $q_i = P(X = i)$ be the associated probability distribution. The probability generating function of X is defined by $G(z) = \sum_i q_i z^i$ for all z for which the sum converges.

Now, consider a network $\mathscr{G}$ (either corresponding to the repo or the collateral market) with the joint distribution for in- and out-degree given by p_{jk} and shock profile $\boldsymbol{w} = \mathbf{1}$. To compute the size of the giant out-component we first need to compute the distribution of the out-degree of the terminal node of a randomly chosen link. Note that it is j times more likely to end up at a node with in-degree j. Therefore the distribution of the out-degree of the terminal node is

$$p_k^+ = \sum_j \frac{j}{\lambda} p_{jk},$$

where the average out (in) degree λ is required to normalize the distribution. The distribution for the in-degree of the terminal node of a randomly chosen link can be defined similarly but is not of interest for us.

Suppose one starts to explore the network from this randomly chosen link via a breadth-first search algorithm. How many banks can one reach by following only out-going links? Given our random network model, this exploration process can be approximated by a standard branching process where the number of offspring (i.e. out-going links) of an individual (bank) are distributed according to p_k^+. The probability generating function for number of banks explored in the 1^{st} generation Z_1 is the given by

$$H(z) = \sum_k p_k^+ z^k$$

Denote the r^{th} iteration of $H(\cdot)$ by H_r, i.e.

$$H_r(z) = \underbrace{H(H(\ldots H}_{r \text{ times}}(z)).$$

It is well known that the probability generating function for the number of offspring in the r^{th} generation Z_r is given by $H_r(z)$.[25] Instead of computing the total number of banks that can be reached by following a link, it is easier to compute the probability that the branching process eventually goes extinct, i.e. $Z_r = 0$ as $r \to \infty$.

Lemma 3. Let f denote the probability that the branching process defined by p_k^+ eventually goes extinct. Then

$$f = \lim_{r\to\infty} f_r = H(f) = \sum_k p_k^+ f^k.$$

This is a standard result for branching processes and can be see as follows. Let f_r denote the probability that the branching process goes extinct by the r^{th} generation. We have $f_r = P(Z_r = 0) = H_r(0)$ and $f_r = H(f_{r-1})$. Clearly f_r is a bounded non-decreasing sequence. Therefore by monotone convergence the fixed point $f = H(f)$ exists. In the context of a random network, the fixed point f determines the probability that following a random link will *not* lead to the giant out-component. In the giant out-component the branching process would continue indefinitely (as the network size grows to infinity). Then, the size of the giant out-component is given by a simple corollary of Lemma 3.

Lemma 4. Given a solution to $f = H(f)$, the fraction of nodes in the giant out-component is

$$s = g(f) = 1 - \sum_{jk} p_{jk} f^k.$$

This follows from the fact that the probability that a random node with k outgoing links is not in the giant out-component is simply f^k.

[25] Of course this only corresponds to the number of banks explored if the breadth-first search does not turn back on itself and does not re-explore parts it has already seen. The assumption that this does not occur is usually referred to as the requirement that the network is "locally tree-like", i.e. that there are no short cycles. Hence the application of the branching process is indeed an approximation. However, given Assumption 3.1, Cooper and Frieze (2004) show that this approximation is indeed valid. Furthermore, Molloy and Reed (1998) show that under assumptions similar to ours, the network is indeed locally tree-like outside the giant component.

Equilibrium liquidity and the mutual giant out-component Now that we have established how to compute the size of the giant out-component for a single network we can proceed to derive the size of the mutual giant component. Here we present a variation of the derivation presented in Buldyrev et al. (2010). From now on we will associate each of the quantities introduced in section B.2.3 with the collateral or repo markets via the subscripts C or R respectively. For example, $H_R(z)$ will be the probability generating function of the out-degree process for the network corresponding to the collateral market.

As before let $\mathcal{G}_R$ and $\mathcal{G}_C$ denote the repo and collateral markets, respectively, and take for now $\boldsymbol{w} = \mathbf{1}$. In this setting, consider the following coupled branching process. Choose a link at random in the repo network and follow it to the bank which receives repo via this link. As shown in section 3.3, this bank will provide repo to its counterparties if and only if it is in a stable component in the collateral market. As we have shown above, this is equivalent to requiring that the bank is in the giant out-component of the collateral network. Since $\mathcal{G}_C$ and $\mathcal{G}_R$ are independent by assumption, the bank we reached will be in the giant out-component of $\mathcal{G}_C$ with probability s_C (the fraction of nodes in the giant out-component in the collateral network). If the bank is not in the giant out-component of $\mathcal{G}_C$ the branching process will not continue further from this bank.

The requirement that all banks in the repo network are in the giant out-component of the collateral network effectively removes those banks from the branching process for which this is not the case. This random removal of $1 - s_C$ banks leads to a “thinning” of the degree distribution. The transformed (thinned) degree distribution is given by

$$\hat{p}_{jk,R}(s_C) = \sum_{l=j}^{\infty} \sum_{m=k}^{\infty} \underbrace{p_{lm,R}}_{A} \underbrace{\binom{l}{j}(1-s_C)^{l-j} s_C^{j}}_{B} \underbrace{\binom{m}{k}(1-s_C)^{m-k} s_C^{k}}_{C}. \tag{5}$$

The transformed degree distribution consists of three terms: A, B and C. A corresponds to the initial probability that a random node has in-degree l and out-degree m. B is the probability that j out of initially l in-links are present after thinning. Finally, C is the probability that k out of initially m out-links are present after thinning. Let us make the following assumption.

An equivalent argument can be made for the collateral market. Again we obtain a transformed degree distribution $\hat{p}_{jk,C}(s_R)$. Similarly, we define the transformed distributions for the out-degree process $\hat{p}^{+}_{k,R}(s_C)$ and $\hat{p}^{+}_{k,C}(s_R)$. Therefore, the number of banks in the giant out-component in the repo market will depend on the number of banks in the giant out-component in the collateral market and vice versa.

So far we have only considered the special case where $\boldsymbol{w} = \mathbf{1}$. Let $1-x$ denote a fraction of banks chosen uniformly at random that receive an adverse shock $w_i = 0$. These banks withdraw from both markets. We call $1-x$ the size of the exogenous shock. The final size of the mutual giant out-component (and thus liquidity in the maximal equilibrium) is then determined by the branching process on the residual networks $\mathcal{G}_R(W)$ and $\mathcal{G}_C(W)$ after a fraction $1-x$ has withdrawn from the markets. Let $\mathcal{L}^*(x)$ be the expected liquidity of the maximal equilibrium averaged over all shock profiles of size $1-x$.

Lemma 5. Given the degree-distributions $p_{jk,\mu}$ for $\mu \in \{R, C\}$ and a shock of size $1-x$, the size of the giant out-component in the repo (collateral) network s_R^* (s_C^*) is the greatest solution to

$$\begin{aligned}
s_R &= x g_R(f_R, s_C) = x\left(1 - \sum_{jk} \hat{p}_{jk,R}(s_C) f_R^k\right),\\
f_R &= H_R(f_R, s_C) = \sum_k \hat{p}^+_{k,R}(s_C) f_R^k,\\
s_C &= x g_C(f_C, s_R) = x\left(1 - \sum_{jk} \hat{p}_{jk,C}(s_R) f_C^k\right),\\
f_C &= H_C(f_C, s_R) = \sum_k \hat{p}^+_{k,C}(s_R) f_C^k.
\end{aligned} \tag{6}$$

Liquidity is then

$$\mathcal{L}^*(x) = s^* = x g_R(x g_C(s^*)).$$

First consider the special case when $\boldsymbol{w} = \mathbf{1}$. To see that this system of equations indeed has a solution first note that $g(f, s) \leq s$. Now consider the sequences s_R^m and s_C^m with $m \geq 0$. Define $s_R^0 = 1 - \sum_{jk} p_{jk,R} f_R^k$ with $f_R = H_R(f_R)$ and $s_C^0 = 1 - \sum_{jk} p_{jk,C} f_C^k$ with $f_C = H_C(f_C)$. Clearly we must have that $s_R^0 \leq 1$. Now compute the next element in the sequence as $s_C^1 = g_C(f_C(s_R^0), s_R^0)$. Since $g(f, s) \leq s$ we must have $s_C^1 \leq s_R^0$. Similarly we must have that $s_R^1 \leq s_C^0$. The next iteration yields $s_C^2 \leq s_R^1 \leq s_C^0$ of the collateral network and similarly for the repo network. Since s is a fraction it is bounded below by 0. Therefore by monotone convergence the sequences s_R^m and s_C^m must converge and the solution to Eq. (6) must exist. By implication, in equilibrium we also must have $s_R^* = s_C^*$. Since the branching process that determines the size of the giant out-component in the repo network only considers nodes that are also in the giant out-component of the collateral network, the size of the mutual giant out-component is just s_R^*. Hence equilibrium liquidity is given by $\mathcal{L}^* = s^* = g_R(g_C(s^*))$.

