Optimal Dynamic Contracts in Financial Intermediation: With an Application to Venture Capital Financing

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November 14, 2013

Abstract

This paper extends the costly state verification model from Townsend [1979] to a dynamic and hierarchical setting with an investor, a financial intermediary, and an entrepreneur. Such a hierarchy is natural in a setting where the intermediary has special monitoring skills. This setting yields a theory of seniority and dynamic control: it explains why investors are usually given the highest priority on projects' assets, financial intermediaries have middle priority and entrepreneurs have the lowest priority; it also explains why more cash flow and control rights are allocated to financial intermediaries if a project's performance is bad and to entrepreneurs if it is good. I show that the optimal contracts can be replicated with debt and equity. If the project requires a series of investments until it can be sold to outsiders, the entrepreneur sells preferred stock (a combination of debt and equity) each time additional financing is needed. If the project generates a series of positive payoffs, the entrepreneur sells a combination of short-term and long-term debt.

*Department of Economics, Stanford University, Stanford, CA 94305. Email: igors@stanford.edu. I am especially grateful to my advisors Martin Schneider and Jeffrey Zwiebel. I am very thankful to Ilya Segal, Monika Piazzesi, Paul Milgrom and Ilya Strebulaev for ongoing support and discussions. I also thank Robert Hall, Pablo Kurlat, Peter Klenow, Joshua Rauh, Pavel Zrumov, Dmitry Orlov, Vilsa Curto, Huiyu Li, Guzman Gonzalez-Torres, Chiara Farronato, Daniel Shen, Diego Perez, Pablo Villanueva, and all participants of the Macroeconomics Finance Reading Group, Macroeconomics Seminar, and Theory Lunch at the Department of Economics and Finance Student Seminar at the Graduate School of Business at Stanford University. I gratefully acknowledge financial support from the Leonard W. Ely and Shirley R. Ely Dissertation Fellowship.

1 Introduction

Many investment settings have the following three characteristics. First, they involve three types of agents: an entrepreneur who can run the project; investors who can provide capital; and financial intermediaries who have specialized monitoring skills that ensure that the entrepreneur takes efficient actions. Second, the agents write long-term contracts that specify how intermediate performance affects the allocation of cash flow and control rights. Third, monitoring by financial intermediaries is costly in time and resources. Particularly significant examples of such a setting are venture capital financing and bank lending. In venture capital financing, investors, such as pension funds and university endowments, form partnerships with venture capitalists, who invest in projects and monitor them by being in close contact with entrepreneurs and participating in board or shareholder meetings. In bank lending, depositors provide capital to bankers, who use this capital to finance businesses and monitor them by doing due diligence. In both cases, contracts are long term[1] and monitoring is costly[2].

In this paper, I develop a dynamic model of this environment and fully characterize the set of contracts that maximize the value of the project net of monitoring costs. The key challenge in this problem is to find such an allocation of cash flow rights that minimizes the number of states that require monitoring. I show that the resulting optimal contracts yield a theory of priority structure and dynamic control rights. The priority structure specifies that the investor has the highest priority on the project's assets, the financial intermediary has middle priority, and the entrepreneur is the residual claimant. The dynamics of the cash flow and control rights are such that more cash flow and control rights are allocated to the financial intermediary if past performance is bad and to the entrepreneur if it is good. Notably, this structure mirrors the observed contracts in venture capital financing and banking and can be replicated with a combination of debt and equity.

In order to study this environment, I extend the costly state verification model from Townsend [1979] to a dynamic setting and I incorporate a hierarchy with an investor, a financial intermediary, and an entrepreneur. The main features of this extension are that the entrepreneur can extract private benefits from the project; the financial intermediary can monitor the entrepreneur at a cost to enforce efficient actions; and there is a possibility of collusion between the entrepreneur and the financial intermediary. Moreover, in order to

[1]See, e.g., Petersen and Rajan [1994] for bank lending and Kaplan and Stromberg [2003] for venture capital financing.

[2]For example, Gorman and Sahlman [1989] document that venture capitalists spend on average 110 hours per year in direct contact with each portfolio company for which they sit on the board of directors, while anecdotal evidence suggests that time is the most scarce resource for venture capitalists (Quindlen [2000]).

incorporate the case of intermediate financing rounds typical for venture capital financing, I allow payoffs to be negative and to have different distributions in different periods.

This focus on dynamic contracts is natural given that the usual life of a venture capital partnership is eight to thirteen years; for a start-up, the time from the original investment to the IPO ranges from three to eight years; and relationships between banks and firms are typically long-term relationships (Petersen and Rajan [1994]). In particular, the contracts in venture capital and banking typically contain numerous covenants that determine how future cash flow and control rights are allocated based on intermediate performance (e.g., Kaplan and Stromberg [2003] and Nini et al. [2012]). The dynamics of such relationships are also expressed through additional securities when firms require more capital or through repayment on a loan before it matures.

I show that the optimal dynamic contracts have a simple structure. In each period, decisions to shut down or monitor the project are governed by simple covenants. If the project's performance is below a certain threshold, the project is closed. If the project continues but its performance is lower than the next threshold, the project is monitored by the financial intermediary. When the project shows good performance, there is no intervention. This structure minimizes the expected costs of monitoring and early termination of the project, as in Gale and Hellwig [1985]. The marginal cash flow rights are given to whoever is in control of the project. When the entrepreneur is in control (not monitored) due to good performance, he needs a sufficient share of the marginal cash flow right in order not to shirk. When the financial intermediary is in control (monitors) in the low states, he needs a sufficient share of the marginal cash flow right in order to monitor the entrepreneur diligently. If the project is terminated, there are no actions to be taken and the entrepreneur and the financial intermediary do not receive any compensation. The dynamics of the optimal contract are expressed through the dependence of these thresholds (strictness of covenants) on the project's past performance. Bad performance makes the thresholds higher (covenants stricter) and good performance makes the thresholds lower (covenants weaker).

The optimal contracts give a theory of priority consistent with the stylized priority structure in financial intermediation, in which investors are paid first, intermediaries are paid second, and entrepreneurs are paid last. In the context of bank lending, this corresponds to the fact that depositors have debt claims on bankers, and bankers have debt claims on businesses. In venture capital financing, the cash flow rights are structured in a similar way, except for the fact that investors and financial intermediaries also have shares of equity in start-ups. The model is able to generate both types of financing depending on how productive the entrepreneur is relative to the cost of effort. If the entrepreneur is highly productive, as in venture capital financing, he can sell a portion of equity without compromising his incentives. If the entrepreneur is less productive, as in bank lending, he needs full equity in order to work efficiently.

The dynamics of the optimal contracts are consistent with the stylized fact that more control and cash flow rights are allocated to financial intermediaries when past performance is bad; and more control and cash flow rights are allocated to entrepreneurs when past performance is good. In the context of venture capital financing, Kaplan and Stromberg [2003] document that bad financial or non-financial performance for a start-up typically leads to additional allocation of shares, board rights, and voting rights to venture capitalists. In the context of bank lending, Nini et al. [2012] show that a violation of a covenant in a loan contract due to low performance leads to higher interest rates and stricter covenants in the renegotiated contract. The optimal contract uses future cash flow rights as an instrument against shirking by the entrepreneur. When the entrepreneur owns a large share of future cash flows due to good past performance, the continuation contract can have little monitoring. On the other hand, if the entrepreneur's share in future cash flows is small due to bad past performance, monitoring is required to force him to work efficiently.

I show that the optimal contracts can be replicated with a combination of debt and equity and simple rules on how these securities can be traded in intermediate periods. Due to differences in the structure of cash flows for start-ups and regular businesses, contracts in venture capital financing and bank lending are replicated separately. A start-up typically requires a series of investments until it can be sold to outside investors. In this case, the entrepreneur initially sells a combination of the start-up's debt and equity to the venture capitalist and the investor. When the project requires more financing he sells more of the start-up's debt. However, if the value of this debt becomes close to the value of the start-up, the venture capitalist stops buying debt and starts monitoring the project. In contrast to a start-up, a regular business typically generates a series of payoffs after the original investment. In this case, in the initial period the entrepreneur sells a combination of short-term and long-term debt to the banker and the depositors. If the project does not generate high enough payoffs in the intermediate stage to repay the short-term debt, the banker monitors the project. If the cash flows are favorable and the entrepreneur repays the short-term debt and has a surplus, he uses this capital to partially repay the long-term debt in order to lower the possibility of monitoring in the future.

This model is able to generate a rich set of comparative statics. For example, it predicts that firms should issue more long-term debt and less short-term debt when they expect more profit in the long term than in the short term. It predicts that firms should issue more short-term debt when the variance of payoffs is higher because such firms require more monitoring (e.g., as documented in Barclay and Smith [1995]). It shows that a decrease in monitoring costs should make monitoring in intermediate periods more attractive and result in higher usage of covenants in those contracts. It also predicts that when investors with expertise in monitoring have more capital, more projects are financed. Finally, it shows that when projects have lower variance, intermediaries have higher leverage (borrow more from investors relative to their own capital) to finance more projects.

This paper is related to two strands of literature. First, I build on models with costly monitoring or verification, as in Townsend [1979], Gale and Hellwig [1985], Holmström and Tirole [1997], and Diamond [1984]. These papers show the optimality of debt contracts and predict the priority structure of cash flows in a financial intermediation hierarchy. My main contribution to this literature is the characterization of the optimal contracts when they are long-term contracts and can dynamically allocate cash flow and control rights contingent on the history of the project's performance. Second, this paper is related to the literature on incomplete contracts, as in Aghion and Bolton [1992], Dewatripont and Tirole [1994], and Hart and Moore [1998]. These papers also predict that control over the project should be conditional on its past performance. The explanation that these papers give is that there is some information in past performance about future productivity. Hence, different actions should be taken depending on the project's past performance and the control is allocated accordingly to implement efficient actions. This contrasts with the cost-minimization rationale presented in this paper. In support of the cost-minimization story, learning models are not known to produce priority structures as described above and may not predict the comparative static results presented in this paper.

The rest of the paper is structured as follows. The next section discusses the related literature. Section Three presents the three-period model. Section Four characterizes the solution to the three-period model. Section Five describes the replication of the optimal contract with the contracts used in venture capital financing; discusses comparative statics; and discusses the role of the intermediary's capital. Section Six shows the replication of the optimal contract with a combination of debt contracts with different maturities; it also provides comparative statics on the optimal debt maturity. Section Seven presents an infinite-horizon version of the model. Section Eight characterizes the optimal contracts when the underlying cash-flow-generating process is in continuous time and the period between the reports goes to zero. Section Nine concludes.

2 Literature Review

The main building block is the model of Costly State Verification (CSV) from Townsend [1979] and Gale and Hellwig [1985]. CSV models are different from other principal-agent models in that an investor, after observing a report from the entrepreneur, can pay a cost to learn the project's true payoff. The main contribution of this literature is that it rationalizes the use of debt contracts. Among other extensions, Winton [1995] gives an example of optimal seniority arrangements when multiple investors finance one project. Wang [2005] and Monnet and Quintin [2005] study CSV models in a dynamic setting with stochastic monitoring. This paper diverges from the existing literature in that it introduces a hierarchy with an investor, monitor, and entrepreneur into the CSV framework; it also provides a dynamic extension of Gale and Hellwig [1985] with a tractable solution, which is a combination of debt contracts with different maturities and seniorities.

More generally, this paper is related to the literature on incomplete contracts, such as Aghion and Bolton [1992], Hart and Moore [1998] and Dewatripont and Tirole [1994]. These papers also predict that bad past project performance leads to greater control by investors. This literature argues that introducing contingent control is efficient when the optimal actions are contingent on a project's performance and agents prefer different actions; say entrepreneurs prefer to continue a project while investors prefer to end it. This paper provides a different explanation for the fact of contingent control. I show that using contingent control minimizes the expected costs of monitoring by the Venture Capitalist (VC) for any given allocation of surplus among the agents.

In its treatment of financial intermediation, this paper is related to the literature on collusion in three-tier hierarchies. A good survey of this literature is given in Laffont and Rochet [1997]. The closest papers in that literature are Tirole [1986], Holmström and Tirole [1997] and Dessi [2005]. These models study a hierarchy of a principal, supervisor and agent, in which the supervisor observes an imperfect signal about the agent's type or effort and can either report to the principal or collude with the agent and share the benefits with him. These papers show that intermediaries need to be compensated if the outcome is good and show that the surplus of a project increases with the amount of capital that intermediaries invest. My paper gives additional predictions about the priority of the compensations in the hierarchy and characterizes the dynamics of the cash flow and control rights.

Methodologically, this paper is closely related to the literature on dynamic moral hazard, examples of which are DeMarzo and Fishman [2007] and DeMarzo and Sannikov [2006]. Several papers in this literature study the effect of monitoring that is costly. Varas [2013] has a model in which the principal can learn at some cost the quality of a chosen unit of production. Piskorski and Westerfield [2012] share a similar trade-off between monitoring and giving the entrepreneur high-powered incentives. The main contribution of this paper to that literature is that it introduces a hierarchy into a dynamic moral hazard setting.

3 The Model

There are three agents in the model: an entrepreneur, a monitor (VC) and an investor. The entrepreneur has access to a project and unique skills to run it, but no capital to cover the initial investment I and possibly negative payoffs in the future. The monitor has some capital or skills that can increase the value of the project and he is able to monitor the project at a cost. However, the amount of capital the monitor can invest is not enough to run the project. This gives a role to the investor with no skills but unlimited capital. All agents are risk neutral and have the same discount rate of zero.

There are three dates in the model, $t = 0$, 1, and 2. At $t = 0$ the agents sign a non-renegotiable contract and make the initial investment. If the initial investment has been made, the project generates an observable for the entrepreneur and the monitor productive opportunity a_t each period. However, a_t is not observed by the investor and the court[3]. Having observed a_t, the entrepreneur chooses how to use this productive opportunity. He can engage in two types of non-verifiable activities. First, he can choose how much private benefit to derive from the project, e.g., by diverting cash or having extra leisure. This activity is denoted $x_t \geq 0$ and it is assumed to give the entrepreneur φx_t utility in monetary units, where $0 \leq \varphi \leq 1$. In addition, the entrepreneur can destroy some of the output without deriving any utility from it. Such activity is denoted $z_t \geq 0$. Later I discuss that destruction may be used by the entrepreneur because unlike stealing it cannot be monitored.

Both private benefit and destructive activities decrease the verifiable payoff of the project, $y_t = a_t - x_t - z_t$. Hence, the entrepreneur can choose among a continuum of actions that either give a high verifiable payoff and little private benefit or low verifiable payoff and high private benefit.

Productive opportunities a_t are unknown before time t and distributed according to some distribution with pdf $f_t(.)$ and cdf $F_t(.)$. Note that the distributions can depend on time. For example, one can think of the first period as the additional investment period and the second period as the project's IPO. For simplicity, a_1 and a_2 are assumed to be independent. There is no restriction on the support of these distributions. However, the following assumption about the hazard rate of the second-period distribution needs to be made:

Assumption 1. *$f_2(a)/(1 - F_2(a))$ is non-decreasing in a.*

Note that common distributions such as the normal, exponential and uniform distributions satisfy this property.

[3]Note that the costly state verification model in Gale and Hellwig [1985] assumes that a_t is not observable for the monitor. Since the monitor is allowed to act based only on verifiable information, this makes no real difference in the model.

The project can be shut down at any point in time. If the project is terminated at date t, both the payoff at t and the future payoffs are canceled; for example, if the project is closed after observing y_1 there is no payoff from the project at all. This can be motivated by the fact that at any point in time investors can stop paying the start-up's bills and invoke limited liability. One can denote the terminal period T as the last period when the project generates a payoff.

3.1 Monitoring and Monitor/Entrepreneur Coalition

Monitoring in this model is the right of the monitor to force the entrepreneur to have no private benefit from the project, i.e., to enforce that $x_t = 0$. This right is contractible and can be conditional on the past and current verifiable information. One can think of monitoring as the control right resulting from bankruptcy, covenant violation, being on the board of directors or having voting rights. Monitoring entails a cost of c_t each period the monitor holds this right, independently of whether any enforcement occurs. This can be motivated by the fact that participating in board meetings or renegotiating a contract after a covenant violation takes time for monitors or bankers. Note that z_t is a type of activity that the monitor cannot control. For example, it is reasonable to think that VCs are able to control how much time entrepreneurs spend on writing programs, but cannot stop them from intentionally writing bugs into their code.

Monitoring itself is verifiable, i.e., whenever this right is contractually given to the monitor, he must monitor the project. However, whether the monitor enforces $x_t = 0$ or not is not verifiable. For example, investors and outsiders may be able to see whether venture capitalists participate in board meetings, but they are not able to assess whether the decisions made at these meetings are in the best interests of the investors[4]. Whether the monitor enforces the efficient action depends on the compensation structure and on the ability of the monitor and the entrepreneur to collude in order to share private benefits. The case when there is no collusion between the entrepreneur and monitor is not interesting because even when the monitor has a fixed compensation for every y_t, he is weakly interested in enforcing $x_t = 0$. In this case, the investor and the monitor are able to act as one agent. Thus, the interesting case is when the monitor is able to collude with the entrepreneur and share the private benefits.

In order to show the effect of collusion, all inefficiencies in the potential bargaining and in the transfers of benefits are assumed away. In particular, following the literature on

[4] Note that Holmström and Tirole [1997] assume that monitoring itself is non-verifiable, but in their model monitoring automatically leads to enforcing the efficient action. This model focuses on the other layer of moral hazard: enforcement when the monitor has control.

collusion in organizations, e.g., Tirole [1986], the entrepreneur is able to have arbitrary side contracts (agreements) with the monitor and share private benefits in such a way that the total utility from private benefits φx_t can be arbitrarily divided between the entrepreneur and monitor. Moreover, the monitor and entrepreneur are able to pledge not only the current private benefits, but also future private benefits or compensations in order to overcome their current liquidity constraints. This may be a rather extreme degree of collusion, but it greatly simplifies the theoretical description of the model. Restricting the agents' ability to collude would weaken the dependence of the monitor's compensation on the project's outcome.

The outcome of such bargaining is the actions x_t and z_t and the compensating transfer from the entrepreneur to the monitor Tr. The specific details of the bargaining model are not essential here. Similar to Tirole [1986], the outcome of the bargaining satisfies the following assumption:

Assumption 2. *1) (Efficient bargaining) The monitor and the entrepreneur choose a side contract that maximizes the combined expected utilities for these two parties.*

2) (Entrepreneur's outside option) The entrepreneur can always guarantee himself the best no-side-contract outcome.

The formal description of the assumption above is given in Equations (3.7)-(3.8). The first assumption is equivalent to stating that the result is Pareto optimal because the agents are able to transfer utilities. The main implication of this assumption is that even when the monitor is in control either he or the entrepreneur needs to be compensated for good performance. The main implication of the second assumption is that when the entrepreneur is in control, he must be compensated for good performance because he is not required to negotiate his actions with the monitor. Note that many bargaining models satisfy these assumptions. For example, a take-it-or-leave-it offer from the entrepreneur, a similar offer from the monitor or any other Nash bargaining solution satisfies these two assumptions. Efficiency of bargaining is a desirable property because it makes the solution much more tractable. However, it does not affect the qualitative results.

Bargaining between the entrepreneur and monitor is the only place in the model where the fact that a_t is observable by the monitor makes the model more straightforward. If a_t is not observed by the monitor at the time of bargaining, the assumptions on the bargaining outcomes above may be more restrictive. In particular, efficiency is not satisfied in many models of bargaining with asymmetric information. Setting this technical element aside, the model is equivalent to the costly state verification model from Townsend [1979]. To interpret the model this way, call a_t the output and y_t the reported output. Based on the reported output the monitor can pay the cost c_t and learn a_t. If $a_t > y_t$ then he can enforce y_t to be equal to a_t.

