A Mean-Variance Benchmark for Household Portfolios over the Life Cycle

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Keywords: Life-cycle portfolio decisions, human capital, housing, stock market participation, growth/value tilts

JEL subject codes: G11, D15

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1 Introduction

The mean-variance portfolio theory of Markowitz (1952; 1959) is a milestone of modern finance and remains pivotal in both investment teaching and practical investment decision making. If investment opportunities are constant over time, the dynamic extension of Merton (1969; 1971) leads, quite remarkably, to the same optimal portfolio as Markowitz' static model. Markowitz' approach is, however, considered unsuitable for households' portfolio decisions over the life cycle. Such problems are studied in Merton-type dynamic models, but realistic specifications are solved by complex numerical solution techniques, which blurs economic insights.

This paper extends Markowitz' traditional mean-variance setting to labor income by incorporating human capital as an innate, illiquid asset. By maximizing the mean-variance utility of total wealth instead of just financial wealth, we can easily derive the optimal financial portfolio, i.e. how the current financial wealth should be allocated to different financial assets. In the absence of portfolio constraints, the optimal portfolio equals the traditional Markowitz-Merton portfolio scaled by the ratio of human capital to financial wealth and then adjusted for the extent to which the human capital resembles the financial assets, which is primarily determined by the income-asset correlations. Life-cycle variations in household portfolios are generated from this simple model by varying the magnitude of human capital relative to financial wealth. Compared to dynamic utility maximization, the mean-variance approach clarifies the economic forces at play, easily accommodates relevant portfolio constraints, and is comprehensible for a wider audience.

The extended Markowitz model accentuates two key determinants of life-cycle variations in individuals' optimal financial portfolios: the size of human capital relative to financial wealth and the correlation of labor income with available financial assets. An individual's human capital is unobservable and often calculated crudely by discounting expected income by the riskfree rate. We explain how to incorporate mortality risk and income uncertainty in a way that still leads to a simple formula for human capital. In an illustration, we estimate that the ratio of human capital to financial wealth for the median US worker is as large as 55 at age 25 and then gradually decreasing over life.

Based on survey or registry data, an existing literature has identified patterns in individuals' labor income growth and volatility over the life cycle and also estimated the correlation of income with the stock market (e.g. Attanasio and Weber, 1995; Campbell and Viceira, 2002; Cocco, Gomes, and Maenhout, 2005; Guvenen, Karahan, Ozkan, and Song, 2019). Our method is well-suited to handle several asset classes, but the correlations of individual income with the returns on those asset classes are unknown from the existing literature. Given both the methodological focus of the present paper and the massive challenges in handling individual income data, an econometrically rigorous large-scale analysis of how income-asset correlations and other relevant model inputs vary across the population is beyond the scope of the paper. We emphasize that the same asset-income correlations have to be estimated whether the portfolio decisions are determined using this paper's extended Markowitz approach or the Merton-style multi-period dynamic programming approach. As an indication, Table 1 shows estimates of the correlation between the annual US aggregate real income growth per capita and annual real returns on various asset classes. Note that, reflecting some of the empirical challenges, the correlation estimates depend somewhat on the sample period and also on whether the annual income growth is calculated using Q4-Q4 or Q1-Q1 changes, and it is not obvious which estimates to prefer.[1] Per capita income has a significant positive correlation with house prices and all the considered stock portfolios—at least if we ignore the right-most column—whereas the correlation with bond returns is close to zero. The standard deviation of annual income changes is 2% for the aggregate measure, which is significantly smaller than the 10-20% estimated for individual labor income in other studies. This reflects a sizeable idiosyncratic component which suggests that the typical correlation between individual income and asset returns is lower than the estimates in the table, although a considerable variation in income-asset correlations across individuals should be expected.[2]

We first derive the optimal unconstrained portfolio in a general setting with human capital and characterize the efficient frontier. Then we consider three applications of the extended mean-variance approach. In the first application, we revisit the classical framework in which the household can invest only in a riskfree asset (a bond) and the stock market index. Without formal modeling, Jagannathan and Kocherlakota (1996) illustrate that the bond-stock allocation depends on the magnitude of the human capital relative to financial wealth and on the extent to which the human capital resembles the bond and the stock. Bodie, Merton, and Samuelson (1992) confirm these insights in a stylized continuous-time model. Carefully modeling life-cycle income patterns, Cocco et al. (2005) solve numerically for the bond-stock allocation through life assuming the household is prohibited from borrowing. In their baseline parametrization, human capital is more bond-like than stock-like so young households—having large human capital relative

[1] The choice between Q4-Q4 and Q1-Q1 changes depend on precisely how the income is reported, when during the quarter the income is available to households, etc. Since the estimation using aggregate income is used only to get an impression of the relevant correlation values, we do not pursue this further. An annual income growth measure is also available in NIPA, but seems to be a simple average of the quarterly growth rates within the year, and thus not ideal in a correlation estimation.