Now consider a general shock of size $1-x$. All arguments made above follow through. However, the realization of the shock bounds the size of the giant out-component, and thereby equilibrium liquidity, from above by x. This is simply because a fraction of $1-x$ of banks withdraw from the markets due to a shock realization of $w_i = 0$. This yields the final expression for equilibrium liquidity $\mathcal{L}^* = s^* = x g_R(x g_C(s^*))$.

Lemma 6. Let $\mathcal{G}_R$ be a random network. Let $\overline{\mathcal{G}}_C$ be a complete network. Given a shock of size $1-x$, the size of the giant out-component in the repo network s_R^* is the greatest solution to

$$\begin{aligned}
s_R &= g_R(f_R, x) = x\left(1 - \sum_{jk} \hat{p}_{jk,R}(x) f_R^k\right),\\
f_R &= H_R(f_R, x) = \sum_k \hat{p}^+_{k,R}(x) f_R^k.
\end{aligned}$$

Liquidity is then

$$\mathcal{L}^*(x) = s_R^* = g_R(f_R, x).$$

To see this, first note that if $\overline{\mathcal{G}}_C$ is complete there will be no contagion through $\overline{\mathcal{G}}_C$. All banks in $\overline{\mathcal{G}}_C$ are active except those that are not in the giant out-component of the repo market. Now let us consider again the sequences s_R^m and s_C^m with $m \geq 0$ that we discussed before. As before we have $s_R^0 = 1 - \sum_{jk} p_{jk,R} f_R^k$ with $f_R = H_R(f_R)$ and $s_C^0 = 1 - \sum_{jk} p_{jk,C} f_C^k$ with $f_C = H_C(f_C)$. Since the collateral market is complete $s_C^0 = 1$. In the next iteration we will obtain $s_C^1 = g_C(s_R^0) = s_R^0$. Now how does the collateral market affect the repo market in the next iteration? Before we assumed that a random fraction of banks withdrew from the repo market due to their earlier withdrawal from the collateral market. However, now we can no longer assume that these banks are a random sample from the repo market. In fact, only those banks that were already outside the giant out-component in the repo market will now have withdrawn from the collateral market. Therefore the degree distribution will *not* thin out. Thus we must have $s_R^1 = s_R^0 = s_R$ and of course $s^* = g_R(x, f_R)$.

The behavior of the size of the giant out-component of a random network is well understood and follows immediately from the properties of the probability generating function $f_R(\cdot)$; see Newman (2002) and Cooper and Frieze (2004).

B.3 Proofs of random network results

We will first prove Proposition 7 (A) since this is a standard result from the literature (see Cooper and Frieze (2004)) and is useful for the subsequent proofs. For the proof we will use standard properties of a generic probability generating function (pgf) that we summarize in the following remark.

Remark 1. A generic pgf $f(s) = \sum_i p_i s^i$ has the following properties. $f(0) = p(0)$, $f(1) = 1$, $f'(1) = df/ds(1) > 0$ (increasing) and for $s > 0$ $d^2 f/ds^2 > 0$ (convex). Therefore $s^* = f(s^*)$ has a solution $s^* < 1$ if $f'(1) > 1$. Otherwise only the trivial solution $s^* = 1$ exists. When $f'(1) = 1$ we have $s^* = 1$ as well. Thus s^* is continuous in the slope $f'(1)$.

We illustrate some graphical intuition for this proof in Fig 5.

Proof. Proposition 7 (A). Recall that before an exogenous shock we have $H(z) = \sum_k p_k^+ z^k$ with $p_k^+ = \sum_k j p_{jk}/\lambda$. Let's define the pgf for the initial degree distribution of of a random bank.

$$G(v, z) = \sum_{jk} p_{jk} v^j z^k. \tag{7}$$

First note that $\lambda = dG/dv(1,1) = dG/dz(1,1)$. Also note that the pgf of the out-degree of a bank reached by following a random link is the normalized derivative of $G(z, v)$, namely

$$H(z) = \frac{1}{\lambda} \frac{dG}{dv}\Big|_{v=1} = \frac{1}{\lambda} \sum_{jk} j p_{jk} z^k.$$

Now consider how the pgf in Eq. (7) changes if a fraction of $1-x$ banks are randomly removed. From Eq. (5) we know the transformed degree distribution. Thus

$$\hat{G}(v,z,x) = \sum_{jk} \hat{p}_{jk}(x) v^j z^k = \sum_{jk} \sum_{l=j}^{\infty} \sum_{m=k}^{\infty} p_{lm} \binom{l}{j} (1-x)^{l-j} x^j \binom{m}{k} (1-x)^{m-k} x^k v^j z^k.$$

Rearranging the sums we obtain

$$\begin{aligned}\hat{G}(v,z,x) &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} p_{lm} \left(\sum_{j=0}^{l} \binom{l}{j} (1-x)^{l-j} (xv)^j \right) \left(\sum_{k=0}^{m} \binom{m}{k} (1-x)^{m-k} (xz)^k \right), \\ &= \sum_{lm} p_{lm} (1-x+xv)^l (1-x+xz)^m = G(1-x+xv, 1-x+xz).\end{aligned}$$

Thus, the exogenous shock leads to a transformation of the pgf under which $v \to 1-x+xv$ and $z \to 1-x+xz$. The mean out-degree under transformed distribution is thus $\hat{\lambda} = x\lambda$. Again we can compute the pgf of the out-degree of a bank reached by following a random link via the normalized derivative of G, i.e

$$\hat{H}(z,x) = \frac{1}{\hat{\lambda}} \frac{d\hat{G}}{dv}\bigg|_{v=1} = H(1-x+xz).$$

From remark 1 we know that $f = H(f) = 1$ if $dH/dz(1) = H'(1) \leq 1$ and $f < 1$ if $H'(1) > 1$. When $f = 1$ the size of the giant out-component vanishes, i.e. $g(1) = 0$. If $f < 1$ the size of the giant out-component is $g(f) > 0$, i.e. the giant out-component exists. Thus we need to ask at which x_c the derivative of the pgf becomes $\hat{H}'(1) = 1$. Note that $\hat{H}'(1) = xH'(1)$. Thus

$$x_c = \frac{1}{H'(1)} = \frac{\lambda}{\sum_{jk} p_{jk} jk}. \tag{8}$$

Since f is continuous in the derivative $\hat{H}'(1)$ it is also continuous in x_c. This concludes the proof. □

Lemma 7. $f = H(f,x)$ is continuous, monotonically decreasing in x for $x \in [0,1]$.

Proof. Lemma 7. $f(x) = H(f(x),x)$ is continuous follows from the proof of Proposition 7. To show that $f(x)$ is monotonically decreasing we use the result from the proof of Proposition 7 that $x \in [0,x_c]$: $f = 1 \implies df/dx = 0$. Now consider what happens when $x \in (x_c,1]$ and $f < 1$.