3.2 Contracts

The contracts specify the allocation of monitoring rights, the termination rule and compensations as functions of the history of the verifiable payoffs, denoted y^t. In particular, monitoring rights at t are denoted $M_t(y^t) \in \{0,1\}$, the termination rule is denoted $D_t(y^t) \in \{0,1\}$ and the compensations to the agents at $t = 2$ are denoted $C^E(y^2)$, $C^M(y^2)$ and $C^I(y^2)$, respectively. Note that since the agents are risk neutral and have zero discounting, consuming at $t = 2$ is weakly optimal. In addition, the contract may contain recommended actions x_t^*, z_t^* and a transfer Tr, although not enforceable in the court of law.

The contracts must satisfy a number of restrictions. First, the initial investment must be sufficient to start the project:

$$E^M + I^I = I, \tag{3.1}$$

where E^M and I^I are the amounts that the monitor and investor invest, respectively. Note that even when $E^M > I$ the equation above holds. In that case the monitor gives $E^M - I$ up front to the investor as a deposit. This deposit can be repaid in the last period.

The compensations need to add up to the project's total payoff:

$$C^E + C^M + C^I = \sum_{t=1}^{T} y_t. \tag{3.2}$$

The compensations of the entrepreneur and monitor need to be positive because they have no capital after investing in the project:

$$C^E, C^M \geq 0. \tag{3.3}$$

The project can be initiated if all agents agree to participate, i.e., if the following individual rationality constraints for the entrepreneur, monitor and investor, respectively, are satisfied:

$$U^E = E[C^E - Tr + \varphi \sum_{t=1}^{T} x_t] \geq 0, \tag{3.4}$$

$$U^M = E[C^M + Tr] - E^M \geq 0 \tag{3.5}$$

and

$$U^I = E[C^I - \sum_{t=1}^{T} c_t M_t] - I^I \geq 0. \tag{3.6}$$

Note that monitoring costs enter the investor's payoff, but not the monitor's utility.

Since monitoring (but not enforcement) is verifiable, the investor can always compensate the monitor for incurring monitoring costs without creating a moral hazard problem. Thus, without loss of generality they are directly attributed to the investor in order to make the equations simpler.

Finally, the entrepreneur and monitor must be willing to choose the recommended actions x_t^* and z_t^*. Hence, Assumption 2 requires the recommended actions to maximize the utilities of the monitor and the entrepreneur and to make the entrepreneur better off than in the no-collusion outcomes:

$$x_t^*, z_t^* \in \arg\max_{x_t, z_t \geq 0}[U^E + U^M], \tag{3.7}$$

$$U^E(x_t^*, z_t^*, Tr) \geq \max_{x_t \in \{0, M_t(y^t)=0\}, z_t \geq 0}[U^E]. \tag{3.8}$$

Note that the set $\{0, M_t(y^t) = 0\}$ is the set consisting of the efficient action and the actions leading to no monitoring, where as before $y_t = a_t - x_t - z_t$. Hence, this is the set of actions for which the entrepreneur does not need to negotiate with the monitor.

To sum up, feasibility constraints (3.1)-(3.3), individual rationality constraints (3.4)-(3.6) and incentive compatibility constraints (3.7)-(3.8) constitute the constraints on the contracting space. The set of contracts satisfying these constraints is denoted Γ.

4 Optimal Contracts

At $t = 0$ the agents want to find a long-term contract that satisfies feasibility, individual rationality, incentive compatibility and Pareto optimality. Note that the set of Pareto optimal contracts includes the best contracts for the entrepreneur, the monitor or the investor. Their relative bargaining powers should determine which contract from the set of Pareto optimal contracts is chosen. All Pareto optimal contracts γ^* are solutions of the following optimization problem for some $\overline{U}^E \geq 0$ and $\overline{U}^M \geq 0$:

$$\gamma^* = \arg\max_{\gamma \in \Gamma} E[\sum_{t=1}^{T}(y_t + \varphi x_t - c_t M_t)] \tag{4.1}$$

$$s.t.$$

$$U^E \geq \overline{U}^E \tag{4.2}$$

$$U^M \geq \overline{U}^M, \tag{4.3}$$

where the objective function is the total surplus of the project, $U^E + U^M + U^I$.

The first step in solving the problem above is observing that it is weakly better not to recommend any private benefits or destructive activities.

Lemma 1. *Recommended actions are $x_t^* = 0$ and $z_t^* = 0$. Side transfers are not necessary, i.e., $Tr = 0$. To implement these actions, contracts must be (let $\overline{C}(y_t) = E_t[C|y^t]$):*

1. Incentive compatible for the entrepreneur:

$$0 \in \arg\max_{x_t \geq 0}[\varphi x_t(1 - D(a_t - x_t))(1 - M(a_t - x_t)) + \overline{C}^E(a_t - x_t)]]. \tag{4.4}$$

2. Collusion proof:

$$0 \in \arg\max_{x_t \geq 0}[\varphi x_t(1 - D(a_t - x_t)) + \overline{C}^E(a_t - x_t) + \overline{C}^M(a_t - x_t)]], \tag{4.5}$$

3. Monotonic:

$$\overline{C}^E \text{ is non-decreasing in } y_t. \tag{4.6}$$

Proof. See Appendix.

It is optimal to have no private benefits and no destruction because both activities decrease the project's total surplus. The key argument is similar to the argument in the Revelation Principle. Any expected utilities that the entrepreneur and monitor can achieve when they choose positive x_t or z_t can be achieved by compensating them accordingly at $x_t = 0$ and $z_t = 0$. This would make the entrepreneur and monitor indifferent and would save the costs of private benefits or value destruction. In the proof I verify that under the new compensation scheme it is incentive compatible to have $x_t = 0$ and $z_t = 0$.

The constraints above are direct implications of choosing $x_t^* = 0$ and $z_t^* = 0$. The entrepreneur's incentive compatibility constraint states that the entrepreneur prefers to choose $x_t = 0$ out of all the outcomes that are not monitored. The constraint on being collusion proof requires the entrepreneur and monitor to prefer the efficient outcome rather than sharing the surplus from private benefits. The last constraint is the necessary and sufficient condition for $z_t^* = 0$. If the entrepreneur's compensation were decreasing in y_t, he would always destroy some output to receive higher compensation.

Note that the incentive compatibility constraints are global. To illustrate this, suppose the lowest possible payoff a_l is not monitored while all other payoffs are monitored. The fact that for any realization the entrepreneur can report a_l implies that his compensation at an arbitrary a needs to be above $C^E(a_l) + \varphi(a - a_l)$, i.e., at any a he receives a rent due to not being monitored at a_l. On the other hand, if a_l is also monitored, his compensation can be constant, i.e., no rent is needed. This observation will play an important role in the structure of the optimal contract.

4.1 Second-Period Contract

I solve the optimal contract by backward induction. For now, I solve the second-period problem as if it were the only period in the model. I denote the expected compensations of the entrepreneur and monitor in that period as $\overline{C}^E$ and $\overline{C}^M$, respectively. Later I show that this solution is part of the optimal contract in the three-period model. The second-period problem is:

$$V_2^{tot} = \max_{\gamma \in \Gamma} E[(a_2 - c_2 M_2(a_2))(1 - D_2(a_2))] \tag{4.7}$$

$$s.t.$$

$$E[C^E(a_2)] = \overline{C}^E, \tag{4.8}$$

$$E[C^M(a_2)] = \overline{C}^M \tag{4.9}$$

where the optimization is taken with respect to four unknown functions: $D_2(a_2)$, $M_2(a_2)$, $C^E(a_2)$ and $C^M(a_2)$. The objective function in the problem above is the expected surplus created in the second period. This surplus is created only in the states that do not lead to termination, i.e., when $1 - D_2(a_2) = 1$. It is equal to the second-period payoff net of monitoring costs. Note that this problem is identical to the problem in Gale and Hellwig [1985], only with an additional incentive compatibility constraint for the coalition of the monitor and entrepreneur.

To solve this problem I perform the following steps. First, I show that for any given allocation of monitoring and termination it is optimal to give as little rent as possible to the entrepreneur and monitor, i.e., that incentive compatibility constraints (4.5)-(4.4) are binding. It is inefficient to give more rent in some states because this requires more costly monitoring or termination in other states in order to keep the expected compensations at a given level. Second, I show that binding incentive compatibility constraints imply that the monitoring and termination functions are threshold rules, i.e., the lowest payoffs are terminated, the middle payoffs are monitored and the highest payoffs are left without intervention. As I described before, leaving some low payoffs unmonitored makes monitoring of high payoffs ineffective. This gives the results stated in the following theorem:

Theorem 1. *The monitoring and termination rules are threshold rules, i.e., there exist* a_2^D *and* a_2^M *such that* $D_2(a_2) = 1[a_2 < a_2^D]$ *and* $M_2(a_2) = 1[a_2^D \leq a_2 < a_2^M]$.

· *The optimal compensations are zero on* $a_2 < a_2^D$ *and continuous on* $a_2 \geq a_2^D$

· *In the monitoring region* $a_2^D \leq a_2 < a_2^M$*, the monitor holds the marginal cash flow rights, i.e., (*C_1 *denotes the derivative of* C*)* $C_1^M = \varphi$ *and* $C_1^E = 0$

· *In the non-monitoring region* $a_2 \geq a_2^M$*, the entrepreneur holds the marginal cash flow rights, i.e.,* $C_1^M = 0$ *and* $C_1^E = \varphi$.

Proof. See Appendix.

When the payoff is low and the monitor is in control, he is compensated with φ dollars for each additional dollar of payoff in order to deter him from colluding with the entrepreneur. The entrepreneur needs no compensation to work efficiently. Note that compensating both of them with $\varphi/2$ for each additional dollar of payoff would also be incentive compatible. However, in that case it would be possible to divide this region into two and compensate the monitor with φ in one of them and the entrepreneur with φ in the other one. Then, the monitor could stop monitoring in the region where the entrepreneur has high-powered incentives and thereby save on monitoring costs.

Similarly, when the payoff is high and the entrepreneur is in control, the monitor's compensation is fixed and the entrepreneur has high-powered incentives to work efficiently. In the termination region both compensations are zero because the actions of the entrepreneur and monitor have no effect on output. Since the expected compensations are fixed, it is efficient to compensate the agents only in the states where their actions matter. Note also that there can be a jump in the compensations at a_2^D from zero to a positive level. Theorem 1 does not specify the size of the jump. However, for given values of a_2^D and a_2^M the size of the jump is determined by the expectations of the compensations given in Equations (4.8)-(4.9). I illustrate the structure of the compensations given in Theorem 1 in Figure 4.1a.

One can show that when the monitor finances the project without the investor and the payoff from the project is always positive, Theorem 1 implies that the optimal contract is identical to the debt contract in Gale and Hellwig [1985]. This observation is used later on to show that in a more general case the optimal contract can be replicated with senior and junior debt for the project.

The result above characterizes the solution of the second-period problem for given levels of a_2^D and a_2^M. Thus, to complete the solution one needs to find these thresholds as functions of the expected compensations $\overline{C}^E$ and $\overline{C}^M$. Although $a_2^M(\overline{C}^E, \overline{C}^M)$ and $a_2^D(\overline{C}^E, \overline{C}^M)$ can be given in a semi-closed form, their functional forms are not important here and are left for the Appendix. Instead, the important properties of the solution are listed below.

Lemma 2. *The optimal monitoring threshold* a_2^M *is a non-increasing function of* $\overline{C}^E$ *and the optimal termination threshold* a_2^D *is a non-increasing function of* $\overline{C}^E + \overline{C}^M$.

Proof. See Appendix.

Intuitively, when the entrepreneur's expected compensation is high, his limited liability constraint is not binding and the investor and monitor can effectively sell the project to him.

When the entrepreneur owns the project he runs it efficiently without being monitored. On the other hand, when the entrepreneur's expected compensation is low, he needs to be monitored to restrict the amount of his rent. Similarly, when $\overline{C}^E + \overline{C}^M$ is high, the project is terminated only when it is efficient to do so. When $\overline{C}^E + \overline{C}^M$ is low, the only way to ensure that the entrepreneur and monitor have low rents is to terminate the project in most states.

Another important property of this solution is that the second-period value of the project is concave and has negative cross-partial derivatives.

Lemma 3. $V_2^{tot}(\overline{C}^E, \overline{C}^M)$ *is increasing, twice differentiable, concave and satisfies:*

$$\frac{\partial^2 V_2^{tot}}{\partial \overline{C}^E \partial \overline{C}^M} = \frac{\partial^2 V_2^{tot}}{\partial^2 \overline{C}^M}. \tag{4.10}$$

Proof. See Appendix.

This property relies on the assumption that the hazard rate of the second-period distribution is non-decreasing. To see this, consider the dependence of V_2^{tot} on $\overline{C}^E$. Increasing a_2^M by Δ costs $f(a_2^M)c_2\Delta$ in terms of monitoring and decreases the rent of the entrepreneur by $(1 - F(a_2^M))\Delta$ because he is paid less by Δ in states $a > a_2^M$. Hence, the hazard rate in the model represents how much it costs in monitoring terms to decrease the entrepreneur's rent by one dollar. Since monitoring costs are subtracted from V_2^{tot}, the increasing hazard rate makes the second-period total value a concave function of $\overline{C}^E$.

4.2 First-Period Contract

In the first period the project's total expected surplus consists of the surplus generated in the first period, $a_1 - c_1 M_1(a_1)$, and the continuation value of the project $V_2^{tot}(\overline{C}^E(a_1), \overline{C}^M(a_1))$, with the compensations expected in the second period being allowed to depend on the first-period payoff realization. Given that the constraints on the first-period contract are also functions of $D_1(a_1)$, $M_1(a_1)$, $\overline{C}^E(a_1)$ and $\overline{C}^M(a_1)$, the optimal first-period contract solves the following problem:

$$V_1^{tot} = \max_{\gamma \in \Gamma} E[(a_1 - c_1 M_1(a_1) + V_2^{tot}(\overline{C}^E(a_1), \overline{C}^M(a_1)))(1 - D_1(a_1))] \tag{4.11}$$

$$s.t.$$

$$E[\overline{C}^E(a_1)] = U^E, \tag{4.12}$$

$$E[\overline{C}^M(a_1)] = U^M + E^M, \tag{4.13}$$

where the optimization is taken with respect to the unknown functions $D_1(a_1)$, $M_1(a_1)$, $\overline{C}^E(a_1)$ and $\overline{C}^M(a_1)$ and where U^E and U^M are the utilities of the agents from participating in the project. Note that from knowing the optimal functions $\overline{C}^E(a_1)$ and $\overline{C}^M(a_1)$ and the optimal second-period thresholds $a_2^M(\overline{C}^E, \overline{C}^M)$ and $a_2^D(\overline{C}^E, \overline{C}^M)$, one can derive the optimal compensations $C^E(a_1, a_2)$ and $C^M(a_1, a_2)$ as functions of the history of payoffs as well as the optimal second-period termination and monitoring thresholds $a_2^D(a_1)$ and $a_2^M(a_1)$ as functions of the first-period payoff. This explicitly characterizes the optimal dynamic contract.

The first-period problem is different from the second-period problem because the compensations now also affect the project's continuation value. However, with the properties of the continuation value described in Lemma 3, the solution to the first-period problem is quite similar to the solution of the second-period problem.

Theorem 2. *The monitoring and termination rules are threshold rules,i.e., there exist a_1^D and a_1^M such that $D_1(a_1) = 1[a_1 < a_1^D]$ and $M_1(a_1) = 1[a_1^D \leq a_1 < a_1^M]$.*

· The optimal compensations are zero on $a_1 < a_1^D$ and continuous on $a_1 \geq a_1^D$

· In the monitoring region $a_1^D \leq a_1 < a_1^M$, the monitor holds the marginal cash flow rights whenever his expected compensation is positive, i.e., $\overline{C}^M = \varphi$ and $\overline{C}_1^E = 0$ when $\overline{C}^M > 0$

· If $\overline{C}^M(a_1^D) = 0$, there is a region where the monitor's compensation is zero and the entrepreneur holds the marginal cash flow rights, i.e., there exist $a_1^H \in [a_1^D, a_1^M]$ such that $\overline{C}^M = 0$, $\overline{C}_1^M = 0$ and $\overline{C}_1^E = \varphi$

· In the non-monitoring region $a_1 \geq a_1^M$, the entrepreneur holds the marginal cash flow rights, i.e., $\overline{C}^M = 0$ and $\overline{C}_1^E = \varphi$.

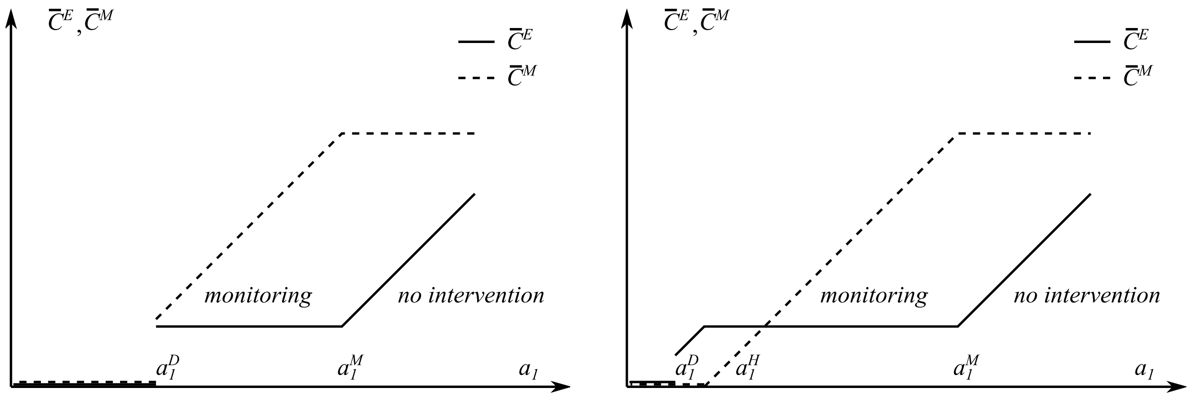
Proof. See Appendix.

Note that the case when $\overline{C}^M(a_1^D) > 0$ is identical to the second-period compensation structure. In this case, whenever the project is monitored, only the monitor's compensation depends on the project's payoff. When there is no monitoring, only the entrepreneur's compensation depends on the payoff. The case of $\overline{C}^M(a_1^D) = 0$ is different because on $a_1 \in [a_1^D, a_1^H]$ the monitor's compensation is already zero and cannot depend on the payoff. Hence, to incentivize the efficient actions the entrepreneur's compensation becomes dependent on the payoff in that region. These compensations are shown graphically in Figure 4.1.

The proof of Theorem 2 follows the main steps of the proof of Theorem 1. First, incentive compatibility constraints should always be binding because giving too much rent to the agents in high states requires more monitoring or termination in low states. Second, leaving some low states unmonitored makes monitoring high states ineffective because it does not affect the entrepreneur's rent. Third, only the monitor is compensated for extra payoffs in the monitoring region. Otherwise one could divide this region into one region with high-powered

Figure 4.1: Optimal Expected Compensations as Functions of the First-Period Payoff

a) Monitor does not hit liability constraint   b) Monitor hits liability constraint



These figures show how the compensations that the entrepreneur and monitor receive after the second period depend on the project's first-period performance. Payoffs $a_1 < a_1^D$ lead to termination, payoffs $a_1^D \leq a_1 < a_1^M$ lead to monitoring and payoffs $a_1 \geq a_1^M$ lead to no additional actions. Figure 4.1a shows the case when the monitor's limited liability constraint is not binding and Figure 4.1b shows the case when the liability constraint is binding on $a_1^D \leq a_1 < a_1^H$.

compensation only to the monitor and another region with high-powered compensation only to the entrepreneur and then stop monitoring in the second region.