[2] For example, Davis and Willen (2000) find that—depending on the individual's sex, age, and educational level—the correlation between aggregate stock market returns and labor income shocks is between -0.25 and 0.3, while the correlation between industry-specific stock returns and labor income shocks is between -0.4 and 0.1.

	Q1-Q1 income change		Q4-Q4 income change	
	1947-2018	1980-2018	1947-2018	1980-2018
House prices	0.36	0.40	0.22	0.32
10Y T-bonds	0.02	-0.01	0.08	0.15
Stock market	0.34	0.29	0.17	-0.03
Growth stocks	0.39	0.27	0.17	-0.04
Neutral stocks	0.35	0.35	0.19	0.03
Value stocks	0.36	0.30	0.26	0.08
Small stocks	0.31	0.27	0.14	-0.12
Medium stocks	0.35	0.27	0.15	-0.14
Large stocks	0.39	0.32	0.21	0.03

Table 1: Correlations between aggregate income and asset class returns. We use the aggregate disposable personal income per capita in the U.S. (in chained 2012 dollars, i.e., inflation adjusted) from the NIPA tables published by the Bureau of Economic Analysis (Table 2.1, line 39) at `https://apps.bea.gov/itable/index_nipa.cfm`. In the left part income growth is measured from Q1 this year to Q1 next year, in the right part growth from Q4 previous year to Q4 this year. For house prices we use the U.S. real home price index published on Professor Robert Shiller's homepage `http://www.econ.yale.edu/~shiller`. Annual nominal returns on 10-year Treasury bonds are downloaded from Professor Aswath Damodaran's homepage `http://pages.stern.nyu.edu/~adamodar/`. Annual nominal returns on the full US stock market as well as on three portfolios sorted on either book-to-market or market capitalization (value-weighted returns) are taken from Professor Kenneth French's homepage `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french`. We deflate returns by the Consumer Price Index published by the Bureau of Labor Statistics. Data were retrieved Jan. 30, 2020.

to financial wealth—should hold 100% of their financial wealth in stocks and only later in life gradually include bonds in their portfolio.[3] The same conclusions follow from our much more transparent mean-variance approach.

The second application expands the investment menu by residential real estate, the largest tangible asset for many households (Guiso and Sodini, 2013; Badarinza, Campbell, and Ramadorai, 2016). Housing is typically excluded from theoretical portfolio studies because of the added complexity, but it can easily be incorporated in the mean-variance setting. Given historical prices and rents, real estate is a fairly attractive investment in itself. The attractiveness is amplified by the ability to take out a loan using the house as collateral, in contrast to stock investments which do not facilitate borrowing to the same extent, if at all. The credit access is particularly important for young and risk-tolerant households who want to lever up their investments in light of their sizeable bond-like human capital. In a numerically solved, stylized multi-period model, Cocco (2005) finds that housing crowds out stock holdings which, together with a sizable stock market

[3] Other multi-period models with human capital were studied by Viceira (2001), Benzoni, Collin-Dufresne, and Goldstein (2007), Munk and Sørensen (2010), and Lynch and Tan (2011), among others.

entry cost, can explain stock market non-participation.[4] Our transparent life-cycle mean-variance model reaches the same conclusions (without stock market costs). We also find that the stock portfolio weight can be increasing or hump-shaped over life. Furthermore, the stock weight can be non-monotonic in risk aversion because agents compare stocks to a (more risky) levered house investment so, when increasing the risk aversion, agents shift from a levered house investment to stocks and eventually to the riskfree asset.