In this case we derive for df/dx:

$$\frac{df}{dx} = \underbrace{\frac{1}{\lambda}\sum_{jk} jkp_{jk}(1-x+xf)^{k-1}}_{dH/df 1/x}\left(f + x\frac{df}{dx} - 1\right),$$

$$\frac{df}{dx} = \frac{dH}{df}(f,x)\frac{1}{x}\left(f + x\frac{df}{dx} - 1\right),$$

$$\frac{df}{dx} = \frac{dH}{df}(f,x)\frac{1}{x}\left(1 - \frac{dH}{df}(f,x)\right)^{-1}(f-1).$$

Note that for supercritical x the derivative of $H(f,x)$ with respect to f evaluated at the intersection with the diagonal is less than one, i.e. for $x \in (x_c, 1]$ $\frac{dH}{df}(f,x) < 1$, where $H(f,x) = f < 1$. This can be seen as follows. Clearly for there to exist a solution $f < 1$ to $f = H(f,x)$, $H(f,x)$ must cross the diagonal. But since $\frac{dH}{df}(1,x) > 1$ for $x \in (x_c, 1]$ and $H(1,x) = 1$, $H(f,x)$ must cross the diagonal from below when approaching the intersection from the right. This implies that $\frac{dH}{df}(f,x) < 1$ at the intersection. This together with $0 < x$, $f < 1$ and $\frac{dH}{df}(f,x) > 0$ implies that $\frac{df}{dx} < 0$. □

Lemma 8. $g(f,x)$ is continuous and monotonically increasing in x for $x \in [0,1]$.

Proof. Lemma 8. The fact that $g(f,x)$ is continuous follows directly from the proof of Proposition 7. $g(f,x)$ is monotonically increasing since

$$\frac{dg}{dx} = -\sum_{jk} p_{jk}k(f(x)x + 1 - x)^{k-1}\underbrace{\left(\frac{df}{dx}x + f(x) - 1\right)}_{\leq 0} \geq 0.$$

□

Let $F(s,x) := xg_R(xg_C(s))$. In order to prove Proposition 6 we first need to establish a couple of facts about $F(s,x)$ which we summarize in the following lemma. We will use the index $\mu \in \{R, C\}$ whenever results apply to both repo and collateral networks.

Lemma 9. For $s \in (0,1]$

1. $F(s,x)$ is continuous in s,
2. $F(s,x)$ is monotonically increasing in s,
3. $F(s,x)$ is bounded from above: $F(s,x) < x$,
4. $F(s,x)$ is concave in s.
5. $\lim_{s\to 0} F(s,x) \to 0$,
6. $\lim_{s\to 0} \frac{\partial F(s,x)}{\partial s} \to 0$.

Proof. Lemma 9. For this proof we invoke results from Lemmas 7 and 8. For $s \in (0,1]$:

1. $F(s,x)$ is continuous: $g_\mu(s)$ is continuous as shown in Lemma 8. $F(s,x)$ is a function of $g_\mu(s)$ and therefore also continuous in s.

2. $F(s,x)$ is monotonically increasing: $g_\mu(s)$ is monotonically increasing as shown in Lemma 8. $F(s,x)$ is therefore also monotonically increasing in s.

3. $F(s,x)$ is bounded from above - $F(s,x) < 1$: Clearly $g_\mu(s)$ is bounded from above since $g_\mu(s) \leq 1$. Furthermore, $x \leq 1$. Therefore $F(s,x) = xg_R(xg_C(s)) < 1$. Also note that the above implies that $F(s,x)$ has a maximum at $s = 1$ which scales with x, i.e. as x is decreased the maximum of $F(x,s)$ decreases by at least the same amount.

4. $F(s,x)$ is concave in s:

$$\frac{\partial^2 F}{\partial s^2}(s,x) = x^2\left(x\frac{d^2 g_R}{ds^2}(s)\left(\frac{dg_C}{ds}(s)\right)^2 + \frac{dg_R}{ds}(s)\frac{d^2 g_C}{ds^2}(s)\right)$$

   Since $\frac{dg_\mu}{ds}(s) > 0$, $\frac{d^2 g_\mu}{ds^2}(s) < 0$ (by assumption) and $x > 0$ we must have $\frac{\partial^2 F}{\partial s^2} < 0$, i.e. $F(s,x)$ concave.

5. $\lim_{s\to 0} F(s,x) \to 0$: Since for $s < s_{c,\mu}$ we have $f_\mu(s) = 1$ and $g_\mu(s) = 0$, where $s_{c,\mu}$ is the threshold for network μ at which the giant out-component vanishes as given in Proposition 7. In other words, there exits a critical $s_{c,\mu}$ at which the giant out-component in one of the intermediation networks vanishes (recall that we assume that $\lambda < \infty$, hence there always exists this critical $s_{c,\mu}$ by Proposition 7).

6. $\lim_{s\to 0} \frac{\partial F}{\partial s}(s,x) \to 0$:

$$\lim_{s\to 0}\frac{\partial F}{\partial s}(s,x) = x^2\frac{dg_R}{dv}(v)\frac{dg_C}{ds}(s) \to 0.$$

   Since for $s < s_{c,\mu}$ we have $f_\mu(s) = 1$ and $g_\mu(s) = 0$. Hence for $s < s_{c,\mu}$ we have $\frac{dg_\mu}{ds}(s) = 0$. In other words, since there exists a critical $s_{c,\mu}$ at which the giant component vanishes in one of the intermediation networks, there is a region for values of $s < s_{c,\mu}$ in which $F(x,s)$ is flat.

These observations show that, under the assumptions made here, $F(x,s)$ can be decomposed into two regions: (i) for small values of s ($s < s_{c,\mu}$) $F(x,s)$ vanishes ($F(x,s) = 0$) and is flat ($\partial F/\partial s = 0$). (ii) for larger values of s ($s > s_{c,\mu}$) $F(x,s)$ is strictly monotonically increasing and concave but bounded from above ($F(x,s) < 1$). □

Proof. Proposition 6. This proof invokes results from Lemma 9 and relies in particular on our observations of the shape of $F(x,s)$ in the interval $s \in [0,1]$. We illustrate the graphical intuition for this proof in Fig. 6.

First note that $s = 0$ is a trivial solution to $s = F(s,x)$ for all x since $g_\mu(0) = 0$. Furthermore as shown in Lemma 9 there exists a region for sufficiently small s in which $F(s,x)$ is constant and equal to zero. As seen in Lemma 9, for all $s > s_{c,\mu}$ the function $F(s,x)$ is strictly increasing and concave provided $g_\mu(s)$ is concave. The fact that $F(x,s)$ is constant and flat close to $s = 0$ implies that in at least some of the interval $s \in [0,1]$, $F(x,s)$ must lie below the diagonal. If for $s > s_{c,\mu}$ the function $F(x,s)$ increases sufficiently fast to cross the diagonal there will exist two solutions in addition to the trivial solution (since $F(x,s) < 1$ and hence cannot remain above the diagonal for the entire interval $s \in [0,1]$).

In Proposition 6 we assume these nontrivial solutions exist. Since we are investigating cascades following a small exogenous shock we are only interested in the largest fixed point s^* of the map $s_n = F(s_{n-1},x)$ with $s_0 = x$. This fixed point will be stable due to the concavity of $F(s,x)$ and because at s^* the slope of $F(x,s)$ is $\partial F/\partial s(s^*,x) < 1$.

Now consider how the largest fixed point s^* changes when the initial exogenous shock $1-x$ is increased. Clearly, when x goes down, s^* goes down as well. This is because for a smaller value of x the curve $F(s,x)$ will have a smaller maximum value. This pushes the entire segment of the curve of $F(x,s)$ for $s > s_{c,\mu}$ downwards. Therefore $F(s,x)$ will intersect the diagonal at a smaller value. When both x and s^* decrease further the curve $F(s,x)$ will ultimately become tangent to the diagonal. This will correspond to some critical value x_c. At this point the largest solution s^* merges with the second largest on the diagonal.

If x is decreased further ($x < x_c$) both non trivial solutions vanish and only the trivial solution at $s = 0$ remains. In summary, if there exists some fixed point of $F(x,s)$, s^*, and some critical exogenous shock $1 - x_c$ such that $F(x,s)$ is tangent to the diagonal ($\frac{\partial F}{\partial s}(s^*,x_c) = 1$), then there will be a region below x_c where only the trivial solution exists ($s^* = 0$) and a region above x_c where a non trivial solution $0 < s^* < 1$ exists.