Combining the results of Theorem 2 with Lemma 2 gives the contingent allocation of control rights.

Corollary 1. *a_2^M and a_2^D are non-increasing functions of a_1.*

Theorems 1-2 and Corollary 1 show that the optimal contracts have the stylized properties of contracts used in practice. The priority structure of cash flow rights can best be seen for the case of $\varphi = 1$. In each period the investor is paid in expectation the same amount in all the states that do not lead to termination and receives all the project's assets if it is terminated, which means that he has the highest priority on the project's cash flow. In the monitored states, the monitor receives in expectation every extra dollar of payoff that the project generates, meaning that he has second priority on the cash flows. Finally, in the states with high payoffs the entrepreneur benefits from the project's extra cash flow.

The contingent allocation of cash flow rights is directly given in Theorem 2. The better the project's performance, the higher is the share of future cash flow rights allocated to the entrepreneur. Corollary 1 adds to this the contingent allocation of control rights. If the performance is good in the current period, the entrepreneur retains control in more states in the future. Thus, cash flow and control rights are used as complements in the model in order to give the entrepreneur incentives to operate the project efficiently.

To complete the characterization of the optimal dynamic contract one needs to find the optimal first-period thresholds a_1^D and a_1^M. The solution for these parameters does not have a simple analytic form. However, the trade-off for these parameters is quite intuitive. Consider the effect of raising the monitoring threshold. Obviously, this increases the expected monitoring costs in the first period. However, this also decreases the variation in the entrepreneur's compensation and increases the variation in the monitor's compensation. Intuitively, having lower variation in the entrepreneur's expected compensation helps to avoid the states in the second period in which there is a lot of monitoring. Mathematically, due to concavity of V_2^{tot} and the other properties listed in Lemma 3, the project's expected continuation value is higher when the variance of the entrepreneur's compensation is lower. Hence, the trade-off for monitoring in the first period is between paying some monitoring costs in that period and the risk of paying higher monitoring costs in the next period.

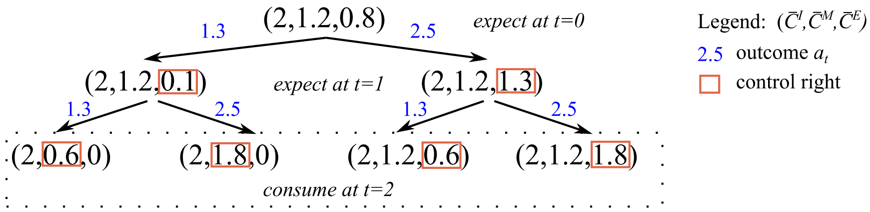
Depending on where the optimal thresholds are, the optimal contract can look like the one depicted in either Figure 4.1a or 4.1b. If the monitor's initial expected compensation is low but there is a lot of monitoring in the first period, the monitor is likely to hit his limited liability constraint. If the monitor's initial expected compensation is high or there is little monitoring in the first period, his expected compensation never becomes zero and the contract looks like the one in Figure 4.1a. There is no simple characterization of the set of parameter values for which each of these cases is relevant. In the case of equal monitoring costs and distributions of payoffs in both periods, it is usually the case that $a_1^H = a_1^D$ and the compensations look like the ones depicted in Figure 4.1a. On the other hand, if the monitor's compensation is low and $c_1 \ll c_2$ then the other case is relevant.

To conclude this section I give a simple numerical illustration of the optimal allocation of cash flow and control rights. In this illustration the project requires $I = 2.9$ for financing and generates uniformly distributed payoffs on $[1, 3]$ in both periods. The monitor has capital $E^M = 1.2$, monitoring costs are $c = 0.6$ in both periods, $\varphi = 1$ and at $t = 0$ the agents choose the contract that is best for the entrepreneur (the capital market is competitive). One can compute that for this case $a_1^D = 0.76$, $a_1^M = 1.22$ and at $t = 0$ the expected compensations are $\overline{C}^I = 2$, $\overline{C}^M = 1.2$ and $\overline{C}^E = 0.8$.[5] As an illustration, in each period I will consider two realizations: $a_t = 1.3$ and $a_t = 2.5$.

Figure 4.2 gives an illustration of the dynamics of the cash flow and control rights. In the first period the entrepreneur retains control for most outcomes ($P(a_1 < a_1^M)$ is small), so his compensation is affected by the project's performance. If $a_1 = 1.3$, which is far below the average, his expected compensation drops close to zero. On the other hand, if

[5]Given these levels one can compute $\overline{C}^E(a_1)$ and $\overline{C}^M(a_1)$ from Theorem 2 and solve for $a_2^D(\overline{C}^E, \overline{C}^M)$ and $a_2^M(\overline{C}^E, \overline{C}^M)$ from Theorem 1.

Figure 4.2: Allocation of Cash Flow and Control Rights in a Numerical Example



In this illustration the project requires $I = 2.9$ for financing and generates uniformly distributed payoffs on $[1, 3]$ in both periods. The monitor has capital $E^M = 1.2$, monitoring costs are $c = 0.6$ in both periods, $\varphi = 1$ and at $t = 0$ the agents choose the contract that is best for the entrepreneur (the capital market is competitive).

$a_1 = 2.5$ his expected compensation increases to 1.3. Moreover, if the entrepreneur's expected compensation is small after the first period, the monitor needs to retain control of the project in the second period in order to enforce efficient actions. In this situation, his compensation becomes dependent on the project's performance in order to prevent him from colluding with the entrepreneur. However, if the entrepreneur's expected compensation is high after the first period, there is no need to control him in the second period, since one can make his compensation depend on the project's performance. After the second period the project's cumulative payoff is used to compensate the agents.

5 Contracts in Venture Capital Financing

The section above showed that the optimal contract mimics some stylized properties of the contracts observed in venture capital financing, such as claim priorities and contingent allocation of control rights. The goal of this section is to show how the optimal contract can be explicitly replicated with the securities used in VC financing. First of all, instead of holding direct claims on the payoffs of start-ups, investors and monitors form partnerships that invest in securities of start-ups. The investors and monitors share the cash flow and control rights over these partnerships. I show how the compensation structure predicted by my model can be replicated by a profit-sharing rule of the VC partnership and the securities that the project sells to the partnership.

To match the distribution of payoffs in a typical start-up, the following set of assumptions is used.

Assumption 3. *The support of the first-period payoff distribution is bounded from above by 0 and the support of the second-period payoff distribution is bounded from below by c_2. Moreover, the project is never terminated in the second period, i.e., $a_2^D(a_1^D)$ is not in the* *interior of the support of the second-period distribution.*

The first part of Assumption 3 states that the first-period payoff is actually an investment, which is typical for start-ups because they usually require multiple rounds of investment and pay investors only when they are sold. According to the assumption above, the second period is when the start-up does not require additional investment and is ready to be sold. Hence, it is reasonable to expect that those start-ups are usually sold rather than terminated. However, this assumption just makes the replication more straightforward and does not affect the main structure. In the replication, I use the following two securities common in VC financing.

Definition. a) *Preferred stock (straight)* gives the owner the highest priority on the assets of a start-up. Until its face value has been paid in full, other claimants receive nothing.

b) *Common stock* gives the owner the residual right on the assets of a start-up.

Note that these securities are defined as pure cash flow rights. The control rights are allocated by covenants in these securities, as discussed below. Using these securities, the optimal contract can be replicated in the following way.

Corollary 2. *The monitor (VC) initially provides E^M to the partnership and the investor provides $I - E^M - a_1^D$.*

-At $t = 0$ the partnership finances I and receives $a_2^M(0)$ of the preferred stock and $1 - \varphi$ of the common stock of the start-up.

-At $t = 1$ the entrepreneur sells additional $a_2^M(a_1) - a_2^M(0)$ of the preferred stock for $-a_1$ in investment; if the partnership cannot provide the capital, the project is terminated.

-At $t = 2$ the investor receives all the assets of the partnership below $a_2^D(a_1^D)$ and $(1 - \varphi)(a_2 - a_2^D(a_1^D) + a_1 - a_1^D)$ above that level; the monitor receives the rest.

Proof. See proof of Corollary 3 and the discussion below.

At $t = 0$ the investor and monitor form a partnership, invest $I - E^M - a_1^D$ and E^M, respectively, and agree on how to share the assets of the partnership at $t = 2$. After that, the partnership invests I in the start-up (project) and receives $1 - \varphi$ of its common stock and $a_2^M(0)$ of its preferred stock. If no investment is required at $t = 1$, i.e., if $a_1 = 0$, then no additional securities are issued. If $a_1 < 0$ and the project needs additional financing, the partnership provides $-a_1$ and receives an additional amount $a_2^M(a_1) - a_2^M(0)$ of the preferred stock.

At $t = 2$ the start-up's payoff is divided among the agents. The start-up pays the partnership $a_2^M(a_1)$ and distributes any remaining payoff among the common stock holders. At this point the partnership has $a_2^M(a_1) + (1 - \varphi)(a_2 - a_2^M(a_1))$ in revenue from the start-up and $a_1 - a_1^D$ in cash left from financing at $t = 1$. Out of these assets the investor receives

$a_2^D(a_1^D)$ before the monitor is paid. The revenue of the partnership in excess of $a_2^D(a_1^D)$ is shared between the investor and the monitor in the following way: the investor receives $(1-\varphi)(a_2 - a_2^D(a_1^D) + a_1 - a_1^D)$ and the monitor receives the rest. One can verify that the monitor's share of the revenue in excess of $a_2^D(a_1^D)$ is between 0 and φ, depending on the monitoring thresholds and payoff realizations.

The compensation structure described here is close to the compensation structure used in venture capital financing. Initially, investors provide most of the endowment of the partnership they form with VCs. Financing of start-ups is usually done in rounds. In each round start-ups sell various securities, the most common of which is preferred stock. In terms of cash flow rights, this security resembles a combination of the straight preferred stock and common stock described above. It allocates most of the value of the start-up to the VC partnership when the value is low and shares the value of the start-up with the entrepreneur when the value is high. Once the portfolio start-ups are sold, partnerships distribute the revenue among their investors and VCs. The investors are paid first until the amount of their investment is repaid. After the initial investment is repaid, the investors share the profit with the VCs.

It should be mentioned, however, that the model does not predict that VCs get a share of extremely high payoffs, as is usually the case in practice. To see this, note that when payoffs are high and there is no monitoring, the entrepreneur gets φ of every extra dollar generated and the investor gets the remaining $1-\varphi$ dollars. This happens in the model because there is no role for the monitor when payoffs are high. In practice, VCs have active roles in developing and selling projects even when performance is exceptionally good. However, to keep the model simple I do not introduce this additional element.

To complete this replication I discuss the allocation of control rights. First of all, within the venture capital partnership, both in practice and in this replication, monitors have full control, i.e., whenever the partnership has control over the start-up, the monitor is in control. Second, in this replication the partnership has control in the first period if a certain performance threshold is not met, i.e., if $a_1 < a_1^M$, and it has control in the second period when its preferred stock is not paid in full. Hence, this gives a good replication of the allocation of control rights in venture capital financing.

The comparative statics of this replication are summarized in Table 1. An increase in the second-period value of the project (its IPO value) decreases the amount of preferred stock that the start-up needs to issue because preferred stock becomes safer. This also implies that in the first period, the entrepreneur has enough equity value to operate the project efficiently without being monitored. If the cost of monitoring in the first period decreases, the VC exerts more control in the first period. This implies that the entrepreneur will be

Table 1: Comparative Statics of VC Contracts

Shock\Effect	Public Securities Sold at $t = 0$	Control by Monitor in First Period
Expected IPO value (second period payoff) ↑	↓	↓
Monitoring costs (first period) ↓	↑	↑
First period variance ↓	↑	↑
Efficiency of private benefits φ ↓	Common stock financing ↑ Preferred stock financing ↓	-

This table shows the comparative statics of the VC contracts with respect to the second-period average payoff h (the second-period payoff is uniform on $[3+h, 7+h]$), the first-period monitoring costs c_1, first-period variance (the first-period payoff is uniform on $[-2-\delta, -0.5+\delta]$), and the efficiency of private benefits φ (degree of moral hazard). These results refer to the case when monitoring happens with positive probability in both periods and when the allocation effect dominates the effect of the change in monitoring costs.

able to sell more preferred stock at the initial stage and be more passive in the first period. Higher first-period variance leads to an increase in the need to monitor the entrepreneur in the first period. Again, with more control in the first period, the entrepreneur sells more preferred stock at the initial stage and becomes more passive in the first period. Finally, if the utility that the entrepreneur gets from private benefits decreases relative to the generated output, i.e., if the degree of moral hazard decreases, the project gets more financing from selling common stock and less financing from selling preferred stock.

The efficiency of the private benefit parameter helps to differentiate the case of VC financing from the case of bank lending. Arguably, start-ups financed by VCs have much higher potential than small businesses financed by banks, but the benefits of diverting resources to some alternative use are comparable. This would give a much lower φ for VC financing than for bank lending. As shown in Table 1, this explains why start-ups sell common stock to VCs and small businesses do not.

Otherwise, the contracts in banking have a lot in common with the compensation structure in the model. Consider the case of $\varphi = 1$, i.e., the case of no inefficiency of private benefits. Repeating the same replication for this special case, renaming the agents as depositors, bankers, and entrepreneurs, respectively, and renaming preferred stock as a loan contract, leads to the following structure. At $t = 0$ depositors and bankers organize a bank and contribute money to it. This bank finances the entrepreneur using a loan contract.

At $t = 1$ the entrepreneur can increase the loan if he needs additional investment and the interest rate is adjusted accordingly. At $t = 2$ the loan is repaid or the bank seizes the business and sells it. After that, depositors receive their investment with a premium and the remaining proceeds are paid to the bankers.

6 Debt Contracts and the Role of Monitors' Capital

Debt contracts are typical for financing of large corporations, which are not the primary motivation for the relationships described in this paper. Reasons for this include tax shielding or using debt as a way to discipline managers. However, the problem of minimizing the costs of transferring control from owners to investors, e.g., bankruptcy costs, is meaningful for large corporations as well. Hence, in the context of large corporations one can interpret the optimal contracts as the contracts that minimize the expected bankruptcy costs. In this section I show how the optimal contracts can be replicated by combining standard debt contracts with different maturities and seniorities.

6.1 Debt Contracts

For the purpose of this replication, debt contracts are defined in the following way.

Definition. a) Senior debt due at period t gives the owner the highest priority over the payoff at period t. If at t the face value is not paid, the project is terminated.

b) Junior debt due at period t gives the owner priority over equity on the payoff at t, i.e., it is paid after the senior debt at that period. If at t the face value is not paid, the project is monitored.

c) Equity gives the residual cash flow rights.

An important feature of the definition of junior debt due at period 1 is that it gives cash flow rights only on the payoff of period 1. Failure to pay the face value of this debt results only in the allocation of the control right to the monitor but does not imply that the rest must be paid in the next period. To make the replication as straightforward as possible, the set of specifications is restricted by the following conditions.

Assumption 4. *The support of the second-period payoff distribution is bounded from below by c_2. If $C^M(a_1^D) = 0$, then $a_1^D = a_1^H$.*

The first assumption ensures that termination in the second period happens only due to limited liability of the agents, and not because it is efficient to close the project when the payoff is low. The second assumption restricts attention to the case depicted in Figure 4.1a, i.e., when there is no interval with $C^M(a_1) = 0$. As was mentioned before, the latter condition is the most relevant case when monitoring costs are equal across periods.

With the assumptions stated above, the optimal contract can be replicated in the following way. For simplicity, debt maturing at $t = 1$ is called short-term debt and debt maturing at $t = 2$ is called long-term debt.

Corollary 3. *For $\varphi = 1$:*

-the investor owns a_1^D in short-term debt and $a_2^D(a_1^D)$ in long-term senior debt

-the monitor owns $a_1^M - a_1^D$ in short-term debt and $a_2^M(a_1^D) - a_2^D(a_1^D)$ in long-term junior debt

-the entrepreneur owns equity

-at $t = 1$ the monitor buys long-term senior debt from the investor and the entrepreneur buys long-term junior debt from the monitor.

For $\varphi < 1$, in addition to senior debt the investor owns a share $1 - \varphi$ of junior debt and equity.

Proof. See Appendix.

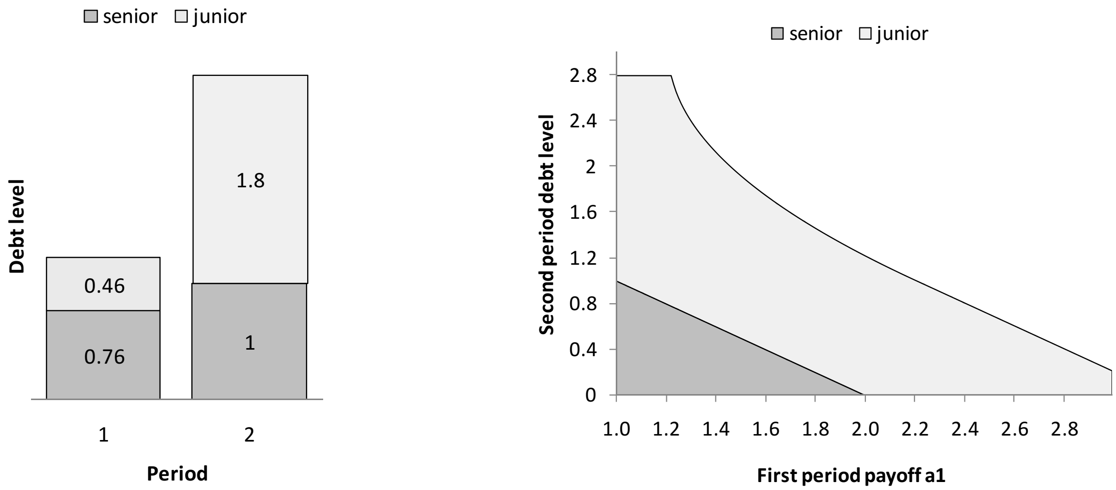
In other words, for $\varphi = 1$, the investor initially holds short- and long-term senior debt while the monitor holds all junior debt and the entrepreneur keeps all the equity. At $t = 1$ the payoff is distributed according to the securities the agents own. However, they do not consume it right away. The monitor uses his payoff to buy long-term senior debt from the investor. As a result, for every dollar that the monitor pays, the value of the investor's long-term senior debt decreases by one dollar and the value of the monitor's long-term junior debt increases by one dollar. Similarly, the entrepreneur uses his first-period payoff to buy junior long-term debt from the monitor. Note that such a trade of debt securities does not require commitment because everyone is weakly better off from it ex post.

The initial allocation of debt determines how the surplus of the project is distributed among the investors and what the cash flow and control rights are in the first period. The long-term securities guarantee that the investor and monitor receive their shares of the payoff. However, long-term senior and junior debt is an expensive source of financing because it triggers costly monitoring or termination. In the first period, whenever the agents have cash, they substitute the more expensive source of financing with a less expensive one. Senior debt is the most expensive and the monitor substitutes it with less expensive junior debt. Equity is the least expensive and whenever the entrepreneur has cash he substitutes debt with equity.