Note that, since there exists some value $s_{c,\mu} > 0$ at which the derivative $\partial F/\partial s(s,x)$ vanishes, $F(x,s)$ must lie below the diagonal close to $s = 0$. Therefore, the non trivial solution must always be greater than zero, i.e. $s^* > 0$ for $x \geq x_c$. Therefore

$$\lim_{\epsilon \to 0} F(s^*, x_c - \epsilon) = 0 \neq F(s^*, x_c) > 0.$$

Hence $F(s,x)$ is discontinuous in x at $x = x_c$. From the above it also follows that, if there exists no $0 < s^* < 1$ such that at some $x = x_c > 0$, $\frac{\partial F}{\partial s}(s^*,x_c) = 1$, then only the trivial solution can exist and $F(s^*,x) = 0 \ \forall \ x < 1$. In this case a minimal disturbance of the network leads always to a complete collapse of the network.

□

Now let us turn to the Proposition 7 (B).

Proof. 7 (B). Let's write $r_c(\mathcal{G}_R, \bar{\mathcal{G}}_C) = r_c$ and $x_c(\mathcal{G}_R, \mathcal{G}_C) = x_c$. Suppose we have $1 - x_c \geq 1 - r_c$ ($x_c \leq r_c$). Note that by definition $g_R(r_c) = 0$. Then $F(s,x_c) = x_c g_R(x_c g_C(s)) < r_c$ since $g_\mu(s) < s < 1$ for all $s < 1$. However, at a fixed point $F(s,x_c) = s$. Hence we have that $x_c g_C(s) < r_c$. But we must have that $g_R(s) = 0$ for all $s < r_c$. This implies that at the fixed point $s^* = 0$. However this contradicts $F(s^*, x_c) > 0$ which is required by Proposition 6. This proves Proposition 7 (B) by contradiction. □

C Calculations for example random networks

C.1 Erdos-Rényi network

Let q denote the probability that a randomly chosen bank is connected to another bank by an outgoing or incoming link. Here, due to the independence of in- and out-degrees the joint degree distribution factorizes into $p_{jk} = p_j p_k$ with $p_j = p_k$ and

$$p_k = \binom{n-1}{k} q^k (1-q)^{n-k-1}.$$

When we hold the average in- and out-degree $\lambda = nq$ fixed and take the limit $n \to \infty$ the generating function for the out-degree distribution of a random node becomes

$$G(z) = e^{\lambda(z-1)},$$

Note that for the Erdős-Rényi network the generating function for the out-degree of a random node is equal to the generating function of the out-degree of the terminal node reached by following a random link (Newman, 2002). Thus, we have $G(z) = H(z)$. As shown in appendix B.2, after an exogenous shock removing a fraction $1-x$ of nodes, the generating functions become

$$\hat{G}(z,x) = \hat{H}(z,x) = G(1-x+zx) = H(1-x+zx) = e^{\lambda x(z-1)},$$

As before, we compute equilibrium liquidity as the size of the giant out-component of the repo network: $\mathscr{L}^*(x) = s^*$. In Figure 7 we solve for s^* numerically.

C.2 Scale free networks

Now let's consider the case where $\mathscr{G}_C$ and $\mathscr{G}_R$ are directed networks with the same power law in- and out-degree distributions (also known as scale free networks). Networks with this degree distribution can be formed for example through a preferential attachment process as outlined in Barabási and Albert (1999). As for the Erdős-Rényi networks we assume that the in- and out-degrees are independent, such that $p_{jk} = p_j p_k$. We also assume that $p_j = p_k = C_\mu k^{-\alpha}$ for $\alpha \in (2,3]$ and $k > 1$. The constant that normalizes the degree distribution is $C = 1/(\zeta(\alpha)-1)$, where $\zeta(\cdot)$ is the Riemann zeta function. Also define the generating functions with their usual meanings

$$G(z) = C\sum_{k>1} k^{-\alpha} z^k = C(\mathrm{Li}_\alpha(z) - z),$$

where $\text{Li}_s(z)$ is the polylogarithmic function defined by:

$$\text{Li}_s(z) = \sum_{k=1} \frac{z^k}{k^s},$$

where s is complex number and z is a complex number with $|z| < 1$, which is clearly valid here. In the following we will only consider real s and z. We also have

$$H(z) = \frac{1}{\lambda} \sum_j p_j j \sum_k p_k z^k = G(z).$$

As before we have that $\hat{G}(z,x) = G(1-x+zx)$ and $\hat{H}(z,x) = H(1-x+zx)$. We can make the substitution $w = 1-x+zx$, i.e. $z = (w+x-1)/x$. Then to find the extinction probability of the branching process we must solve

$$(w+x-1)/x = H(w) = C(\text{Li}_\alpha(w) - w)$$

Again we compute equilibrium liquidity as the size of the giant out-component of the repo network: $\mathscr{L}^*(x) = s^*$. In Figure 8 we solve for s^* numerically.

D Correlations between repo and collateral networks

It is useful to write the equations (6) slightly differently. In particular let us introduce

$$\begin{aligned}
\tilde{s}_R &= g_R(f_R, x - x(1-\tilde{s}_C)),\\
f_R &= H_R(f_R, x - x(1-\tilde{s}_C)),\\
\tilde{s}_C &= g_C(f_C, x - x(1-\tilde{s}_R)),\\
f_C &= H_C(f_C, x - x(1-\tilde{s}_R)),
\end{aligned}$$

where $s_R = x\tilde{s}_R$ and $s_C = x\tilde{s}_C$. Clearly $x - x(1-\tilde{s}_C)$ is simply the fraction of nodes remaining after the initial shock $1-x$ minus the number of nodes that are not in the giant component of $\mathscr{G}_C$ but remain in the network after the initial shock $1-x$. We can make a crude, but simple, approximation to the effect of overlap as follows. Only banks which do not lie outside the giant component in $\mathscr{G}_R$ can withdraw upon their withdrawal in $\mathscr{G}_C$. The number of nodes that are not in the giant component in $\mathscr{G}_R$ and not in the giant component of $\mathscr{G}_C$ is approximately $(1-\tilde{s}_C)(1-\omega)$. Thus we obtain

$$\begin{aligned}
\tilde{s}_R &= g_R(f_R, x - x(1-\tilde{s}_C)(1-\omega)),\\
f_R &= H_R(f_R, x - x(1-\tilde{s}_C)(1-\omega)),\\
\tilde{s}_C &= g_C(f_C, x - x(1-\tilde{s}_R)(1-\omega)),\\
f_C &= H_C(f_C, x - x(1-\tilde{s}_R)(1-\omega)),
\end{aligned}$$

Note that this formulation reduces to the centralized market benchmark for $\omega = 1$ and the usual two network case for $\omega = 0$.

Recall from section C that the generating functions for the Erdős-Rényi network are given by

$$\hat{G}(z,x) = \hat{H}(z,x) = G(1-x+zx) = H(1-x+zx) = e^{\lambda x(z-1)},$$

Then it can be shown that

$$\tilde{s}_R = 1 - e^{-\lambda_R x(1-\tilde{s}_C)(1-\omega)\tilde{s}_R},$$
$$\tilde{s}_C = 1 - e^{-\lambda_C x(1-\tilde{s}_R)(1-\omega)\tilde{s}_C}.$$

If we take $\lambda_R = \lambda_C$, due to the symmetry of the expressions above we must have $\tilde{s}_R = \tilde{s}_C$, hence we can reduce the above to a single equation

$$s = 1 - e^{-\lambda x(1-s)(1-\omega)s}. \tag{9}$$

We know that there exits a regime for ω for which we observe a continuous transition at the critical exogenous shock (e.g. $\omega = 1$) as well as a regime with a discontinuous transition (e.g. $\omega = 0$). The critical value of ω at which the transition switches from continuous to discontinuous is often referred to as the tri-critical point. We can follow the standard procedure to determine the tri-critical point at which the transition becomes discontinuous, cf. Son et al. (2012). Let us first define the deviation measure

$$h(s) = s - (1 - e^{-\lambda x(1-s)(1-\omega)s}).$$

Suppose we are in a regime of ω in which the transition is continuous. Close to the critical exogenous shock we have $\epsilon = s \approx 0$ and we can expand around $h(0)$ to approximate $h(\epsilon)$, i.e.

$$h(\epsilon) = h'(0)\epsilon + \frac{1}{2}h''(0)\epsilon^2 + \frac{1}{6}h'''(0)\epsilon^2 + O(\epsilon^4).$$

Suppose for now that the first and second derivatives are non zero. At a solution of Eq. (9) we must have $h(\epsilon) = 0$. If we ignore higher order terms and solving for ϵ we obtain

$$\epsilon \approx \frac{2h'(0)}{h''(0)},$$

At the critical point $\epsilon = 0$. Thus, provided $h''(0) \neq 0$, at the critical point we must have $h'(0) = 0$. It can be shown that $d\epsilon/dx$ does not diverge at the critical point in this case. Now suppose that $h''(0) = 0$. When solving for ϵ we now need to include higher order terms. Thus

$$\epsilon \approx \sqrt{\frac{6h'(0)}{h'''(0)}},$$

By applying the chain rule we find that $d\epsilon/dx = \partial\epsilon/\partial h'(0)\partial h'(0)/\partial x + R$, where R corresponds to the remaining terms of the derivative. Note that $\partial\epsilon/\partial h'(0) \propto 1/\sqrt{h'(0)}$. Thus, when $h'(0) =$

$h''(0) = 0$, the derivative $d\epsilon/dx$ diverges and a discontinuous transition emerges. Solving for the value of ω at which the first and second derivatives go to zero, we obtain that $\omega_c = 2/3$. Thus, for coupled Erdős-Rényi networks there exists a discontinuous transition as long as approximately one third of the links differ between the two networks.

E Figures

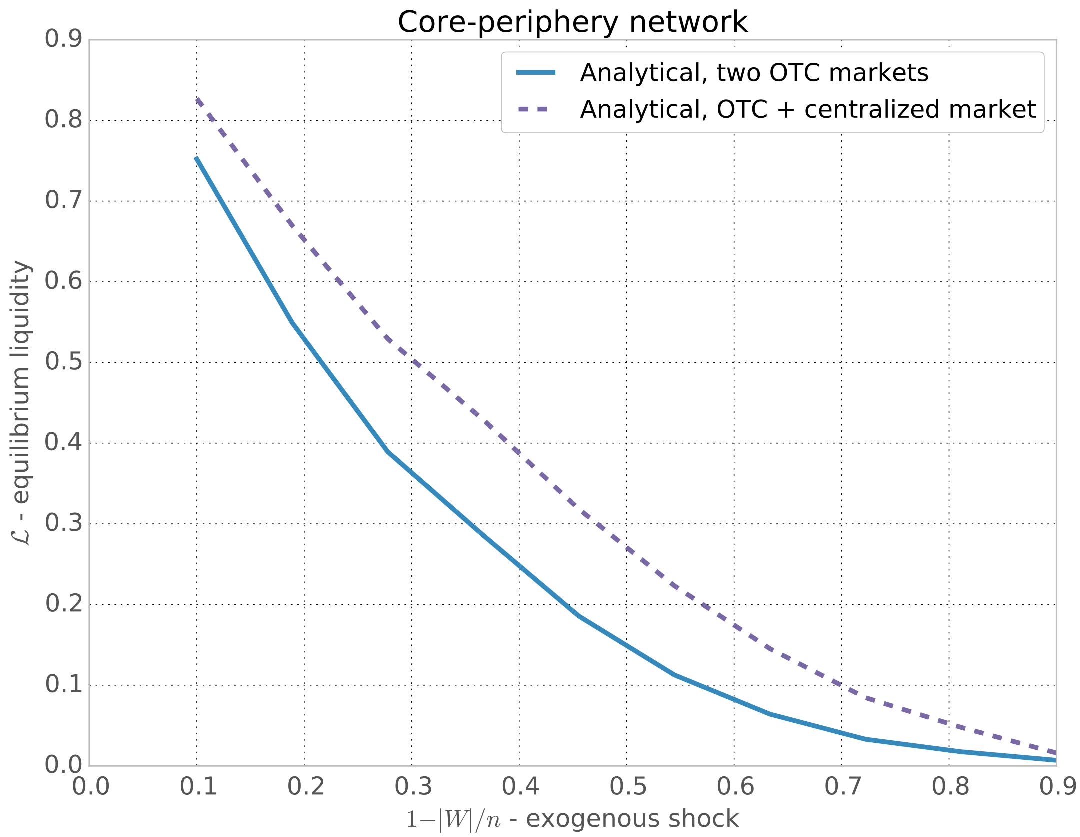


Figure 4: Equilibrium liquidity as a function of the fraction of banks $1-|W|/n$ that withdraw from the repo and collateral markets following an exogenous shock in a core periphery network. The core periphery network is determined by the fractions ($n_{cc} = 0, n_{cp} = 2, n_{pc} = 2, n_{pp} = 50$). Continuous line: analytical approximation for the case of repo and collateral OTC networks. Dashed line: analytical approximation for the case of OTC repo market and centralized collateral market. Dots: Numerical solution based on Algorithm 1.

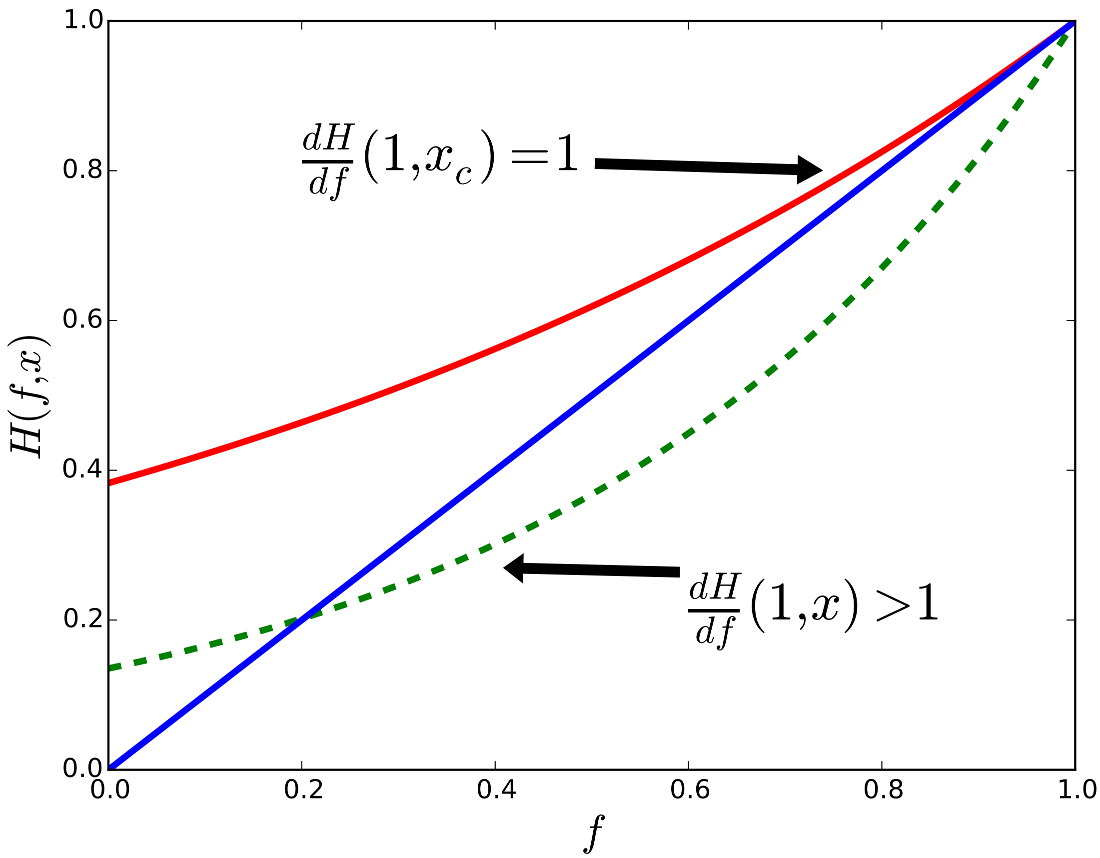


Figure 5: Graphical intuition for proof of Proposition 7. We are interested in fixed points $f^* = H(f^*, x)$ with $f^* < 1$. We plot $H(f,x)$ for two choices of x. Note that the value of x determines the slope of $H(f,x)$ at $f = 1$. The dashed green line corresponds to the case when x is such that $dH(1,x)/df > 1$ while the continuous red line corresponds to the case when x is such that $dH(1,x)/df = 1$. Due to the convexity of $H(f,x)$ in f, $dH(1,x)/df \leq 1$ implies that there will be no fixed point apart from $f^* = 1$ in the interval $[0,1]$. Thus $dH(1,x_c)/df = 1$ determines a critical value of x at which $f^* < 1$ merges with $f^* = 1$.