The case of $\varphi < 1$ is a simple modification of the case $\varphi = 1$. Since for $\varphi < 1$ the monitor and the entrepreneur receive only φ dollars for each dollar of payoff, their compensations

Figure 6.1: Numerical Replication of the Optimal Contract with Debt

a) Initial maturity structure b) Second-period debt and first-period payoff



These figures show the replication of the optimal contract by combining debt contracts with different maturities and seniorities. Figure 6.1a shows the debt maturity structure at the time of initial financing. Figure 6.1b shows how after the realization of the first-period payoff the entrepreneur and monitor repay some of the second-period debt. If $a_1 \in [1, 1.22]$ *then only the investor and monitor receive cash in the first period. The monitor uses it to purchase second-period debt from the investor. If* $a_1 > 1.22$ *then the entrepreneur uses his cash to purchase second-period junior debt from the monitor and the monitor uses his cash to purchase second-period senior debt from the investor.*

can be replicated with just a share φ of the junior debt and equity, respectively, i.e., the monitor holds a share φ of junior debt and entrepreneur holds a share φ of equity. The amounts of debt issued are still a_1^D, $a_1^M - a_1^D$, $a_2^D(a_1^D)$ and $a_2^M(a_1^D) - a_2^D(a_1^D)$, respectively. The rule of substituting more senior claims with more junior claims still holds. The cash that is allocated in the first period to junior claim holders (the monitor and investor) is used to buy long-term senior debt and the cash allocated to equity holders (the entrepreneur and investor) is used to buy long-term junior debt.

To illustrate the described capital structure, I compute the optimal contract for the following set of parameters: $I = 2.7$, $E^M = 1.2$, $c_t = 0.6$ and a_t uniformly distributed on $[1, 3]$ in each period. For these parameters the project is never terminated and its initial senior debt levels are $a_1^D = 0.76$ and $a_2^D(a_1^D) = 1$. The initial monitoring thresholds are $a_1^M = 1.22$ and $a_2^M(a_1^D) = 2.8$, which makes the face value of the short-term junior debt equal to 0.46 and that of the long-term junior debt equal to 1.8.

This capital structure is shown graphically in Figure 6.1. In the region of payoffs $a_1 \in [1, 1.22]$ the monitor pays the investor and exchanges the senior debt for junior debt. For payoffs $a_1 > 1.22$ the entrepreneur buys junior debt from the monitor and the monitor uses this cash to buy senior debt from the investor. Note that the initial level of second-period debt is higher than the level of first-period debt. This prediction is robust across a wide range of parameters because the actual amount of debt that the project has after the first period is much lower. Figure 6.1b shows that for the medium realization of the first-period payoff ($a_1 = 2$) the amount of debt in the second period becomes about 1.2, i.e., close to the first-period debt level.

This numerical example is also used to study the effect of the underlying parameters on the optimal debt maturity. The main variable of interest here is the ratio of the short-term debt to the long-term debt, $a_1^M / a_2^M(a_1^D)$. Figures A.1-A.2 in the Appendix show the comparative statics of this ratio with respect to the following three variables: the first-period variance parameter δ (the first-period distribution is uniform on $[2-\delta, 2+\delta]$), the mean of the second-period payoff (the second-period payoff is uniform on $[1+h, 3+h]$) and the first-period monitoring costs. Table 2 gives a summary of the comparative statics results. In these examples, the expected utilities of the investor and monitor from participation in the project are kept constant, while the expected utility of the entrepreneur is allowed to vary. Note also that only parameters related to a single period are affected at a time. The effect of the uniform change in parameters across periods is discussed in the infinite-horizon version of the model.

The effect of an increase in the mean of the second-period payoff is straightforward. The higher the expected second-period payoff, the more long-term debt the project issues. However, the overall effect on the maturity ratio is ambiguous. On the one hand, the project issues less short-term debt because long-term debt becomes less risky. On the other hand, due to lower risk the face value of long-term debt decreases as well.

Similarly, changes in the first-period monitoring costs have two competing effects. The dominating effect is that a decrease in the cost of the short-term debt leads to an increase in the amount of short-term debt and a decrease in the amount of long-term debt, therefore increasing the share of short-term debt. However, this effect can be offset by the effect on the overall level of debt. A decrease in monitoring costs leads to a decrease in the total amount of debt issued, which may result in a decrease in the maturity ratio.

The most interesting prediction of the model is that an increase in the variance of the first-period distribution mainly leads to an increase in the relative amount of short-term debt. This prediction may seem counter-intuitive because a higher first-period variance makes short-term debt more costly in the sense that for the same expected payment higher monitoring costs are required. However, the relevant comparison is between providing incentives through monitoring versus providing them through future payoffs. Interestingly, compensating the entrepreneur with future payoffs becomes even more costly! To understand this

Table 2: Comparative Statics of Debt Contracts

Shock\Effect	Short-Term Debt	Long-Term Debt
Second-period average payoff ↑	↓	↑
First-period monitoring costs ↓	↑	↓
First-period payoff variance ↓	↑	↓
Share of financing by monitor E^M/I ↑	Junior debt ↑	-
	Senior debt ↓	

This table summarizes the comparative statics results of the debt contracts used to the replicate the optimal contract with respect to the second-period average payoff (the second-period payoff is uniform on $[1+h, 3+h]$), the first-period monitoring cost c_1, the first-period variance (the first-period distribution is uniform on $[2-\delta, 2+\delta]$) and the share of financing by the monitor E^M/I. The results in the table refer to the case when the project issues risky debt in both periods and when the substitution effect (allocation of debt between periods) dominates the level effect (more debt is needed to compensate for higher monitoring costs).

effect, consider a hypothetical distribution for first-period payoffs with two outcomes $\{a_l, a_h\}$ that are equally probable. If outcome a_l is monitored, the entrepreneur's compensation can be constant because the entrepreneur cannot misreport a_h. If a_l is not monitored, the entrepreneur's compensation at a_h needs to be $C^E(a_h) = C^E(a_l) + \varphi(a_h - a_l)$, which makes the expected continuation value lower due to the concavity of the value function. The higher the distance between a_h and a_l, the higher the variance of C^E and the lower is the expected continuation surplus due to its concavity. However, the cost of monitoring a_l does not depend on the distance between a_h and a_l. Hence, the higher the distance between a_h and a_l (higher variance), the more attractive monitoring of a_l becomes.

A strong positive effect of the payoff variance on the relative amount of short-term debt is reported in Barclay and Smith [1995]. This testable prediction potentially separates this model from other models of the trade-off between short-term and long-term debt, such as the signaling model in Diamond [1991] or moral hazard model in Myers [1977].

6.2 The Role of Monitors' Capital

So far the presence of three agents in the model was taken as given and was motivated by the fact that we observe these types of agents in VC financing. However, having solved for the optimal contracts between the agents one can also characterize the situations in which the investor and entrepreneur benefit from the presence of the monitor. The first step in this characterization is given in the following lemma.

Lemma 4. *The project's value is non-decreasing in E^M. If $E^M = 0$ then the presence of the monitor does not affect the project's surplus. When $E^M \to \infty$, the presence of the investor* *does not affect the surplus.*

Proof. See Appendix.

The fact that the project's value is non-decreasing in E^M is a trivial consequence of the fact that monitoring is an option. Hence, the contracts that are feasible when the monitor has little capital are still feasible when the monitor invests a large amount. Similarly, when E^M goes to infinity there is no need to seek financing from the investor, as the same contracts are feasible under financing from the monitor. Finally, when the monitor does not contribute to the project his expected compensation needs to be zero as well. However, due to the limited liability of the monitor this implies that his compensation needs to be zero in all states and that $a_t^D = a_t^M$ because he never has incentives to enforce efficient actions.

Although the lemma above states that monitors who do not contribute to the project should not participate, the monitor's contribution can be interpreted broadly. For example, venture capitalists do not always contribute their own capital to the partnerships they run, but may instead contribute their expertise in selecting good projects and developing them, which has a significant market value. In bank lending, bankers can justify their compensation by lowering transaction costs between depositors and small businesses.

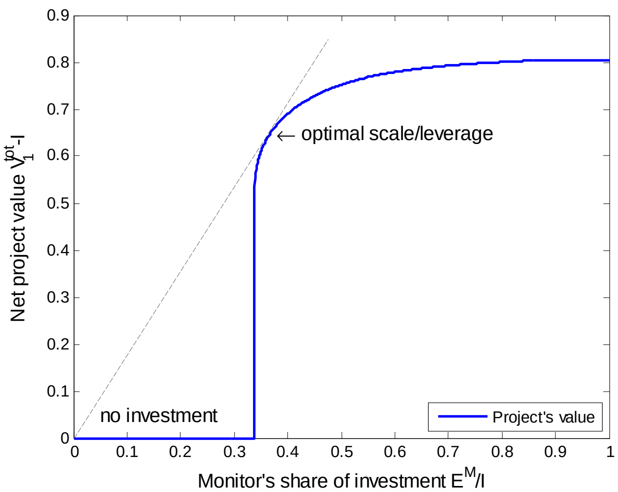
To illustrate the role of the monitor's capital, I calculate the surplus of the project for different shares of financing made by the monitor E^M/I and the following set of parameters: $I = 1.5$, $c_t = 0.6$, and $a_t \sim N(1.5, 1)$. The investor and monitor are assumed to receive competitive compensations, which maximizes the project's total surplus. The result is shown in Figure 6.2a. Overall, the qualitative predictions of the role of monitors' capital on the project's surplus are similar to the predictions of the model of financial intermediation in Holmström and Tirole [1997].

Note that unless the monitor is able to provide 34% of the original investment, the project is not initiated. At that level, the project's total surplus jumps to a relatively high level. This happens because the Pareto frontier does not include low levels of U^E (an illustration of that is given in Figure 7.2). To guarantee a low expected compensation for the entrepreneur, the project needs to be either terminated or monitored in too many states. In this case it is better for the investor and monitor to give the entrepreneur higher compensation and monitor or terminate less. After the jump, the effect of marginal investment by the monitor is decreasing.

Given the scarcity of the monitor's capital, it may be important to find the optimal scale of the project for a given E^M, where scaling the project changes the initial investment cost, monitoring costs and other parameters of the project proportionally. Since the surplus is also proportional to the scale of the project, Figure 6.2a in this case shows the normalized surplus V_1^{tot}/I. The trade-off between starting a large project and a small project is the following. If

Figure 6.2: Monitors' Capital Net Project's Value and Optimal Scale

a) Project's value and optimal scale



b) Comparative statics of the optimal scale

Effect/ Shock	Scale I, Leverage I/E^M
$c\downarrow$	$\uparrow$
$\mu\uparrow$	$\uparrow$
$\sigma\downarrow$	$\uparrow$

This figure displays the numerical simulation of the net expected value of the project as a function of the share of investment contributed by the monitor (VC). When the contribution is low the project is not started; when the contribution exceeds a certain amount the project is started and generates strictly positive surplus. It also shows the optimal scale of the project, I, when one holds E^M fixed. The table on the right shows that the optimal scale for a given E^M increases with the profitability of the project, μ, and decreases with the monitoring costs c and payoff variance σ.

the project is large, the share of the monitor's financing is small and the normalized surplus is small. Figure 6.2a shows that very large projects are never implemented. If the project is small, the share of the monitor's financing is high and the normalized surplus is also relatively high. However, because of the small scale in absolute terms, the surplus may be small.

The problem of finding the optimal scale of the project is equivalent to the problem of determining the optimal leverage for a financial intermediation firm. Note that the book value of the partnership between the investor and monitor is I and the book value of the equity held by the monitor in the partnership is E^M. Hence, the leverage of this partnership is I/E^M. To find the optimal scale or optimal leverage one needs to find the scale that maximizes the surplus created per dollar of the monitor's capital. Figure 6.2 shows the optimal leverage point graphically. Higher than optimal leverage leads to a high probability of termination and lower than optimal leverage leads to operating the project on too small a scale.

The table shown in Figure 6.2b gives a summary of the comparative statics of the optimal scale of the project with respect to the main parameters of the model. The optimal scale increases when the monitoring costs decrease because the project becomes more profitable in expectation. A similar effect happens when its average payoff increases: the agents want to implement a larger project even if it implies less monitoring and more termination. Finally, when the variance of the payoffs is lower, there is less need to monitor the project and the monitor contributes less capital.

7 Infinite-Horizon Model

Although the three-period model described above shows most of the qualitative results, the infinite-horizon model makes comparative statics analysis more meaningful and allows one to compare the results to those of a large variety of infinite-horizon models used in contract theory and corporate finance. In particular, this section describes a model in which the underlying payoff process is a Brownian motion and the stopping time is a Poisson process.

Consider the framework of the three-period model with an infinite sequence of periods $t = 0, 1..\infty$ and some market interest rate r used by all agents to discount future payoffs. After the initial investment I is made, in each period the entrepreneur and the monitor decide whether to continue the project, in which case it generates an iid payoff $a_t \sim N(\mu, \sigma)$, or to try to sell it to outside investors. If they try to sell the project, they succeed with probability $P_{IPO} > 0$ and receive some verifiable IPO value V_{IPO}. If they do not succeed, the project generates the iid payoff $a_t \sim N(\mu, \sigma)$ described above. The IPO value can be random, but since it is verifiable without loss of generality its value is treated as a constant. For simplicity, I restrict attention only to the case when $V_{IPO} \geq \mu/r$ and it is always optimal to sell the project.

Note that this model is quite flexible and includes several important cases. First, for $P_{IPO} \to 0$ and $\mu > 0$ this project resembles a stationary small business. Second, for $P_{IPO} \gg 0$ and $\mu < 0$ this project resembles a start-up because it requires a sequence of investments until it can be sold to outside investors.

As before, the contracts specify the termination rule $D_t(y^t)$, the monitoring rule $M_t(y^t)$ and compensations at the terminal period $C_t^E(y^t)$, $C_t^M(y^t)$ and $C_t^I(y^t)$. Note that the subscript t in the compensation functions denotes the period at which the compensations are paid, i.e., the period when the project is terminated or sold. As before, compensating the agents only in the terminal period is optimal because they are risk neutral, discount future consumption with the market interest rate and the assets of the project earn the same market interest rate. Assuming compensations occur only in the terminal period is appropriate for VC financing because most projects do not generate revenue until they are sold. Allowing for risk aversion or higher discounting by the entrepreneur or monitor to generate consumption in the intermediate periods would not change the main qualitative results.

Selling the project is always efficient. However, it requires consent or participation in the process by both the entrepreneur and the monitor. This participation does not cost anything, but is non-verifiable and requires the entrepreneur and monitor to prefer the IPO to be sooner rather than later. Hence, in the optimal contract their compensations must satisfy:

$$C_t^E \geq E_t[C_{\tau>t}^E] \tag{7.1}$$

and

$$C_t^M \geq E_t[C_{\tau>t}^M] \tag{7.2}$$

where the left-hand side terms are the compensations when the project is sold and the right-hand side terms are the expected compensations if the project continues.

Besides the changes in the number of periods and the constraints shown above, the optimal contract solves the same problem as in the three-period model. To avoid introducing superfluous notation, the surplus of the project for the infinite-horizon model is not shown here. The problem will be presented shortly as a dynamic program.

As in the three-period model, it is optimal to implement zero private benefits and no destruction of output. The proof of this result applies here without any changes. Hence, incentive compatibility constraints (4.5)-(4.4) and monotonicity (4.6) apply for the infinite-horizon model as well. Moreover, the constraints above are always binding in the optimal contract.

Lemma 5. *Constraints 7.1-7.2 are always binding.*

Proof. See Appendix.

The intuition behind the proof of this result is that if the project continues, the entrepreneur's and monitor's compensations increase the value of the project. On the other hand, if the project is sold, there is no moral hazard and the surplus does not depend on their compensations. Hence, one would want to compensate the entrepreneur and monitor as much as possible if the project continues, but would need to give them enough compensation if the project is sold to satisfy constraints (4.5)-(4.4). A convenient implication of the above result is that the expected compensations of the entrepreneur and monitor are equal to what they would receive if the project were sold in the current period.

I will first formulate the dynamic program and then argue that the solution to this problem gives the solution to the original problem. The objective function of the dynamic program is the expected surplus of the project starting from some period t, i.e., the sum of the project's payoffs starting from period t net of monitoring costs. The state variables for this problem are the expected compensations of the entrepreneur and monitor. These variables summarize all the important information concerning the past performance of the project. Due to Lemma 5, the expected compensations are equal to the compensations if the project is sold in the current period, C_{t-1}^E and C_{t-1}^M. Hence, to keep the notation simple, C_{t-1}^E and C_{t-1}^M will be used to denote the state variables. The solution to the dynamic program is composed of the termination and monitoring rules and the expected compensations, all as functions of the current payoff a_t and the state variables $D_t(a_t, C_{t-1}^E, C_{t-1}^M)$, $M_t(a_t, C_{t-1}^E, C_{t-1}^M)$, $C_t^E(a_t, C_{t-1}^E, C_{t-1}^M)$ and $C_t^M(a_t, C_{t-1}^E, C_{t-1}^M)$. This program is stated formally as:

$$V^{tot}(C_{t-1}^E, C_{t-1}^M) = \max_{\gamma \in \Gamma} E[(1 - D_t(a_t))(a_t - cM(a_t) + \frac{1}{1+r} P_{IPO} V_{IPO} + \frac{1}{1+r}(1 - P_{IPO}) V^{tot}(C_t^E(a_t), C_t^M(a_t)))], \quad (7.3)$$

$$s.t.$$

$$E[C_t^E(a_t)] = C_{t-1}^E, \quad (7.4)$$

$$E[C_t^M(a_t)] = C_{t-1}^M. \quad (7.5)$$

Note that this problem is almost identical to the first-period problem of the three-period model. The only difference is that in the three-period model the value functions on the right- and left-hand sides refer to different time periods and in the infinite-horizon model these functions are identical.

The value function that solves the dynamic program is a fixed point of the operator above. One can easily verify that this operator satisfies Blackwell's monotonicity and discounting conditions and thus is a contraction. Since it is a contraction mapping in a complete metric space, a solution exists and is unique. Also, using a standard argument one can show that the solution to the dynamic problem above is the solution to the original problem. Intuitively, the optimal contract needs to be optimal after any history. Hence, the state variables are sufficient statistics for the solution. However, to prove this claim formally, one can assume that the solution at some period t is different from the solution to the dynamic program and show that this implies that the surplus can be increased.

Since the dynamic problem above is almost identical to the first-period problem, the solution technique and its intuition are very similar. In the first-period problem an important condition was that the second-period value of the project was concave and satisfied the condition (4.10). For the infinite-horizon model a similar condition is conjectured and verified later.

Conjecture 1. *The continuation value function* $V^{tot}(C^E, C^M)$ *is concave, twice differentiable and satisfies:*

$$\frac{\partial^2 V^{tot}}{\partial^2 C^M} \leq \frac{\partial^2 V^{tot}}{\partial C^E \partial C^M} \leq 0. \tag{7.6}$$

This conjecture is different from the properties stated in Lemma 3 due to the inequality sign instead of equality between the second derivatives. This inequality accounts for the interaction between monitoring and termination costs in the multi-period model. The conditions for this conjecture to hold are different from the non-decreasing hazard rate of the payoff distribution used in the three-period model. In the infinite-horizon model, the sufficient statistical properties of the payoff distribution are hard to characterize because the complete solution for the optimal contract can only be found numerically. Thus the conjectured properties are also verified numerically. In all numerical simulations for normally distributed payoffs these properties are satisfied.

The structure of the optimal contract is also similar to that of the three-period model.

Theorem 3. *The monitoring and termination rules are threshold rules. The optimal compensations are continuous and have the following derivatives (denoted* C_1*).*

- $a_t^D < a_t \leq a_t^M$, $C_1^M(a_t) = \varphi(1-\xi)$ *and* $C_1^E(a_t) = \varphi\xi$
- $a_t^M < a_t$, $C_1^M(a_t) = 0$ *and* $C_1^E(a_t) = \varphi$

and on $a_t < a_t^D$ *the compensations are zero, where*

$$\xi = \frac{V_{22}^{tot} - V_{12}^{tot}}{V_{11}^{tot} - 2V_{12}^{tot} + V_{22}^{tot}} \in [0, 1]$$

when $(C^E(a_t), C^M(a_t))$ *is in the interior of* $C^E \geq 0$, $C^M \geq 0$ *and* $C^E \leq C^E(a_t^M)$*;* $\xi = 0$ *on* $C^E = 0$ *or* $C^E = C^E(a_t^M)$*; and* $\xi = 1$ *on* $C^M = 0$.