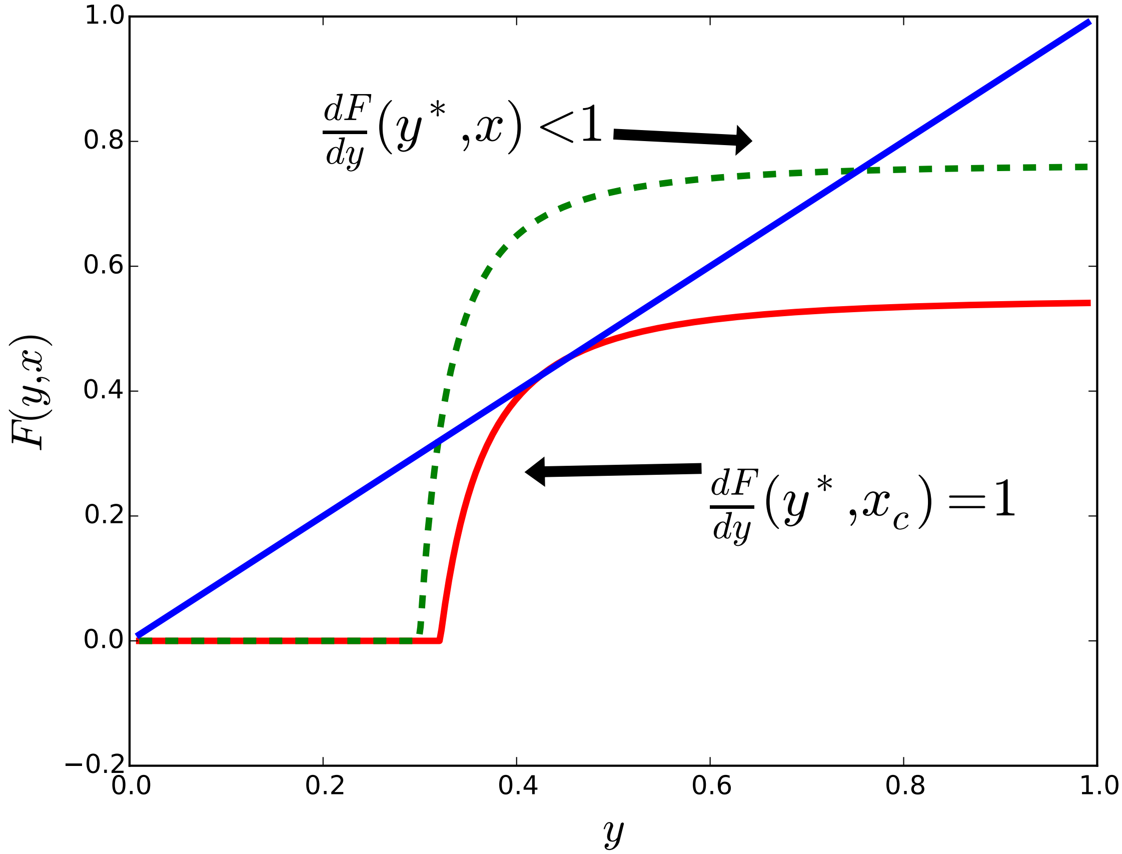


Figure 6: Graphical intuition for proof of Proposition 6. We are interested in the greatest fixed point $y^* = F(y^*, x)$ with $y^* > 0$. We plot $F(y, x)$ for two choices of x. Note that the value of x determines the slope of $F(y, x)$ at y^*. The dashed green line corresponds to the case when x is such that $dF(y^*, x)/dy > 1$ while the continuous red line corresponds to the case when x is such that $dF(y^*, x)/df = 1$. As x is decreased $F(1, x)$ and y^* decrease. At some critical x_c the curve $F(y, x)$ will become tangent to the diagonal. If x_c is decreased any further, $y^* > 0$ disappears and only the trivial fixed point $y^* = 0$ remains.

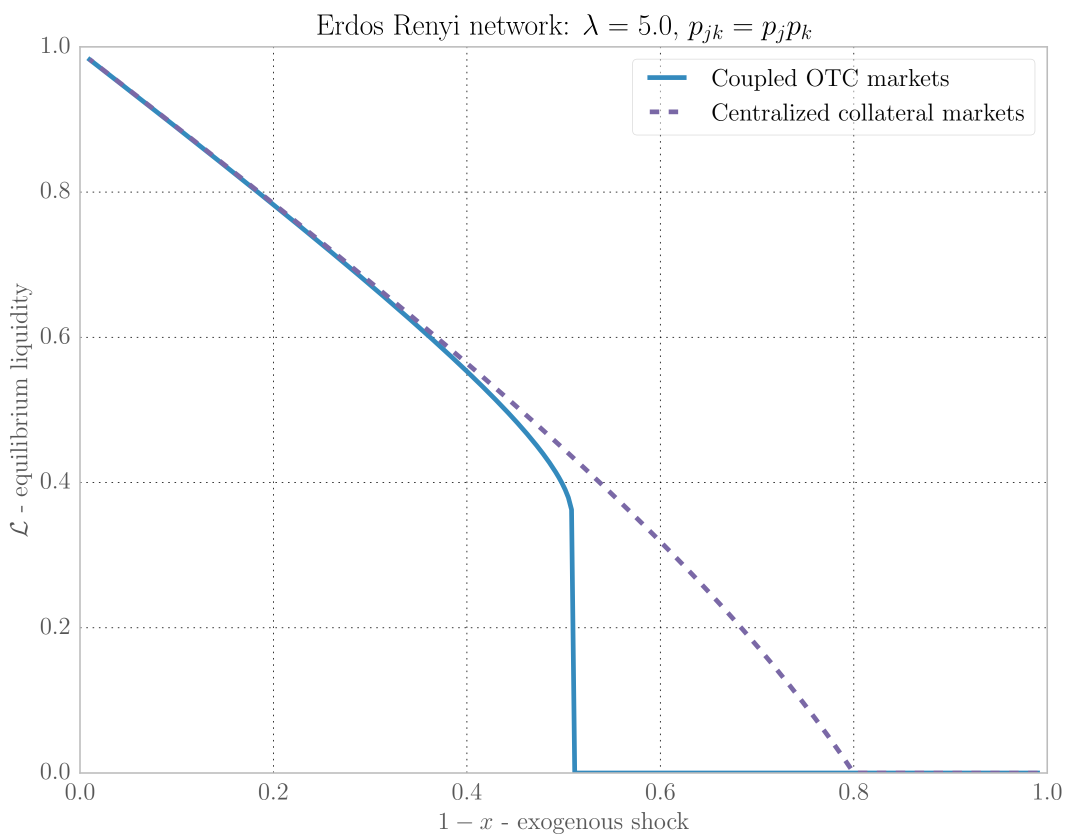


Figure 7: Equilibrium liquidity s^* as a function of the fraction of banks $1-x$ that withdraw from the repo and collateral markets following an exogenous shock in an Erdős-Rényi network with average degree $\lambda = 5$.