Proof. See Appendix.

Note that Theorem 3 coincides with Theorem 2 from the three-period model when $V_{22}^{tot} = V_{12}^{tot}$. When $V_{22}^{tot} < V_{12}^{tot}$, it is no longer efficient to minimize the variance of the entrepreneur's compensation in the monitoring region and the compensation takes a more general form. However, for reasonable parameter values V_{12}^{tot} is very close to V_{22}^{tot} and the compensation structure of the three-period model gives a very good approximation of the optimal contract.

The optimal compensation structure coincides with the structure of the three-period model when the monitor is able to finance the project himself, i.e., when $E^M \to \infty$.

Corollary 4. *When* $E^M \to \infty$ *the optimal entrepreneur's compensation is* $C_1^E = 0$ *on* $a_t \in [a_t^D, a_t^M]$ *and* $C_1^E = \varphi$ *on* $a_t \geq a_t^M$.

Proof. See Appendix.

This result is a simple corollary of Theorem 3 given that for $E^M \to \infty$ the surplus of the project does not depend on C^M, implying that $V_2^{tot} = 0$ and $V_{22}^{tot} = V_{12}^{tot} = 0$. Since the structure of the optimal contract is the same, the replication of the optimal contract with debt contracts of different maturities applies as well for the infinite-horizon model with financing by the monitor. To complete the characterization of the optimal contract, it is left to show how the thresholds $a_t^D(C_{t-1}^E, C_{t-1}^M)$ and $a_t^M(C_{t-1}^E, C_{t-1}^M)$ depend on the state variables. This is the purpose of the numerical simulations in the next subsection.

7.1 Numerical Simulation of the Optimal Contract

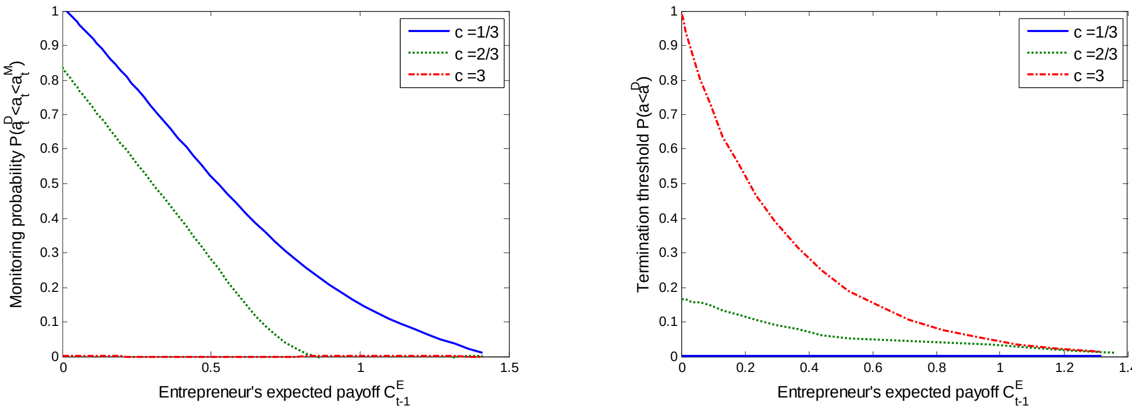
I start by presenting numerical results for the case $E^M \to \infty$, i.e., when both monitoring and financing can be done by one agent. This simplifies both computing and presenting the optimal contracts because the unknown thresholds become functions of only one state variable, $a_t^D(C_{t-1}^E)$ and $a_t^M(C_{t-1}^E)$. A simulation of the general case is presented afterwards.

Throughout this section, the basic numerical example uses the following set of parameter values: $\mu = -2$, $\sigma = 1$, $c = 1/3$, $r = 0.05$, $\varphi = 0.5$, $P_{IPO} = 0.068$ and $V_{IPO} = 40$. For these parameter values the expected time of an IPO arrival is 14.3 periods and the first-best value of the project is 6. The numerical solution of the dynamic program above is relatively straightforward. I start with a guess for V^{tot} and compute the objective function (equation 7.3) for a finite grid of values of a_t, a_t^M, a_t^D and C_{t-1}^E. I then use the first dimension of the grid to numerically compute the expectation of the objective function for each a_t^M and a_t^D and find the thresholds that maximize the expected value. This optimal value is then used as a new iteration for V^{tot}. The procedure is repeated until the objective function reaches a fixed point.

The resulting optimal thresholds and their comparative statics with respect to monitoring cost c are shown in Figure 7.1. Note that instead of absolute levels of thresholds they are shown as probability measures of the respective regions; for example, $P(a_t^D \leq a_t < a_t^M) = 0.5$ means that the project is monitored with probability 0.5. This probability can be interpreted as strictness of the covenants in the contract between the entrepreneur and the monitor. The stricter the covenants, the more likely they will be violated and the control will be shifted to the monitor. Alternatively, this probability can be interpreted as a measure of short-term debt, because in the replication of the optimal contract with debt, a_t^M is the amount of short-term debt for the project. Note also that since the entrepreneur's claim is effectively an equity claim, the entrepreneur's expected compensation can be interpreted as the value of the firm's equity.

Figure 7.1: Optimal Contract and Monitoring Cost c

a. Monitoring probability $P(a_t^D \leq a_t < a_t^M)$ b. Termination probability $P(a_t < a_t^D)$



These figures show the probabilities of the payoffs that lead to monitoring and termination, respectively, as functions of the entrepreneur's expected compensation (value of his equity). The parameter values used are: $\mu = -2$, $\sigma = 1$, $r = 0.05$, $\varphi = 0.5$, $P_{IPO} = 0.068$ *and* $V_{IPO} = 40$.

Using this interpretation, one can state, based on Figure 7.1a, that covenants become stricter as bad performance decreases the value of equity. The use of covenants depends on the monitoring costs. When the costs are low the monitor uses covenants to renegotiate future payments and avoid default. When the costs are high, the monitor does not use covenants and terminates the project whenever the value of equity becomes zero. Note that the contract without covenants resembles public debt because this debt usually has few covenants and is rarely renegotiated. Figure 7.1b shows the optimal termination probability. When monitoring costs are low, the project is never terminated because it is more efficient to monitor the project in every state, effectively giving the monitor full control over the project, than to terminate it. On the other hand, when the costs are high, it is more efficient to shut down the project than to monitor it.

Figure 7.2a shows the model's Pareto frontier. When C^E is high, an increase in the entrepreneur's compensation leads to a decrease in the monitor's compensation because they share the project's surplus. However, when the value of the entrepreneur's equity is low, the amount of monitoring that is required is so high that the monitor would rather decrease the amount of debt than monitor the project. Thus, initially the compensations of the entrepreneur and monitor are never within that region. However, low payoffs can decrease the value of equity and drive the compensations to the Pareto inefficient region. Being able to commit to such contracts ex ante benefits both the entrepreneur and monitor.

The comparative statics of the strictness of covenants with respect to the variance of payoffs, σ, are shown in Figure 7.2b. This figure shows that the covenants are stricter when

Figure 7.2: Optimal Contract and Monitoring Cost c

a. Monitor's value $C^M_{t-1} = V^{tot} - C^E_{t-1}$ b. Monitoring probability and payoff variance

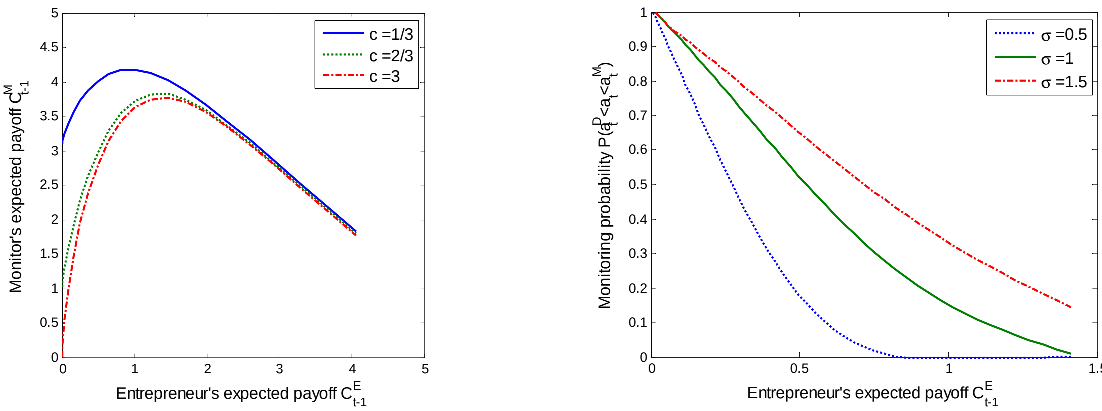
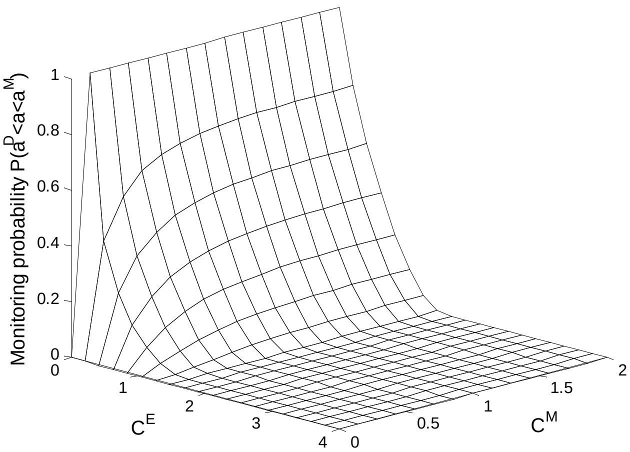


Figure 7.2a shows the monitor's (VC's) expected compensation as a function of the entrepreneur's expected compensation for the infinite-horizon model. Expected compensations always lie within these bounds. Figure 7.2b shows that a higher payoff variance leads to more monitoring for a given entrepreneurial compensation.

the variance of the payoffs is high. Alternatively, one can say that the project has more short-term debt when the variance of payoffs is high. This is a similar result to the result shown for the three-period model when only the variance of the first-period payoff was affected. To understand this prediction, consider the case when there is no short-term debt and variations in payoffs affect only the value of equity. If the variance is very low, it is unlikely that the value of equity will become zero and termination or monitoring will ever be needed. If the variance is high, it is very likely that the value of equity will become zero and the monitor will have to run the company and therefore bear high monitoring costs from that point on. Hence, monitoring more today to avoid running the company in the future is efficient only when the variance is high. Comparative statics with respect to other parameters, e.g., μ and P_{IPO}, are given in the Appendix in Figures A.3-A.5. These parameters affect the firm's surplus, but not the use of covenants.

It is also interesting to know how monitoring in the optimal contract is related to monitoring in a number of reasonable benchmark contracts. First, one may imagine a contract that minimizes the current monitoring costs, i.e., the contract that postpones monitoring as much as possible. This type of contract may arise when the VC avoids monitoring start-ups and does it only when he has exhausted other instruments. Second, one can consider a contract that does not use past performance to allocate cash flow and control rights. This contract replicates the optimal static contract into an infinite sequence. The simulated monitoring probabilities for these contracts relative to the optimal contract are shown in Figure

Figure 7.3: Optimal Monitoring Probability $P(a_t^D \leq a_t < a_t^M)$



This figure shows the optimal monitoring probability when the monitor's capital is limited. The lower the monitor's expected compensation, the smaller is the monitoring region for the same entrepreneur's compensation. When the monitor's compensation is zero monitoring is suboptimal.

A.6 in the Appendix. The main implication of this figure is that the contract that minimizes current monitoring costs starts with too little current monitoring, while the static contract starts with too much monitoring.

The numerical solution for the general case, i.e., when $E^M < \infty$, is computed similarly to the iterative procedure described above. First, one guesses $V^{tot}(C^E, C^M)$ and solves for the optimal thresholds $a^D(C^E, C^M)$ and $a^M(C^E, C^M)$. Using these thresholds one updates the value function and proceeds to the next iteration. Because this mapping is a contraction, the procedure always converges to the solution of the dynamic program. However, due to the high dimensionality of the problem and a more complicated form of ξ in Theorem 3, several approximations are used to compute the optimal contract. These approximations are described in the Appendix.

The set of parameter values used in this numerical simulation is the same as the basic set of parameter values used above. Figure 7.3 shows the optimal monitoring threshold $a^M(C^E, C^M)$. The optimal termination threshold and the resulting value function $V^{tot}(C^E, C^M)$ are shown in the Appendix. The computed optimal monitoring threshold for the high value of C^M coincides with the case $E^M \to \infty$. However, as the monitor's compensation becomes smaller, the amount of monitoring decreases. When $C^M = 0$, the monitor stops monitoring completely. The reason for this is that the monitor's compensation needs to depend on the payoff when there is monitoring. Otherwise the monitor would not enforce efficient actions. Hence, the lower the monitor's expected compensation, the lower is his ability to effectively monitor the project.

The dynamics of the entrepreneur's and monitor's compensations are shown in Figure 7.3. The coordinates of that figure represent the state variables C^E and C^M. Hence, the state of the project can be represented as a point in that space. Given some initial point, the compensations are updated each period in accordance with the draw of a_t and the monitoring threshold. The lines in Figure 7.4 show the likely directions in which the compensations are updated. For example, when C^E is high, there is virtually no monitoring and only the entrepreneur's compensation depends on a_t. Hence, the state jumps up vertically if the payoff is above average and down if the payoff is below average. On the other hand, when the project is monitored in many states, i.e., when C^E is close to zero, only the monitor's compensation depends on the payoff. Hence, the state jumps horizontally to the right when the payoff is high and to the left when the payoff is low.

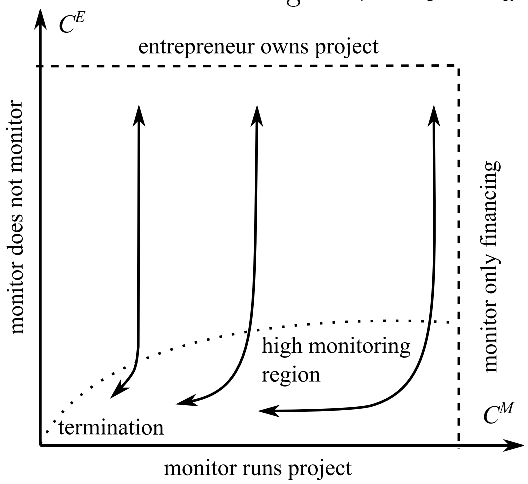
The boundaries of the state space are special cases of $E^M \to \infty$. The right boundary is exactly the case when only the monitor finances the project. The left boundary is the case when $E^M \to \infty$, but monitoring costs are very high. The top boundary corresponds to the case when $C^E \to \infty$ and the entrepreneur does not need outside financing or has high equity value and runs the project efficiently. Finally, the bottom boundary is when the monitor runs the project, but takes financing from the investor. Note also that due to limited liability the project must be closed when $C^E = 0$ and $C^M = 0$. Hence, the bottom left corner in the figure is the point where the project is terminated.

8 Frequent Reporting

In this section I analyze the optimal contracts when the time period between the reports shrinks to zero. In doing so, I assume that there is a continuous payoff flow process and this process is a Brownian motion with drift μ and variance σ. If the period of reporting is Δ, this implies that per-period payoffs are iid normal $N(\mu\Delta, \sigma\sqrt{\Delta})$. Similarly, the arrival of an IPO is given by a continuous-time Poisson process, which implies that the per-period IPO probability is $P_{IPO} = (1 - e^{-\lambda\Delta}) \approx \lambda\Delta$. Naturally, the cost of monitoring is assumed to be proportional to the time period, i.e., per-period costs of monitoring are $c\Delta$, and the per-period interest rate is $r\Delta$. For simplicity, this section also assumes that the per-period value of the project is higher than the cost of monitoring, i.e., that $\lambda V_{IPO} + \mu > c$.

To discuss the limit of the optimal contracts when Δ goes to zero it is convenient to normalize the termination and monitoring thresholds to $x^D = (a^D - \mu\Delta)/\sigma\sqrt{\Delta}$ and $x^M = (a^M - \mu\Delta)/\sigma\sqrt{\Delta}$, respectively. Also, throughout this section I use $f(x)$ and $F(x)$ to denote the standard normal distribution's pdf and cdf, respectively. Using this notation, the first

Figure 7.4: General Contract and Boundary Contracts



This figure shows the dynamics of the entrepreneur's and monitor's compensations implied by the optimal contract. When the entrepreneur's compensation is high and the probability of monitoring is low, only the entrepreneur's compensation depends on the payoff. When the entrepreneur's compensation is low and the probability of monitoring is high, only the monitor's compensation depends on the payoff. The boundaries in this state space represent the special cases that can be reduced to the case when the monitor finances the project.

result can be formalized as follows.

Lemma 6. *For $C^E + C^M > 0$:*

$$\lim_{\Delta \to 0}[1 - (1 - F(x^D))^{1/\Delta}] = 0,$$

i.e., the project is terminated with probability zero over any finite time interval when $\Delta \to 0$.

Proof. See Appendix.

Note that $1 - F(x^D)$ is the probability of not terminating the project in a given period. Hence, taking this probability to the power of $1/\Delta$ gives the probability of not terminating the project in a unit time interval. The main intuition for this result is that when payoffs become small, the payoff itself stops affecting the decision to terminate the project. Terminating the project with some probability becomes equivalent to randomizing the compensations between 0 and some value, which is suboptimal because the continuation value of the project is concave. Hence, it is optimal to continue the project as long as one can make reporting incentive compatible, i.e., as long as $C^E + C^M > 0$.

The main implication of Lemma 6 is the evolution of $C^E + C^M$. Recall that over the non-termination region, $C^E(a_t) + C^M(a_t)$ depends linearly on a_t. Given that its expectation grows with the interest rate, this leads to:

$$C_t^E(a_t) + C_t^M(a_t) = (C_{t-1}^E + C_{t-1}^M)(1 + r\Delta) + \varphi(a_t - \mu\Delta) + o(\Delta), \tag{8.1}$$

where $o(\Delta)$ is the error due to truncation at x^D and $\lim_{\Delta \to 0}[o(\Delta)/\Delta] = 0$. In other words, when reporting is frequent, the joint compensation of the entrepreneur and monitor is a linear function of the cumulative cash flow generated by the project.

The equation above suggests a simple interpretation of the contract between the investor and the coalition of the entrepreneur and monitor. Consider the case when $\lambda \to 0$ and $\varphi = 1$, i.e., when there is no IPO and no loss in cash diversion. It is easy to verify that if the investor gives a cash position of $U^E + U^M$ to the entrepreneur and monitor and requires them to pay a fixed coupon $\mu\Delta$ each period, their cash position always coincides with their expected compensations in the optimal contract. Hence, if the entrepreneur and monitor share the project's assets if it is sold and receive nothing if the project is terminated due to their inability to pay the coupon, their compensations replicate the compensations in the optimal contract. Corollary 5 extends this replication to the general case.

Corollary 5. *a) (Public debt contract) For $\varphi = 1$, $\lambda \to 0$ and $\Delta \to 0$, the contract between the investor and the coalition of the entrepreneur and monitor can be replicated as:*

-The investor initially gives the entrepreneur and monitor a cash position of $U^E + U^M$ and requires a fixed payment of $\mu\triangle$ each period

-The project is terminated whenever the payment cannot be made

b) (VC contract) For arbitrary φ and λ, and for $\Delta \to 0$, the same contract can be replicated with:

-The investor initially gives the entrepreneur and monitor a cash position of $(U^E + U^M + E^M)/\varphi - E^M$ and requires a fixed payment of $\mu\Delta$ each period. In addition, the investor owns project debt with face value V_{IPO} and $(1 - \varphi)$ of its equity.