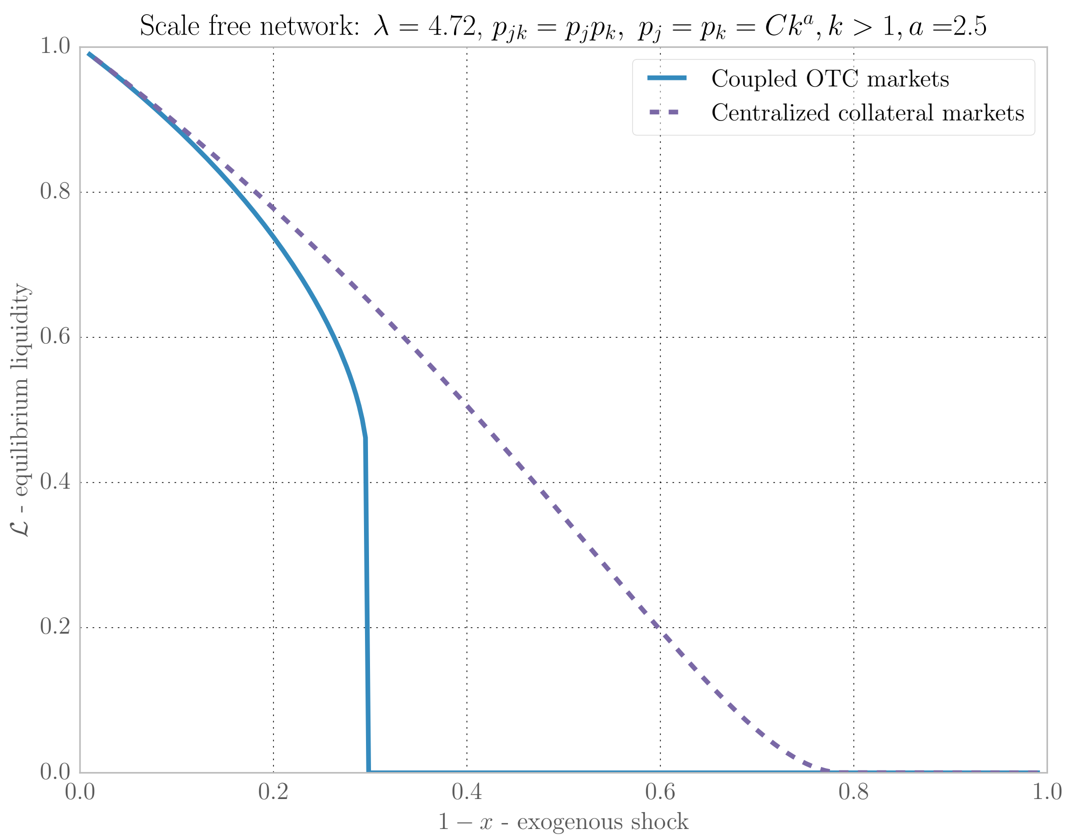


Figure 8: Equilibrium liquidity s^* as a function of the fraction of banks $1-x$ that withdraw from the repo and collateral markets following an exogenous shock in a scale free network.

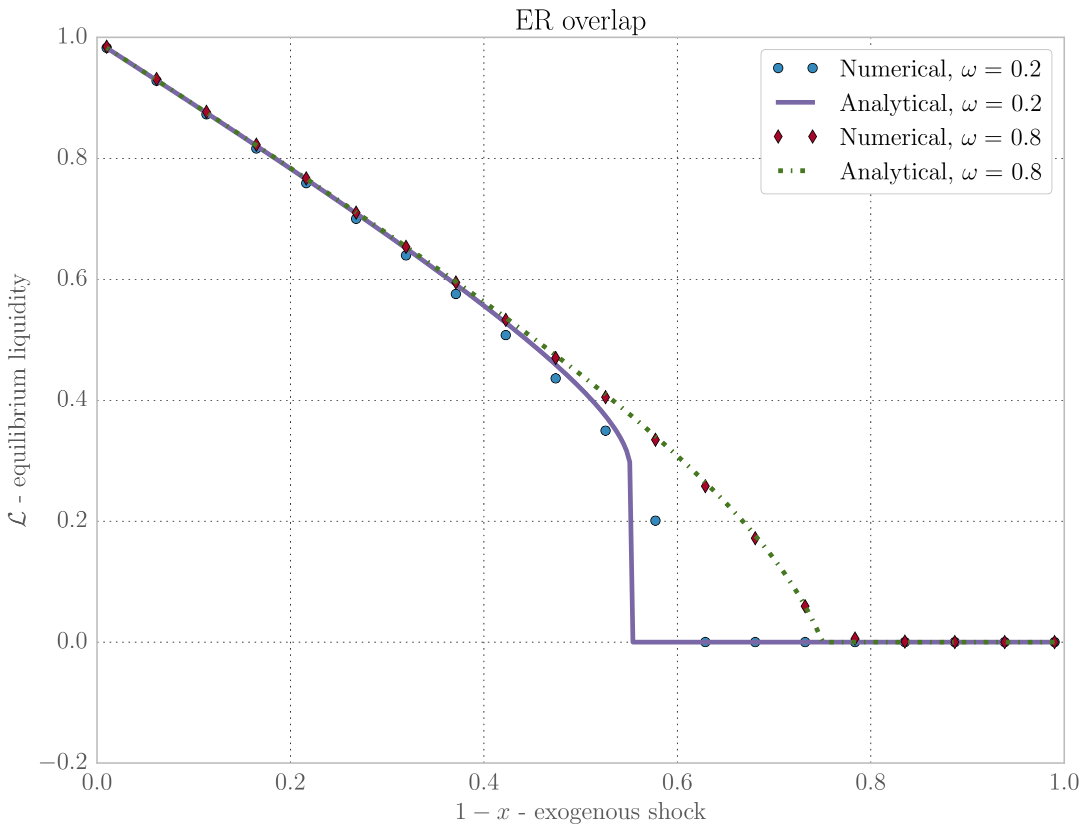


Figure 9: Equilibrium liquidity s^* as a function of the fraction of banks $1-x$ that withdraw from the repo and collateral markets following an exogenous shock in an Erdős-Rényi network with different levels of network overlap.

F Online appendix

F.1 Interpolating between scale-free and core-periphery networks

Many financial networks can be characterized as core-periphery networks. In Section 3.3.2, we showed how illiquidity spirals unfold in stylized core-periphery networks. We showed that for such perfect core-periphery networks, a discontinuous transition as observed in Section 4 does not occur. However, for scale-free networks a discontinuous transition is observed. Arguably, real financial networks are neither perfectly core-periphery, nor perfectly scale-free but somewhere in between these extremes.

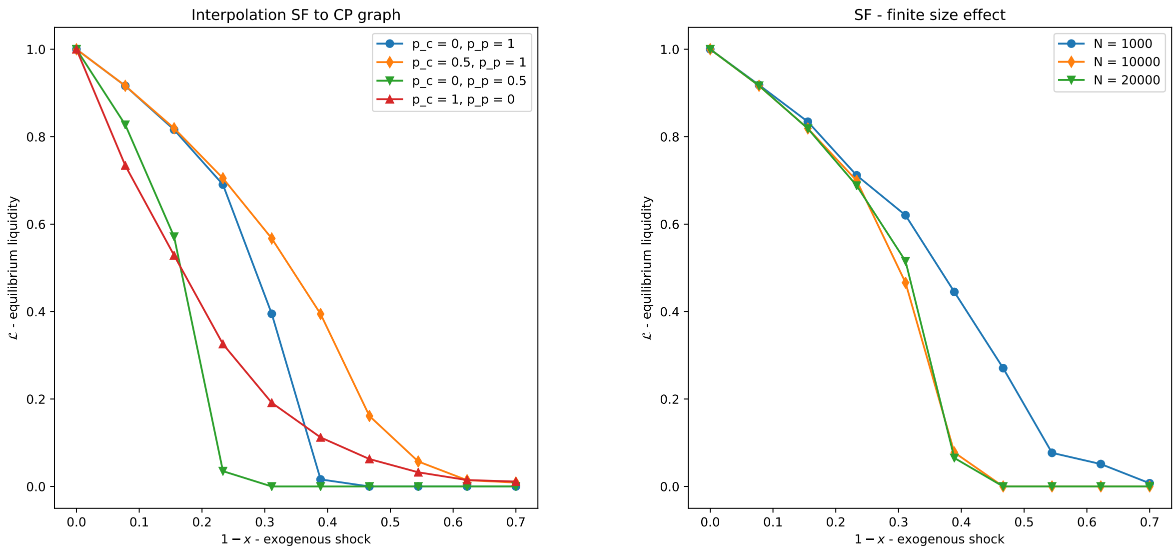
In the following, we propose a very simple approach to interpolate between scale-free and core-periphery networks. We begin by constructing a scale-free network as outlined in Appendix C with tail exponent $\alpha = 2.5$ and N nodes using the configuration model. We then designate the N_C nodes with the highest degree as the core. With probability p_C we connect to core nodes that are not yet connected. Similarly, with probability $1 - p_P$, we remove an existing link between two periphery nodes. Clearly for $p_c = 0$ and $p_P = 1$ we leave the scale-free network unchanged. For $p_C = 1$ and $p_P = 0$ we obtain a perfect core-periphery network. We repeat this procedure to generate two coupled but independent networks.