-The project is terminated whenever the payment of $\mu\Delta$ cannot be made.

The second set of results characterizes the limit of the contracts between the entrepreneur and monitor. To focus attention only on this side of financial intermediation, the inefficiency due to moral hazard between the monitor and investor is assumed away by setting E^M to infinity. To show the main idea of this analysis, consider the second-order Taylor approximation of the right-hand side of the dynamic program 7.3:

$$V^{tot}(V_n^E) = \max_{x^M}[-F(x^M)c\Delta + \mu\Delta + (1-r\Delta)(\lambda\Delta V_{IPO} + (1-\lambda\Delta),$$
$$(V^{tot} + V_1^{tot}E[C_t^E - C_{t-1}^E] + \frac{1}{2}V_{11}^{tot}E[(C_t^E - C_{t-1}^E)^2] + o(\Delta)))], \quad (8.2)$$

where the term $o(\Delta)$ consists of the approximation error of $(1+r\Delta)^{-1}$, P_{IPO} and the expected error of the Taylor approximation. The expectation of $C_t^E - C_{t-1}^E$ is simply the drift of the expected value:

$$E[C_t^E - C_{t-1}^E] = (1+r\Delta)C_{t-1}^E - C_{t-1}^E = r\Delta C_{t-1}^E.$$

The second moment of $C_t^E - C_{t-1}^E$ is decreasing in x^M and is proportional to Δ. The closed-form representation of this moment is given in the Appendix. I denote the moment $\theta(x^M)\Delta$ here. Rearranging terms in the problem above and writing it in operator form yields:

$$V^{tot} = T^0V^{tot} + o(\Delta), \quad (8.3)$$

where the operator T^0 is defined as:

$$T^0V^{tot} = \max_{x^M}[-F(x^M)c\Delta + \mu\Delta + (1-r\Delta)V^{tot} + \lambda\Delta V_{IPO} + V_1^{tot}rC_{t-1}^E\Delta + \frac{1}{2}\theta(x^M)\Delta V_{11}^{tot})]. \quad (8.4)$$

Intuitively, as Δ becomes small, the solution to the original fixed point problem becomes very close to the solution of the problem without $o(\Delta)$, i.e., the solution of $T^0V^{tot} = V^{tot}$. Moreover, using the definition of T^0 and rearranging terms in equation $T^0V^{tot} = V^{tot}$ yields the following linear second-order differential equation:

$$V^{tot}(C^E)(r+\lambda) = \max_{x^M}[-F(x^M)c + \mu + \lambda V_{IPO} + V_1^{tot}(C^E)rC^E + \frac{1}{2}\theta(x^M)V_{11}^{tot}(C^E)]. \quad (8.5)$$

Thus, the solution of the original problem should converge to the solution of the differential equation (8.5). This result is stated formally in the following theorem.

Theorem 4. *The optimal continuation value of the project $V^{tot,\Delta}$ when the period of reporting is Δ converges to the solution of the differential equation (8.5) as Δ goes to zero,*

$$\lim_{\Delta\to 0}\sup_{C^E\geq 0}|V^{tot,\Delta} - V^{tot}| = 0.$$

Moreover, the solution of (8.5) exists, is unique and concave.

Proof. See Appendix.

The differential equation (8.5) shows the main trade-off in monitoring. Monitoring more increases the costs of monitoring in the current period. Monitoring less increases the variance of the promised compensation $\theta(x^M)$. Since $V_{11}^{tot} < 0$, this decreases the expected total continuation value of the project. Note that the theorem also states that the continuation value is concave. The proof of the theorem shows that this result does not depend on the distribution of payoffs, whereas in the three-period model, this result requires a non-decreasing hazard rate of the distribution.

9 Conclusion

In this paper, I present a model that rationalizes a number of stylized facts about the allocation of control and cash flow rights in VC financing. First, it explains why the securities used in VC financing and the asset-sharing rules of VC partnerships are such that investors highest priority on the assets of start-ups, venture capitalists receive middle priority, and entrepreneurs are residual claimants with the lowest priority. Second, it explains why the allocation of cash flow and control rights is contingent on a project's performance. I show that in the model, more cash flow and control rights are allocated to entrepreneurs if the performance is good and to venture capitalists if the performance is bad.

I also demonstrate that the optimal contracts can be replicated with standard securities. First, I show a replication with preferred stock and common stock that fits well with the allocation of securities used in VC financing. The comparative statics of the efficiency of cash diversion also help to explain why in VC financing, start-ups sell common stock among other securities, whereas in bank lending, businesses exclusively use loan contracts. Second, I apply this framework to large corporations and show that when investors have different monitoring abilities, the optimal long-term contracts can be replicated with a combination of debt securities with different maturities and seniorities. This replication gives a new model of the optimal maturity structure and helps to explain, for example, why a higher variance of profits is associated with higher usage of short-term debt.

The optimal contracts derived in this model can also be used for a variety of other applications. For example, the paper gives predictions on the optimal leverage of financial intermediaries, i.e., on the optimal ratio between the banker's capital and investor's capital. This framework can also be used to introduce long-term financial contracts, an empirically important source of financing, into a model of business cycles, as in Bernanke and Gertler [1989].

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A Graphs and Numerical Procedures

First, consider the numerical procedure for the case of monitor only financing, i.e., $E^M = 0$. In this case the value function and the optimal thresholds become functions of one state variable, C^E. Moreover, only the dynamics of the entrepreneur's compensation needs to be computed, the general form of which is given in corollary 4.

The procedure starts with a guess for $V_0^{tot}(C^E)$. One can start with the value function for the constant debt level, i.e., a static contract replicated to infinity that gives the specified equity value. This function is concave. Then, for each combination of C_{t-1}^E, a^D and a^M on a specified grid, the next period compensation as a function of a is computed. Using these compensations one can compute $V^{tot}(a)$ and numerically take the expected value of the right hand side of the dynamic program 7.3. The procedure then maximizes the expected value for each C_{t-1}^E with respect to a^D and a^M. This gives the optimal threshold functions $a^D(C^E)$ and $a^M(C^E)$ and the next iteration of $V_1^{tot}(C^E)$.

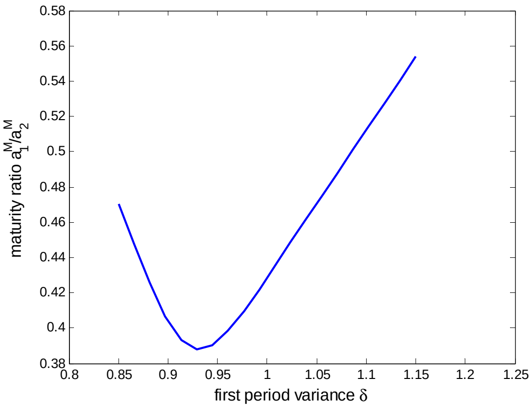
Repeating the procedure until $\sup_{C^E}|V_n^{tot} - V_{n+1}^{tot}| < \varepsilon$ for some arbitrary small ε gives the numerical solution for the dynamic problem. Since the mapping from V_n^{tot} into V_{n+1}^{tot} is a contraction, this procedure always converges to the optimal contract.

Solving numerically for the optimal contract in the case of $E^M < \infty$ is less straightforward for two reasons. First, adding another dimension makes the computations much slower. Second, the slope of the entrepreneur's compensation, ξ, now depends on the second derivatives of V^{tot}. This makes calculating the level of $C_t^E(a_t)$ for a given expectation C_{t-1}^E a complicated task. Moreover, the numerical second derivatives of V^{tot} are very unstable. Hence, using them in the iterative algorithm is problematic.

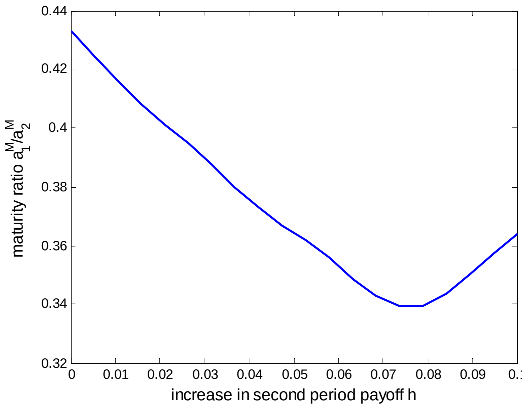
In order to overcome the difficulties described above, two approximations are made. First, as long as the realization of $-V^{tot} + c$ is very unlikely, i.e., when $\mu - 3\sigma > -V^{tot} + c$, it is suboptimal to terminate the project when the project can be continued. Thus, when μ and σ are small enough relative to V^{tot}, one may assume that termination happens only due to incentive compatibility reasons, i.e., when $C^E(a_t^D) + C^M(a_t^D) = 0$. This equation is used to find $a_t^D(C_{t-1}^E, C_{t-1}^M)$. Second, similar to the case of $E^M = \infty$ one can assume that $\xi = \varphi$. Although theoretically true only in the limit, this approximation is very close to the actual contract for all C^M. After solving for the optimal contract one can compute ξ and verify that it is close to φ in all states. Using the approximations above, to solve numerically for the optimal contract, the procedure iterates V^{tot} until convergence.

Figure A.1: Debt Maturity and Payoff Distributions

a) Optimal maturity and payoff variance



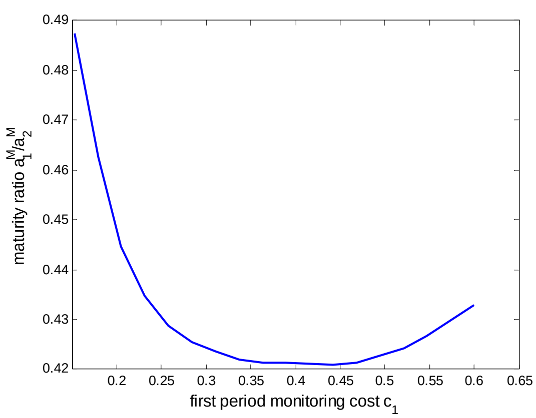
b) Optimal maturity and average payoff



These graphs show the comparative statics of the initial ratio of short term debt to long term debt, $a_1^M/a_2^M(a_1^D)$ with respect to the variance parameter of the first period payoff distribution and the second period average payoff for the three-period model. In Figure A.1a the distribution of payoffs in the first period is uniform on $[2-\delta, 2+\delta]$ and in the second period uniform on $[1,3]$. The decreasing part of the curve corresponds to the case when the project does not issue risky short term debt. In Figure A.1b the first period distribution is uniform on $[1,3]$ and the second period distribution is uniform on $[1+h, 3+h]$. Similarly, the increasing part of the curve corresponds to the case when there is no risky short term debt.

Figure A.2: Debt Maturity and First-period Monitoring Cost

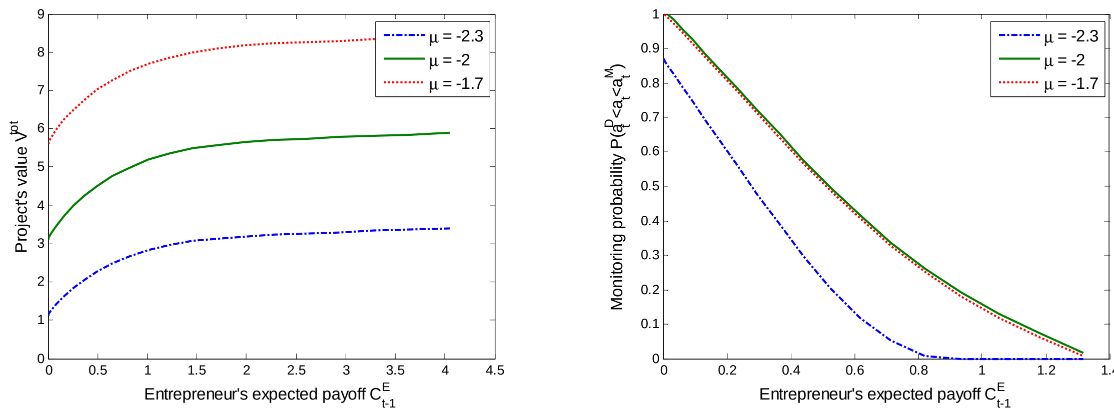
Optimal maturity and first period monitoring costs



This figure shows the comparative statics of the initial ratio of short term debt to long term debt, $a_1^M/a_2^M(a_1^D)$ with respect to the first period monitoring costs. In both periods the payoff distribution is uniform on $[1,3]$. The decreasing part of the curve corresponds to the case when the amount of short term debt decreases and the amount of long term debt increases. The increasing part of the curve corresponds to the case when both the amounts of short term and long term increase with c_1.

Figure A.3: Optimal Contract and Average Investment Requirement $-\mu$

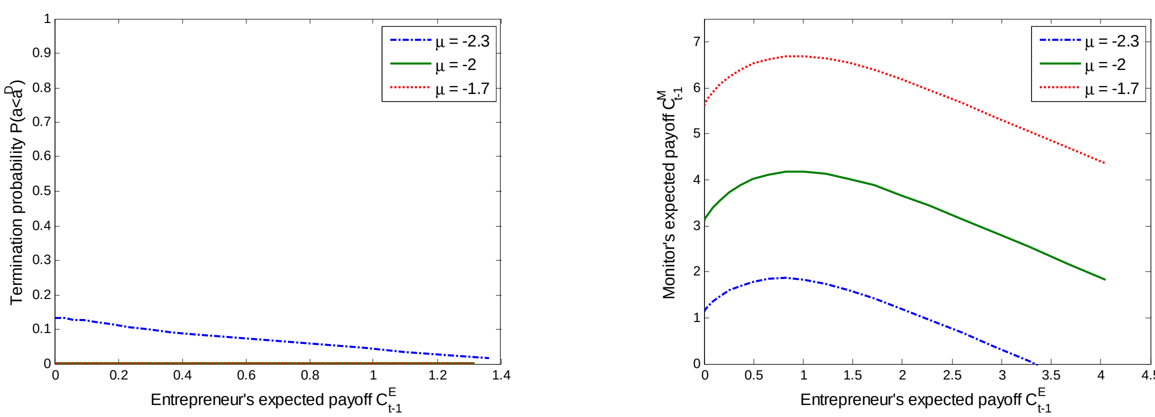
a. Project's value V^{tot} b. Monitoring probability $P(a_t^D \leq a < a_t^M)$



These figures show the project's total value and the probability of monitoring as functions of the entrepreneur's expected compensation. The parameter values assumed for the infinite horizon model are: $c = 1/3$, $\sigma = 1$, $r = 0.05$, $\varphi = 0.5$, $P_{IPO} = 0.068$ and $V_{IPO} = 40$. The total value is concave and increasing in C_{t-1}^E. The probability of monitoring is increasing in μ because termination becomes more costly than monitoring.

Figure A.4: Optimal Contract and Average Investment Requirement $-\mu$

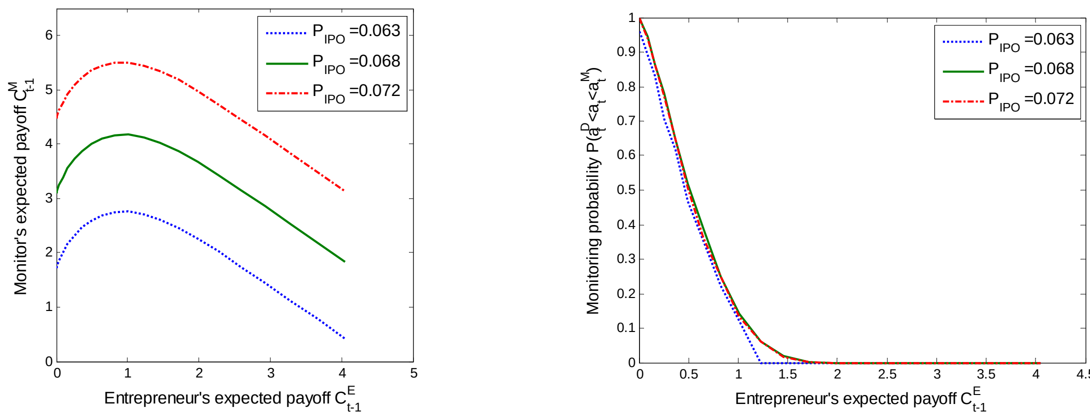
a. Termination probability $P(a_t < a_t^D)$ b. Monitor's expected payoff $C_{t-1}^M = V^{tot} - C_{t-1}^E$



These figures show the probability of termination and the monitor's expected compensation as functions of the entrepreneur's expected compensation. The parameter values for the infinite horizon model are: $c = 1/3$, $\sigma = 1$, $r = 0.05$, $\varphi = 0.5$, $P_{IPO} = 0.068$ and $V_{IPO} = 40$. The probability of termination is decreasing in μ because termination becomes more costly. The frontier of the compensations C^M and C^E shifts upwards as the total surplus of the project increases with μ.

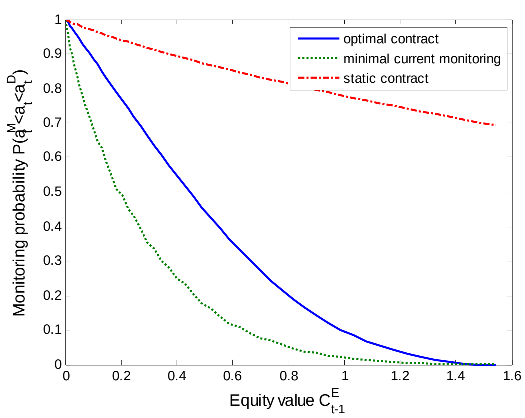
Figure A.5: Optimal Contract and IPO Arrival Rate λ^{GP}

a. Total expected continuation value V^{tot} b. Monitoring probability $P(a_t^D \leq a_t < a_t^M)$



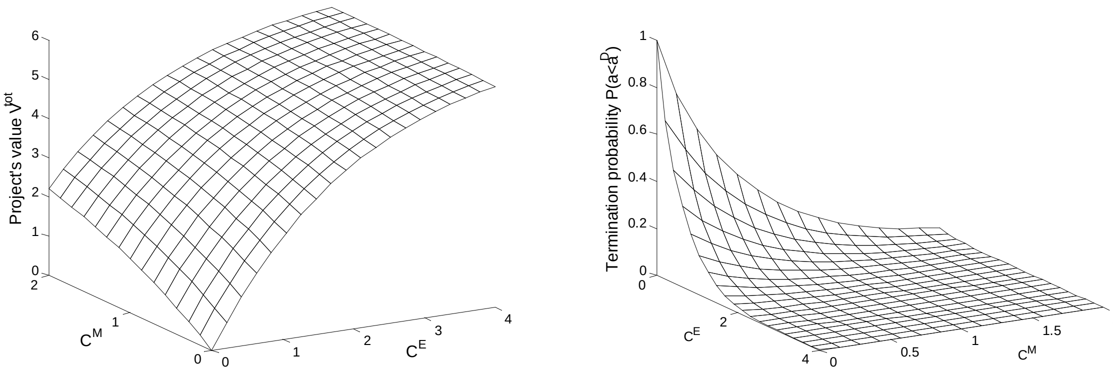
These figures show the monitor's expected compensation and the probability of monitoring as functions of the entrepreneur's expected compensation. The parameter values for the infinite horizon model are: $c = 1/3$, $\sigma = 1$, $r = 0.05$, $\varphi = 0.5$, $\mu = -2$ *and* $V_{IPO} = 40$. *The frontier of the compensations* C^M *and* C^E *shifts upwards as the total surplus of the project increases with* P_{IPO}. *The probability of monitoring is not affected significantly.*

Figure A.6: Monitoring Probability and Alternative Contracts



This figure shows the numerical simulation of the monitoring probability in the optimal contract and two alternative contracts for the basic set of parameter values: $c = 1/3$, $\sigma = 1$, $r = 0.05$, $\varphi = 0.5$, $\mu = -2$, $P_{IPO} = 0.063$ and $V_{IPO} = 40$. *The first alternative contract is the contract that minimizes current monitoring costs, i.e., the contract that postpones current monitoring as much as possible. The second contract is the replication of the optimal static contract into infinite horizon, i.e., this is the contract for which the thresholds do not depend on the state variables. The figure shows that the monitoring minimizing contract has too little current monitoring and the static contract starts with too much monitoring.*

Figure A.7: Total Value of the Project V^{tot}



These figures show the total value of the project and the probability of termination as functions of both the entrepreneur's and monitor's compensations in the case when the monitor cannot finance the project without the investor. The parameter values for the model are: $c = 1/3$, $\sigma = 1$, $r = 0.05$, $\varphi = 0.5$, $\mu = -2$, $P_{IPO} = 0.063$ and $V_{IPO} = 40$. The total value function is concave and satisfies the properties in conjecture 1.

B Proofs

Proof of Lemma 1. In the first step I show that the optimal recommendations are $x_t = 0$ and $z_t = 0$.

Suppose that in the original contract the actions chosen are $x_t^1(a_t)$ and $z_t^1(a_t)$, the respective compensations are $C^E(y_t)$ and $C^M(y_t)$ and the termination rule is $D_t(y_t)$. Incentive compatibility constraints (3.7)-(3.8) require these actions to satisfy:

$$x_t^1(a_t),\ z_t^1(a_t) \in \arg\max_{x_t, z_t \geq 0} [C^E(y_t) + C^M(y_t) + \varphi x_t(1 - D_t(y_t))],$$

where $y_t = a_t - x_t - z_t$.

Assign the new compensations such that under no private benefit and destructive activities they give the same expected compensations to the entrepreneur and monitor:

$$C_t^{E,*}(a_t) + C_t^{M,*}(a_t) = C_t^E(a_t - x_t^1 - z_t^1) + C_t^M(a_t - x_t^1 - z_t^1) + \varphi x_t^1(a_t)(1 - D_t(a_t - x_t^1 - z_t^1)),$$

And terminate the projects in the same states:

$$D_t^*(a_t) = D_t(a_t - x_t^1 - z_t^1).$$

These compensation and termination rules give the same utility to the entrepreneur and monitor, but are weakly better for the investor. Hence, it is left to verify that $x_t = 0$ and $z_t = 0$ are incentive compatible,

$$0 \in \arg\max_{x_t, z_t \geq 0} [C^{E,*}(y_t) + C^{M,*}(y_t) + \varphi x_t(1 - D_t^*(y_t))].$$

Suppose there is $x_t^2(a_t)$ and $z_t^2(a_t)$ such that

$$C^{E,*}(a_t - x_t^2 - z_t^2) + C^{M,*}(a_t - x_t^2 - z_t^2) + \varphi x_t^2(1 - D_t^*(a_t - x_t^2 - z_t^2)) > C^{E,*}(a_t) + C^{M,*}(a_t).$$

Substituting the definition of $C^{E,*} + C^{M,*}$ implies:

$$\begin{aligned} C_t^E(a_t - \widetilde{x}_t - \widetilde{z}_t) + C_t^M(a_t - \widetilde{x}_t - \widetilde{z}_t) + \varphi \widetilde{x}_t(1 - D_t(a_t - \widetilde{x}_t - \widetilde{z}_t)) > \\ C_t^E(a_t - x_t^1 - z_t^1) + C_t^M(a_t - x_t^1 - z_t^1) + \varphi x_t^1(1 - D_t(a_t - x_t^1 - z_t^1)), \end{aligned}$$

where $\widetilde{x}_t = x_t^1(a_t - x_t^2 - z_t^2) + x_t^2$ and $\widetilde{z}_t = z_t^1(a_t - x_t^2 - z_t^2) + z_t^2$. However, this inequality contradicts the fact that $x_t^1(a_t)$ and $z_t^1(a_t)$ were originally chosen.

In the second step I show that the incentive compatibility constraints become equations (4.5)-(4.4).

Since there is no non-verifiable activity, side transfers Tr are not necessary and can be replicated with verifiable compensations. The first equation is a trivial consequence of (3.7). Also, monotonicity of C^E is trivial because otherwise the entrepreneur would want to engage in destructive activity z_t. Similarly, the assumption that the entrepreneur can guarantee himself the best no-side-contract outcome implies that $x_t = 0$ should maximize his utility among $x_t \in \{0, M_t(y_t) = 0\}$. Combining this with the fact that on $\{x_t \in M_t(y_t) = 1\}$ the entrepreneur's compensation needs to be monotone, gives the second incentive compatibility constraint, equation (4.4).■

Proof of Theorem 1, Lemma 2 and Lemma 3. First, one needs to show that incentive compatibility is binding. The result is a special case of theorem 3. However, the steps of the proof are given here as well.

First, whenever the incentive compatibility constraints are not binding, they can be made binding without violating any other constraint. By making the incentive compatibility constraints binding one decreases the compensations in good states and increases the compensation of the entrepreneur and monitor in bad states. Since there is no continuation value, the compensations do not affect the value of the project. However, as in Gale and Hellwig [1985] this would allow one to decrease monitoring of bad states by making the entrepreneur's compensation in bad states depending on the project's performance.

Second, for the binding incentive compatibility constraints once some payoff a_l is not monitored, monitoring $a > a_l$ does not affect the entrepreneur's compensation. Given that monitoring is costly, it is suboptimal to monitor above a_l. Similarly, terminating payoffs higher than some non-terminated payoff does not affect the total compensation of the entrepreneur and monitor. Since terminating lower payoffs is better than higher payoffs, only the lowest payoffs are terminated. This establishes the fact that monitoring and termination are threshold rules.

Third, it is always optimal to keep the entrepreneur's compensation constant in the monitoring region. If it is not constant, it is always feasible to make it constant and decrease the monitoring threshold, keeping compensations everywhere else intact.

Finally, both compensations in the termination region must be zero because this increases the compensations in the non-termination region. Higher compensations in the non-termination region cannot decrease the total value of the project.

This proves the results stated in Theorem 1. To prove the other properties of the solution it is convenient to introduce the following notation for the information rent:

$$\mu(x) = \varphi \int_x^\infty (a - x) dF(a).$$

This is the rent that the agent receives when he has control for payoffs higher than x. Using this notation, the expected compensations of the agents are:

$$\overline{C}^E = (1 - F(a_2^D))C^E(a_2^D) + \mu(a_2^M) \tag{B.1}$$

and

$$\overline{C}^M = (1 - F(a_2^D))C^M(a_2^D) + \mu(a_2^D) - \mu(a_2^M). \tag{B.2}$$

Given that there is no continuation value of the project, finding the optimal levels of a_2^M and a_2^D is relatively simple. First, $a_2^D \geq 0$ because terminating negative payoffs increases the total payoff. Similarly, there is no monitoring when the payoff is below c_2 because in this case termination is less costly than monitoring. Hence, $a_2^M - a_2^D > 0$ only when $a_2^M > c_2$. Moreover, providing any positive compensation to the entrepreneur when he is not in control and $a_2^M > 0$ is suboptimal, because one could lower this payment and decrease the termination or the monitoring threshold. This implies that the lowest compensation for the entrepreneur should be zero, $C^E(a_2^D) = 0$, when $a_2^M > 0$. Similarly, it is never optimal for the monitor to receive a lowest compensation above zero when $a_2^D > c_2$, because a profitable deviation would be to decrease this level and decrease termination threshold.

Hence, $C^M(a_2^D) = 0$ in that region.

Using the properties listed above and equations (B.1)-(B.2) one can derive the optimal thresholds as functions of $\overline{C}^E$ and $\overline{C}^M$ for different regions:

1. $\overline{C}^E \geq \mu(0)$, $a_2^M = a_2^D = 0$
2. $\mu(0) > \overline{C}^E \geq \mu(c)$, $a_2^M = a_2^D = \mu^{-1}(\overline{C}^E)$
3. $\overline{C}^E + \overline{C}^M \geq \mu(c) > \overline{C}^E$, $a_2^M = \mu^{-1}(\overline{C}^E)$ and $a_2^D = c$
4. $\overline{C}^E + \overline{C}^M < \mu(c)$, $a_2^M = \mu^{-1}(\overline{C}^E)$ and $a_2^D = \mu^{-1}(\overline{C}^E + \overline{C}^M)$

The first region is when the expected compensation of the entrepreneur is large enough that no monitoring is necessary. In the second region the threshold is low enough, so that it is better to terminate those payoffs than to monitor them. The third region is when the entrepreneur's expected compensation is low, so that there should be monitoring, but the monitor's expected compensation is high. The last region is the case when the entrepreneur's and monitor's compensations are low enough that both monitoring and termination boundaries are determined by the incentive compatibility constraints. These equations fully characterize the optimal contract in the last period.

Note that since μ^{-1} is a decreasing function of x, the statement in Lemma 2 can be shown for each of the regions above. Finally, to show the properties stated in Lemma 3 one can differentiate the continuation value function.

The total expected value of the project in the second period is given by:

$$V_2^{tot}(\overline{C}^E, \overline{C}^M) = \int_{a_2^D}^{\infty} a_2 dF(a_2) - c_2(F(a_2^M) - F(a_2^D)).$$

Since the functional forms of the threshold levels are known, one can analyze the properties of this function by differentiating it with respect to each promised value ($A = E,\ M$):

$$\frac{\partial V_2^{tot}}{\partial \overline{C}^A} = -c\frac{\partial a_2^M}{\partial \overline{C}^A} f(a_2^M) + (c_2 - a_2^D)\frac{\partial a_2^D}{\partial \overline{C}^A} f(a_2^D),$$

where $\partial a_2^M / \partial \overline{C}^A$ and $\partial a_2^D / \partial \overline{C}^A$ can be determined for each region separately. Note also that from the definition of $\mu(x)$, $\partial \mu^{-1}(C)/\partial C = -(1 - F(x))^{-1}\varphi^{-1}$, computing these derivatives for each region gives:

1. $$\frac{\partial V_2^{tot}}{\partial \overline{C}^E} = 0 \text{ and } \frac{\partial V_2^{tot}}{\partial \overline{C}^M} = 0$$

2.
$$\frac{\partial V_2^{tot}}{\partial \overline{C}^E} = \frac{f(a_2^M)}{(1-F(a_2^M))}\frac{a_2^M}{\varphi} \text{ and } \frac{\partial V_2^{tot}}{\partial \overline{C}^M} = 0$$

3.
$$\frac{\partial V_2^{tot}}{\partial \overline{C}^E} = \frac{f(a_2^M)}{(1-F(a_2^M))}\frac{c_2}{\varphi} \text{ and } \frac{\partial V_2^{tot}}{\partial \overline{C}^M} = 0$$

4.
$$\frac{\partial V_2^{tot}}{\partial \overline{C}^E} = \frac{f(a_2^M)}{(1-F(a_2^M))}\frac{c_2}{\varphi} + \frac{f(a_2^D)}{(1-F(a_2^D))}\frac{(a_2^D - c_2)}{\varphi} \text{ and } \frac{\partial V_2^{tot}}{\partial \overline{C}^M} = \frac{f(a_2^D)}{(1-F(a_2^D))}\frac{(a_2^D - c_2)}{\varphi}$$

Note first\ that the transition from one region to another is smooth. In the first region the function is trivially concave. In the second region the function is concave when the first derivative is decreasing in $\overline{C}^E$. Since a_2^M is decreasing in $\overline{C}^E$, concavity is equivalent to this derivative being increasing in a_2^M. Hence, it is sufficient that the probability distribution has a non-decreasing hazard rate, defined as:

$$h(x) = \frac{f(x)}{(1-F(x))}.$$

In the third region this condition becomes also necessary. Finally, in the last region, a non-decreasing hazard function is sufficient for each second derivative to be non-positive ($a_2^D > c$ in that region). Using the same property of the hazard rate one can verify that:

$$\frac{\partial^2 V_2^{tot}}{\partial^2 \overline{C}^E} < \frac{\partial^2 V_2^{tot}}{\partial \overline{C}^E \partial \overline{C}^M} \tag{B.3}$$

and

$$\frac{\partial^2 V_2^{tot}}{\partial^2 \overline{C}^M} = \frac{\partial^2 V_2^{tot}}{\partial \overline{C}^E \partial \overline{C}^M}, \tag{B.4}$$

where the first inequality holds because the two derivatives are different by a non-increasing first term and the second equality holds due to the fact that the derivatives of a_2^D with respect to $\overline{C}^E$ and $\overline{C}^M$ are equal. Thus, the second principal minor of the Hessian matrix is positive and the function is concave.■

Proof of Theorem 3. Step 1. (Binding incentive compatibility constraints) For this proof and the proofs that will follow it is convenient to consider V_2^{tot} not as a function of C^E and C^M, but as a function of C^E and $C^E + C^M$. Formally, define $\widetilde{V}^{tot}$ in the following way:

$$\widetilde{V}^{tot}(C^E, C^E + C^M) = V_2^{tot}(C^E, C^M).$$

Also, instead of using subscripts to denote the time period it is used to denote the variable with respect to which the function is differentiated, e.g. V_1^{tot} is the derivative of V^{tot} with respect to C^E.

Using lemma 3 one can find the second derivatives of $\widetilde{V}^{tot}$:

$$\widetilde{V}_{11}^{tot} = V_{11}^{tot} - 2V_{12}^{tot} + V_{22}^{tot} = V_{11}^{tot},$$

$$\widetilde{V}_{12}^{tot} = V_{12}^{tot} - V_{22}^{tot} = 0,$$

$$\widetilde{V}_{22}^{tot} = V_{22}^{tot}.$$

In other words, function $\widetilde{V}^{tot}$ is concave and the effects of each variable are separable from the effects of the other variables in the sense that the marginal effect of one variable does not depend on the value of the other variable. Similarly to the case of one incentive compatibility constraint, if the means of the compensations are given and need to be monotone, minimizing the expectation of V^{tot} is equivalent to minimizing the variance of each of the variables.

Thus, for any given $D_1(a_1)$ and $M_1(a_1)$ the optimal contract minimizes the variances of C^E and $C^E + C^M$. Note, that C^E is non-decreasing due to lemma 1 and $C^E + C^M$ is non-decreasing due to the fact that if it were decreasing, the entrepreneur and monitor would divert cash. Hence, the contract would not be incentive compatible. The incentive compatibility constraints must be binding because if they were not binding one could decrease the variance of the compensations by making them binding. ■

Proof of Theorem 2. Step 2. The previous lemma showed that the incentive compatibility constraints are binding. This means that if some payoff a_l is not monitored, the entrepreneur's compensation for $a > a_l$ is $C^E(a_l) + \varphi(a - a_l)$, i.e. he receives the same information rent independently on whether there is monitoring or not for $a > a_l$. Since monitoring is costly, there is no monitoring of states higher than a_l.

Similarly, if some payoff a_k is not terminated, the combination of the compensations of the entrepreneur and the monitor is $C^E(a_k) + C^M(a_k) + \varphi(a - a_k)$ independently on whether states higher than a_k are terminated. Since terminating higher payoffs is more costly than terminating lower payoffs, one it is optimal to shift all termination above a_k to lower states.

Combining the two results above gives that the monitoring and the termination rules are threshold rules, i.e. there are a_1^D and a_1^M such that $D_1(a_1) = 1[a_1 < a_1^D]$ and $M_1(a_1) = 1[a_1^D \leq a_1 < a_1^M]$. The result that the compensations of the entrepreneur and the monitor are zero on $a_1 < a_1^D$ follows from the fact that it is weakly better to give the entrepreneur and the monitor higher compensations when the project is continued.

Note that the incentive compatibility constraints binding implies:

$$C^E(a_1) + C^M(a_1) = C^E(a_1^D) + C^M(a_1^D) + \varphi(a_1 - a_1^D),$$

and the level of $C^E(a_1^D) + C^M(a_1^D)$ is determined by the expected compensations. Hence, the sum of the compensations is fully characterized.

The entrepreneur's compensation in the non-monitoring region follows from the incentive compatibility constraints, i.e. it must be the case that $C_1^{E\prime}(a_1) = \varphi$ on $a_1 > a_1^M$. It is left to show that the entrepreneur's compensation is constant in the monitoring region whenever possible. This result is a simple consequence of the fact that the optimal compensation minimizes the variance of the entrepreneur's compensation. Trivially, a constant minimizes the variance of a random variable. ■

Proof of Corollary 3. Consider the case of $\varphi = 1$ first. In the last period the compensations can be replicated with debt contracts because the support of the payoff distribution is bounded from below by c_2. As was shown above, for this case the monitor's compensation at $a_2^D(a_1)$ is zero and increases by one dollar for each dollar of payoff until the payoff becomes $a_2^M(a_1)$, after which his compensation settles at $a_2^M(a_1) - a_2^D(a_1)$. Similarly, the entrepreneur's compensation is zero at $a_2^M(a_1)$ and increases by one dollar for each dollar of payoff. Combining this with the rule that payoffs $a_2 < a_2^D$ are terminated and payoffs $a_2^D \leq a_2 < a_2^M$ are monitored gives exactly the cash flow and control rights specified in the definitions of the debt contracts.

Similarly, in the first period at $a_1 = a_1^D$ the compensations of the monitor and the entrepreneur consist of junior debt $a_2^M(a_1^D) - a_2^D(a_1^D)$ and equity of the second period respectively (as was shown before, their compensations consist of the second period junior debt and equity for all a_1). For $a_1^D \leq a_1 < a_1^M$ each additional dollar of payoff increases the monitor's compensation by one dollar through increasing the value of his second period debt by one dollar by decreasing a_2^D. This means that in the first period the monitor buys the second period senior debt from the investor. In turn, for $a_1 \geq a_1^M$ the entrepreneur's compensation increases by one dollar for each dollar of payoff through increasing his second period equity value by decreasing a_2^M. In other words, the entrepreneur in the first period buys second period junior debt from the monitor. Note also that the monitor in turns uses the cash to buy second period senior debt from the investor.

Intuitively, using senior debt is more costly for the project than using junior debt because termination is more costly than monitoring. Similarly, using junior debt is more costly than using equity because monitoring is costly. Hence, the monitor and entrepreneur buy more costly long term securities and substitute them with cheaper ones.

The case of $\varphi < 1$ is a straightforward extension. Instead of compensating the monitor and entrepreneur with one dollar for each dollar of payoff, they are compensated with φ dollars, meaning that they hold only share φ of the junior debt and equity defined by the same thresholds a^D and a^M and the investor holds the rest $1-\varphi$ respectively. ■

Proof of Lemma 4. Most of the arguments were given in the main text. What is left to argue is that the entrepreneur's and the investor's compensations are maximized when the monitor's individual rationality constraint is binding, i.e., $U^M = 0$ when $E^M = 0$. Consider a contract with $U^M > 0$, U^E for $E^M = 0$. One can modify this contract such that $\widetilde{U}^E = U^E + U^M$ and $\widetilde{U}^M = 0$ and whenever $C_1^M = \varphi$ (first derivative with respect to a_t), set $C_1^E = \varphi$ instead. In other words, instead of giving incentives to the monitor this contract gives incentives to the entrepreneur. This modification is feasible because in the original contract $C^E + C^M$ satisfied all the properties of C^E in the new contract. However, the new contract does no require monitoring at all, i.e. one can decrease a_t^M to a_t^D and avoid the monitoring costs. Since this modification increases the surplus, it increases the sum of the utilities of the investor and entrepreneur. ■

Proof of Theorem 3. Binding IC constraints. First note that if the respective reports trigger termination, then the objective function does not depend on compensations $C^E(a)$ and $C^M(a)$ as long as their mean is the same. In that region compensation is a pure redistribution. Hence, changing the slope of the compensations when it is higher than φ cannot decrease the objective function.