Figure 10 shows that the transition is smoothed out (as opposed to being discontinuous) as the core becomes more connected. If the core remains unchanged, but peripheral links are removed, the transition is less smooth and liquidity evaporates quicker for smaller shocks. We conjecture that there will be some critical p_C and p_P at which the discontinuous transition disappears. This can be found via a grid search over these parameters. Also note that the discontinuous transition is smoothed when the network is smaller, see Figure 10.

In Figure 11 we study how the equilibrium liquidity measure depends on the size of an exogenous shock for different core-periphery networks. We vary $p_c \in [0, 0.02]$, i.e. we slightly increase the number of links within the core. This has a sizeable impact on the resilience to a shock. The high sensitivity of the system to the existence of additional links within the core is an important insight from our analysis with implications for policy makers tasked with safeguarding financial stability: a market freeze does not have to be complete to leave a system of coupled core-periphery networks much more vulnerable to exogenous shocks. It matters where previously existing links are cut.

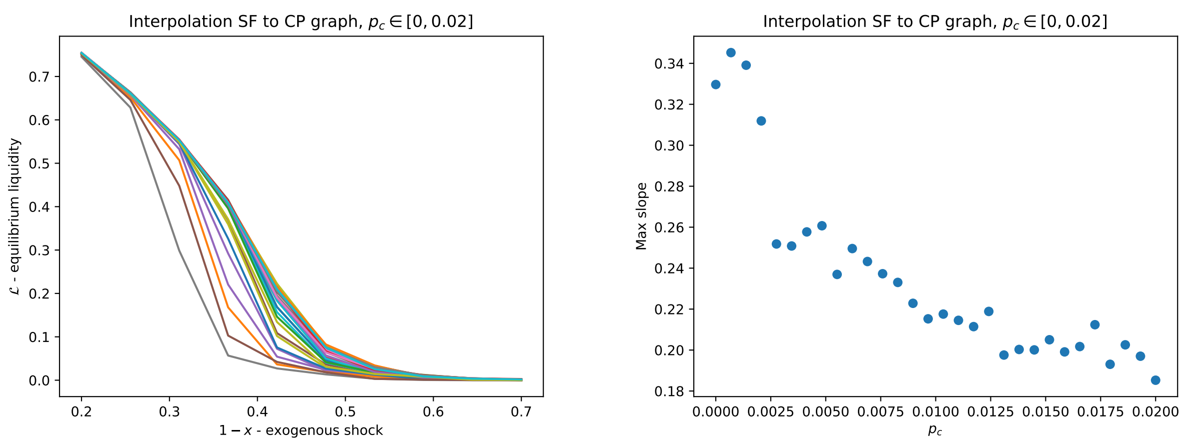
F.2 Cascade sizes in small networks

Suppose we are interested in understanding how fragile a particular network of relatively small size $N \approx 100$ is to the removal of a single node. The fragility of the network depends on the structure of the network. So far, we have always studied networks generated according to some canonical model, such as core-periphery networks, Erdős-Rényi networks or scale-free networks. In the set up we have been studying small shocks typically resulted in relatively small cascades. For some networks we then show that a discontinuous transition occurs for suffi-



(a) (b)

Figure 10: (a) Interpolating between scale-free and core-periphery networks. $N = 10000$, $N_C = 1000$. (b) Finite size effects for scale-free networks.

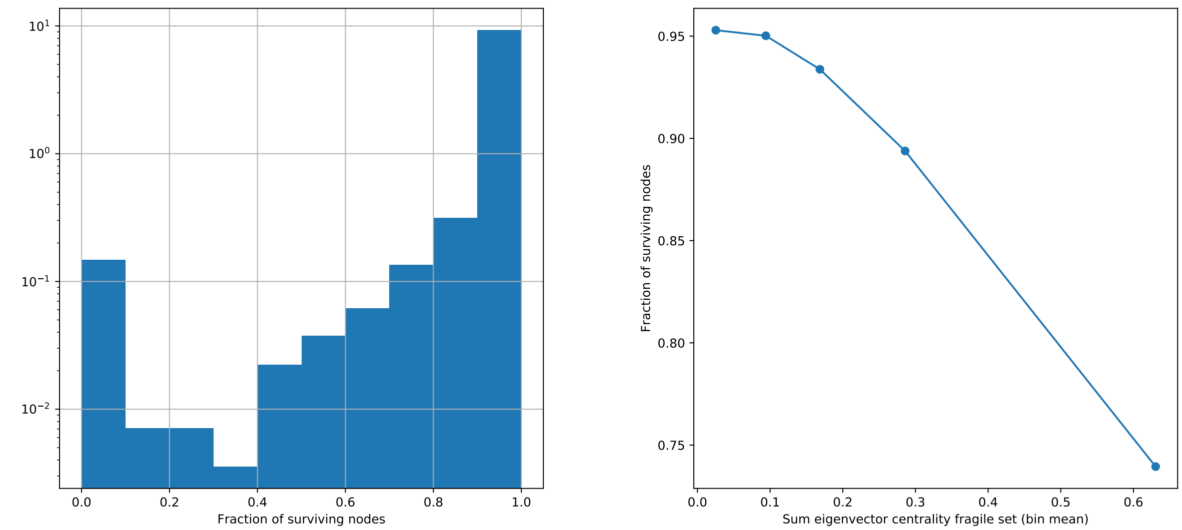


(a) (b)

Figure 11: Interpolation between scale-free and core-periphery networks (a) shows how the equilibrium liquidity measure depends on the size of the exogenous shocks for core-periphery networks with varying probability p_c of two previously unconnected core nodes becoming connected. (b) shows the maximum slope of the curves shown in (a) for every value of p_c.

ciently large shocks.

It is worth noting that, ex-ante, we do not know whether a particular financial network is in a regime where small shocks lead to small changes in liquidity or in a fragile regime where



(a) (b)

Figure 12: (a) Small network cascade sizes. (b) Small network cascade sizes vs. fragile set centrality.

small shocks can have drastic consequences. To understand this statement, note that any network with N nodes can be interpreted as the remains of a larger network M following an exogenous shock of size x.

In the following, we consider the size of cascades caused by the removal of a single node from two "critical" [Do we introduce this notion anywhere? What exactly is a critical network? I thought that a system of two networks can be critical, but not an individual network] coupled scale-free networks. The critical coupled scale-free networks are generated by initializing two coupled scale-free network with $N = 100$ nodes in the usual way. Then, we randomly remove a fraction $1-x = 0.35$ of the nodes and iterate the best response algorithm until the best responses have converged. This leaves us with two coupled networks that are close to the discontinuous transition of scale-free networks as $N \to \infty$. Ex-ante we have no reason to believe that this critical network is less likely than other network configurations.

We then remove a single node at random and study the size of the cascade. Figure 12 shows a histogram of cascade sizes. The distribution is bimodal. In the majority of cases, the removal of a single node does not lead to any cascade at all. However, for a significant fraction of cases, the removal of a single node is catastrophic and the resulting cascade leads to a complete evaporation of liquidity.

What determines whether the removal of a node is catastrophic? One way of studying this question is by studying the "fragile set" of a particular node i. The fragile set of node i is the set of nodes whose best response to the withdrawal of node i is to withdraw from both markets. Intuitively, the extent of the cascade following removal of node i increases with the size of a node's fragile set and interconnectedness of nodes in the fragile set. Figure 12 shows the the fraction of surviving nodes conditional on the sum over the eigenvector centralities of the nodes in the fragile set of node i. To produce the plot we aggregate the results of 500 runs into 5 bins based on the sum of the eigenvector centralities of the nodes in the fragile set. Intuitively, eigenvector centrality is a measure of a node's influence in a network. The more influence the nodes in i's fragile set have, the larger is the size of the ensuing cascade. These results are particularly relevant for supervisory authorities tasked with safeguarding financial stability since they capture two aspects of systemic risk: the probability of a systemic event and it's extent.