The idea behind the proof for the reports (a_1, a_2) not triggering termination is that, due to the convexity of the optimization problem, the constraints that are not binding should not affect the optimal compensation functions. However, in the model without the incentive compatibility constraints, the solution is in the region where they would bind. Hence, they must be binding for the optimal contract.

Besides the incentive compatibility constraints that are tested in this lemma, the relevant part of the dynamic program being solved is:

$$\max_{C^E(a),C^M(a)} [\int_{a_1}^{a_2} V^{tot}(C^E, C^M) f(a) da] \tag{B.5}$$

$$s.t.$$

$$\int_{a_1}^{a_2} C^E(a) f(a) da = \overline{C}^E \tag{B.6}$$

$$\int_{a_1}^{a_2} C^M(a)f(a)da = \overline{C}^M, \tag{B.7}$$

where $\overline{C}^E$ and $\overline{C}^M$ are the respective expectations that the contract provides to the agents when the investment requirement is in (a_1, a_2). Note that the IPO part of the objective function is a pure redistribution and the investment and monitoring expenses are fixed.

As one can see, the objective function above is concave and the set of admissible compensation functions is convex, due to convexity resulting from the incentive compatibility constraints and expectation rationality above. For such problems if a particular constraint is not binding, it can be ignored.

Consider the case in which both $\partial C^E/\partial a > \varphi$ and $\partial C^E/\partial a + \partial C^M/\partial a > \varphi$. This means that the incentive compatibility constraints are not binding. Thus, the optimal contract should be the contract solving problem (B.5)-(B.7). However, Jensen's inequality states that the solution of this problem is $C^E(a) = const$ and $C^M(a) = const$. This violates the assumption that the incentive compatibility constraints are not binding.

Consider the case with $\partial C^E/\partial a \leq \varphi$ and $\partial C^E/\partial a + \partial C^M/\partial a > \varphi$. It implies that there are no binding constraints on $\partial C^M/\partial a$. Hence, one can keep the entrepreneur's compensation as given and solve for the monitor's optimal compensation. Since there are no relevant inequality constraints, one can use the Lagrange method:

$$L = \int_{a_1}^{a_2} V^{tot}(C^E(a), C^M(a))f(a)da - \lambda(\int_{a_1}^{a_2} C^M(a)f(a)da - \overline{C}^M).$$

Differentiating this function with respect to $C^M(a)$ gives:

$$\frac{\partial L}{\partial C^M(a)} = [V_2^{tot}(C^E(a), C^M(a)) - \lambda]f(a) = 0.$$

Differentiating this equality with respect to a gives the slope of the optimal monitor's compensation:

$$\frac{\partial C^M}{\partial a} = -(\frac{V_{21}^{tot}}{V_{22}^{tot}})\frac{\partial C^E}{\partial a} \leq 0$$

However, this condition contradicts the starting inequalities.

The case with $\partial C^E/\partial a > \varphi$ and $\partial C^E/\partial a + \partial C^M/\partial a = \varphi$ is described in the proof of the next theorem. Using the Lagrange method one can show that $\partial C^E/\partial a \leq \varphi$ due to the conjectured properties of V^{tot}.■

Proof of Theorem 3. The first step of the proof is to show that monitoring and termination are threshold rules. Given that the incentive compatibility constraints are binding, the argument for threshold rules is analogous to that of the three period model (theorem 2). Similarly, the compensations of the entrepreneur and the monitor in the termination region are zero because this increases the surplus when the project is not terminated. In the non-monitoring region binding incentive compatibility implies that the entrepreneur's compensation is linear and the monitor's compensation is constant. It is left to show the distribution of the compensations between the monitor and entrepreneur in the monitoring region.

Due to the binding IC constraints in the monitoring region, the sum of compensations $C^E + C^M$ is given. Hence, it is more convenient to use the following objective function:

$$\widetilde{V}^{tot}(C^E, C^E + C^M) = (1+r)^{-1}(1 - P_{IPO})V^{tot}(C^E, C^E + C^M)$$

This function is a convex function of the two variables C^E and $C^E + C^M$ with second order derivatives:

$$\widetilde{V}^{tot}_{22} = (1+r)^{-1}(1 - P_{IPO})V^{tot}_{22}$$

$$\widetilde{V}^{tot}_{12} = (1+r)^{-1}(1 - P_{IPO})(V^{LP}_{12} - V^{tot}_{22})$$

$$\widetilde{V}^{tot}_{11} = (1+r)^{-1}(1 - P_{IPO})(V^{tot}_{11} - 2V^{tot}_{12} + V^{tot}_{22}).$$

This transformation makes the optimization problem one with one unknown function C^E. The lemma above states that incentive compatibility is satisfied and the only relevant constraints are the limited liability constraints:

$$C^E, C^M > 0$$

and the monotonicity of C^E.

Consider an arbitrary candidate contract with a given mean of C^E in the region $(a^D, a^M]$ and the end points $C^E(a^D)$ and $C^E(a^M)$. It is easy to verify that any modification of this contract that preserves the conditions above and has a slope between 0 and φ is feasible. This implies that all feasible contracts lie within the box given by the four equations:

$$C^E(a) \leq C^E(a^M) \tag{B.8}$$

$$C^E(a) \geq C^E(a^D) \tag{B.9}$$

$$C^E(a) \leq C^E(a^D) + \varphi(a - a^D) \tag{B.10}$$

The first inequality constraint refers to the monotonicity of the optimal contracts at a^M. The last two constraints refer to the limited liquidity of the agents. Constraint (B.9) is binding only when $C^E(a^D) = 0$ and constraint (B.10) is binding only when $C^M(a^D) = 0$.

Thus, the problem can be written as:

$$\max_{C^E} \int_{a^M}^{a^D} \widetilde{V}^{tot}(C^E(a), C^E(a^M) + C^M(a^M) + \varphi(a - a^M))f(a)da$$

$$s.t.$$

$$\int_{a^M}^{a^D} C^E(a)f(a)da = \overline{C}^E \tag{B.11}$$

$$constraints\,(B.8) - (B.10).$$

The Kuhn-Tucker condition for the corresponding optimization problem is:

$$\frac{\partial \widetilde{V}^{tot}}{\partial C^E(a)} - \mu - \lambda_i(a) = 0 \tag{B.12}$$

where μ is the multiplier associated with equality constraint (B.10) and $\lambda_i(a)$ with $i = 1...4$ are the multipliers associated with inequality constraints (B.8)-(B.10). The complementarity slackness condition requires that $\lambda_i(a) = 0$ when the corresponding inequality constraints are not binding. In other words, in the interior of the box given by equations (B.8)-(B.10) the marginal value of $V^E_{n+1}(a)$ needs to be the same along the optimal contract line. Differentiating (B.12) with respect to a when all $\lambda_i(a)$ are zeros gives the slope of the optimal contract in the interior region:

$$\frac{\partial C^E}{\partial a} = \varphi \frac{\widetilde{V}^{LP}_{12}}{\widetilde{V}^{LP}_{11}} = \varphi \frac{V^{LP}_{22} - V^{LP}_{12}}{V^{LP}_{11} - 2V^{LP}_{12} + V^{LP}_{22}}.$$

According to conjecture 1, this slope is between 0 and φ. Hence, monotonicity in the interior of $(a^D, a^M]$ is never binding. ■

Proof of Corollary 4. This corollary trivially follows from the main theorem and the fact that $V^{tot}_2 = 0$. The last fact follows from the fact that V^{tot} is increasing in C^M and bounded by the first best value. ■

Proof of lemma 6. The first step is to show that there is no termination when $C^E + C^M > 0$. To see this, consider an arbitrary concave continuation value function and the termination level $a^{D,\triangle}$. If the threshold level is changed to $a^{D,\triangle} + \varepsilon$, the continuation value of the entrepreneur and monitor in $[a^{D,\triangle}, a^{D,\triangle} + \varepsilon)$ is decreased to zero and the continuation value in $[a^{D,\triangle}+\varepsilon,\infty)$ is increased by some amount to keep the expected values equal. Due to the concavity of the total value, this affects the total expected value of the project negatively. Note also the payoff and the continuation value in $[a^{D,\triangle}, a^{D,\triangle}+\varepsilon)$ are lost and if there were monitoring in that region, the costs of monitoring are saves. Hence, this deviation increases the value of the project only if:

$$-a^{D,\triangle} - V^{tot} + c\triangle \geq 0.$$

Normalizing it to the standard normal distribution (cdf $F(.)$ and pdf $f(.)$) yields:

$$\frac{a^{D,\triangle} - \mu\triangle}{\sigma\sqrt{\triangle}} < \frac{-V^{tot} + c\triangle - \mu\triangle}{\sigma\sqrt{\triangle}}.$$

Since $(c\triangle - \mu\triangle)/\sigma\sqrt{\triangle} \to 0$ as $\triangle \to 0$, it is sufficient to show that for an arbitrary constant $\alpha > 0$,

$$\lim_{\triangle\to 0} \frac{F(-\alpha/\sqrt{\triangle})}{\triangle} = 0.$$

Given that both the nominator and the denominator go to zero, one can use L'Hospital's rule to find the limit:

$$\lim_{\triangle\to 0} \frac{F(-\alpha/\sqrt{\triangle})}{\triangle} = \lim_{\triangle\to 0} f(-\alpha/\sqrt{\triangle})\frac{1}{2\triangle^{3/2}} = \lim_{\triangle\to 0} \frac{1}{\sqrt{2\pi}}e^{-\frac{\alpha^2}{\triangle}}\frac{1}{2\triangle^{3/2}} = 0,$$

where the last equality is due to the fact that the exponent decreases faster than any power function.

The fact that the project is terminated with probability zero can be shown in the following way. Consider a unit time interval and any history with $V_n^E + V_n^{GP} > 0$ in each period and take the maximum $a^{D,\triangle}$ among all the periods. This gives the upper bound on the probability of termination. Denoting $x^{\triangle} = (a^{D,\triangle} - \mu\triangle)/\sigma\sqrt{\triangle}$ its normalized value, the probability that the project is terminated during this unit time interval is:

$$P^{tot} = 1 - (1 - F(x^{\triangle}))^{1/\triangle}$$

Substituting $F(x^{\triangle}) = o(\triangle)$ and performing straightforward manipulations with it gives

$$P^{tot} = \frac{F(x^{\triangle})}{\triangle} + o(\frac{F(x^{\triangle})}{\triangle}) \to 0.$$

To show the limit of the first best value of the project consider the equation defining the first best value:

$$V^{fb,\triangle} = (1 - F(x^{\triangle}))(E[a|a > a^{D,\triangle}] + e^{-r\triangle t}(P_{IPO}V_{IPO} + (1 - P_{IPO})V^{fb,\triangle}).$$

Solving it for $V^{fb,\triangle}$ and substituting the conditional expectation of a normally distributed random variable gives:

$$V^{fb,\triangle} = \frac{(1 - F(x^{\triangle}))\mu\triangle t + \sigma\sqrt{\triangle t} f(x^{\triangle}) + e^{-r\triangle t} P_{IPO}V_{IPO}(1 - F(x^{\triangle}))}{1 - e^{-r\triangle t}(1 - P_{IPO})(1 - F(x^{\triangle}))}.$$

Again, using the fact that $F(x^{\triangle}) = o(\triangle)$ one obtains the first best value of the project:

$$\lim_{\triangle \to 0} V^{fb,\triangle} = \frac{\lambda V_{IPO} + \mu}{\lambda + r}.$$

■

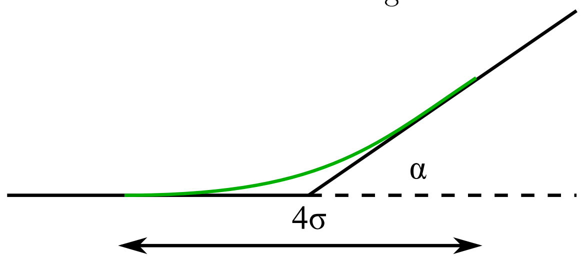
Proof of Corollary 5. First, consider the case $\varphi = 1$ and $\lambda \to 0$. The lemma above shows that the project is not terminated unless $C^E + C^M = 0$. Hence, if the investor provides an initial cash position of $C_0^E + C_0^M$ to the monitor and the entrepreneur and requires them to pay $\mu\Delta$ each period, their cash position always replicates their expected compensations. Thus, it is also optimal to terminate the project when the monitor and the entrepreneur run out of liquidity.

When $\varphi < 1$, both the termination and the compensation rules can be replicated by letting the monitor and the entrepreneur split a share φ of a larger pool with initial balance $(C_0^E + C_0^M)/\varphi$. Out of this pool the investor is paid $\mu\Delta$ each period and the project is terminated when this payment cannot be made.

If in addition $\lambda \gg 0$, i.e., IPO is likely, the investor also owns all the IPO proceeds. In other words, the investor holds debt of the pool with face value V_{IPO} payable at an IPO and a coupon $\mu\Delta$, and a share $1 - \varphi$ of the equity of the pool. Note that in this replication the monitor and the entrepreneur effectively own a share φ of the cash position of the project before the IPO. ■

Proof of Theorem 4. The proof of Theorem 4 consists of several steps. First, lemma 7 shows that the value function is continuous and has a bounded derivative. Lemma 8 shows that the value function has to be twice differentiable when $\Delta \to 0$ in the sense that approximating the value function with a twice differentiable function has $o(\Delta)$ error. A Taylor approximation of this function gives equations (8.3)-(8.4). Lemma 9 proves that the value function converges to the solution of the HJB equation. Finally, lemma (7) shows the properties of the value function.

Figure B.1: Local approximation of V^{tot}



$$\theta(x^M)\Delta = E[(C_t^E - C_{t-1}^E)^2] = \Delta[(1 - F(x^M))Var[x|x > x^M] - E[x - x^M|x > x^M]^2] + o(\Delta).$$

Lemma 7. *$V^{tot}(C^E)$ is continuous and for $C^E > 0$ has bounded derivatives*

Proof. First, I show continuity. Consider an arbitrary jump at V_0 and without loss of generality in the proof, assume that it is right continuous. For $C^E = V_0 - \varepsilon$ one can terminate the project with probability ε/V_0 and continue the project with probability $(V_0 - \varepsilon)/V_0$. This strategy satisfies all the constraints and yields V^{tot} arbitrarily close to $V^{tot}(V_0)$ for small ε. Hence, the jump in the total continuation value of the project is suboptimal, implying that the function is continuous. Note also that this argument implies that the derivative of the continuation value function exists.

To show the continuity of the derivative of V^{tot}, I make a similar argument. Denote the thresholds strategies for $V_0 + \varepsilon$, $a^{M,+}$ and $a^{D,+}$, and the strategies for V_0, $a^{M,0}$ and $a^{D,0}$. The direction of the change (for all histories of payoffs) is $(a^{M,0} - a^{M,+})$ and $(a^{D,+} - a^{D,0})$. Then, there exists an $\alpha > 0$ such that $a^{M,0} + \alpha(a^{M,0} - a^{M,+})$ and $a^{D,0} + \alpha(a^{D,+} - a^{D,0})$ imply value $V_0 - \varepsilon$ for the entrepreneur. Note also that for $\varepsilon \to 0$, $\alpha \to 0$. Hence, due to the fact that all distributions and costs are twice differentiable, the lower bound on the continuation value at V_0 is differentiable.

This argument also shows that the reason for the discontinuity of the slope of the continuation value is the discontinuity of the monitoring and the termination strategies.■

As the lemma above shows, it is possible that the total continuation value function has discontinuities in the first derivative, i.e. kinks. However, when $\Delta \to 0$ these kinks are not possible, as they would imply an infinite drift of the total continuation value at that point. An infinite drift cannot arise because the drift of the continuation value is always rV^{tot}. To see this, consider a local approximation of a function with a kink with a quadratic function as shown in Figure B.1.

Lemma 8. *The approximation error of V^{tot} with a quadratic function goes to zero as Δ goes* *to zero.*

Proof. Consider an arbitrary distance proportional to $\sigma\sqrt{\Delta}$, e.g., $4\sigma\sqrt{\Delta}$ and approximate V^{tot} smoothly as shown in figure B.1. If $\alpha \nrightarrow 0$ when $\Delta \to 0$, then the second derivative of this approximation goes to infinity. This implies that the expected rate of change of V^{tot} increases to infinity, a contradiction. Thus, $\alpha \to 0$. Moreover, the approximation error goes to zero as $\alpha \to 0$. ■

Lemma 9. *Let $T^{\triangle}$ be a sequence of contraction mapping operators with modulus $\beta^{\Delta} = 1 - r\Delta + o(\Delta)$, $1 > r > 0$ and fixed points denoted x^{Δ}. If this sequence converges to some operator T^0 in the sense $|T^{\triangle}x - T^0x| = o(\Delta)$, i.e. it converges faster than Δ, and operator T^0 has a fixed point x^0, then:*

$$\lim_{\Delta \to 0} |x^{\Delta} - x^0| = 0,$$

i.e. the fixed points of these operators converge to x^0.

Proof. $|x^{\Delta} - x^0| = |T^{\Delta}x^{\Delta} - T^0x^0| \leq |T^{\Delta}x^{\Delta} - T^{\Delta}x^0| + |T^{\Delta}x^0 - T^0x^0| \leq \beta^{\Delta}|x^{\Delta} - x^0| + |T^{\Delta}x^0 - T^0x^0|$, where the first inequality is a triangular inequality and the second inequality uses the contraction mapping property. Rearranging terms gives:

$$|x^{\Delta} - x^0| \leq \frac{1}{1 - \beta^{\Delta}}|T^{\Delta}x^0 - T^0x^0| \to 0.$$

■

Note that this also shows that operator T^0 cannot have two fixed points, because this would violate the convergence result above. Thus, it is left to prove that equation (8.5) has a unique solution.

Lemma 10. *The solution to (8.2) exists, is unique and concave, i.e. $V_{11}^{tot} < 0$.*

Proof. Restrict the set of monitoring strategies to $x^M < \overline{x}$, so that $\theta(x^M)$ is bounded from below by $\theta(\overline{x})$, and rewrite equation (8.5) so that V_{11}^{tot} is on the left hand side:

$$V_{11}^{tot}(C^E) = 2 \min_{x^M < \overline{x}} [\frac{F(x^M)c - \mu - \lambda V_{IPO} + V^{tot}(C^E)(r + \lambda) - V_1^{tot}(C^E)rC^E}{\theta(x^M)}].$$

The function on the right hand side has bounded derivatives with respect to x^M, V^{tot} and V_1^{tot}. Hence, it is Lipschitz continuous and the solution to the problem above with some initial conditions $V^{tot}(0)$ and $V_1^{tot}(0)$ exists and is unique.

Due to the fact that monotone value functions with the properties $V^{tot}(C^E) < V^{fb} = (\mu + \lambda V_{IPO})/(r + \lambda)$ and $V_1^{tot} \geq 0$ converge uniformly to the solution of this differential equation, this solution must have this properties as well. Using these properties and substituting $x^M = -\infty$ in the equation above yields that the function is concave everywhere:

$$V_{11}^{tot}(C^E) \leq \frac{2}{\sigma^2}(r+\lambda)(-\frac{\mu + \lambda V_{IPO}}{r+\lambda} + V^{tot}(C^E) - \frac{V_1^{tot}(C^E) r C^E}{r+\lambda}) < 0.$$