The third application investigates how households should invest in value and growth stocks over the life cycle. Despite the immense focus on value investing among practitioners, the optimal value/growth portfolio tilts have not been studied in a formal life-cycle model, but only in a few papers ignoring both human capital and housing. Jurek and Viceira (2011) consider a discrete-time setting allowing for return-predicting variables. They find that short [long] horizon investors tilt their portfolios towards value [growth] stocks, and they attribute this horizon dependence to intertemporal hedging motives and the observation that growth stocks hedge bad times better than value stocks. In a related continuous-time model Larsen and Munk (2012) report that the hedging demands are small so that the optimal value/growth/market allocation is almost constant across horizons. However, both papers ignore human capital which is a key driver of life-cycle variations in portfolios. In a rich Swedish data set Betermier, Calvet, and Sodini (2017) find that value investors tend to be older than the average participant and have low human capital, low income risk, low leverage, and high financial wealth. We show that our simple life-cycle mean-variance model can support these findings, but also that optimal portfolios are highly sensitive to the assumed values of correlations. Even small differences between the (hard-to-estimate) correlations of two individuals' income with different financial assets may help explaining why the individuals choose very different financial portfolios.

The mean-variance approach has its limitations, of course. It suggests that households determine their investment strategy over life by solving at regular time intervals a simple one-period optimization problem. The problem solved at any given date refers to future periods only through the human capital. Since the dynamic programming principle is not invoked, the derived investment strategy is generally not maximizing the life-time expected utility. However, when the relevant return moments are time invariant and portfolios are unconstrained, the mean-variance strategy does coincide with the dynamically optimal strategy for a power utility investor if human capital is either riskfree or perfectly spanned by risky assets. In the more realistic case of a risky and imperfectly spanned human capital, our simple strategy leads to a very small utility loss (measured by certainty equivalent of

[4] Other multi-period models with housing were studied by Yao and Zhang (2005), Kraft and Munk (2011), Fischer and Stamos (2013), Corradin, Fillat, and Vergara-Alert (2014), and Yogo (2016), among others.

wealth) compared to the unknown optimal strategy.[5]

The extended mean-variance approach accommodates time-varying moments as, for example, implied by stock return predictability simply by applying different values of the moments at different points in time. While the approach disregards the intertemporal hedging demand typically generated by such variations in investment opportunities (Merton, 1971, 1973), many studies conclude that the intertemporal hedging demand is typically very small compared to the speculative mean-variance demand (see, e.g., Aït-Sahalia and Brandt, 2001; Ang and Bekaert, 2002; Chacko and Viceira, 2005; Gomes, 2007; Larsen and Munk, 2012), and this is especially so if parameter uncertainty and transaction costs are taken into account (see, e.g., Barberis, 2000; Pastor and Stambaugh, 2012).

Only few papers include human capital or housing in a mean-variance setting. Mayers (1972) derives an equation for the optimal financial portfolio of a mean-variance investor with a nonmarketable asset such as human capital. While very similar to the equation we derive below, his equation does not directly show the importance of the relative size of human capital to financial wealth. Based on a log-linearized approximation of the budget constraint Weil (1994) obtains a related expression for the optimal investment in a single risky asset for a power utility investor with human capital. Both Mayers and Weil focus on the asset pricing consequences of human capital. Neither of them consider the implications for household portfolios and the life-cycle variations therein, which is the aim of this paper.

Flavin and Yamashita (2002) and Pelizzon and Weber (2009) include housing in a mean-variance framework, but assume the housing investment position is exogenously given—as the human capital in the current paper—and ignore human capital. In reality households change their housing investment and consumption in response to significant changes in labor income or financial wealth, and the inclusion of human capital is crucial to understand life-cycle variations in portfolios and consumption. Finally, note that while the title of this paper resembles that of Cochrane (2014), the focus is very different. Cochrane shows how a mean-variance approach to payoff streams leads to an interesting characterization of the optimal payoff stream. However, his approach is generally not explaining which investment strategy that generates the optimal payoff stream, and he is not explicitly addressing life-cycle variations in household portfolios.

The paper is organized as follows. Section 2 sets up the general mean-variance framework with human capital, presents a general explicit formula for the optimal unconstrained portfolio, and derives properties of the efficient frontier. Section 3 explains how to value human capital and how large a share human capital represents of total wealth at different

[5]Some papers have considered dynamic portfolio problems with a mean-variance criterion. Such problems suffer from time inconsistency issues and lead to rather complicated optimal investment strategies, cf., e.g., Basak and Chabakauri (2010) and Björk and Murgoci (2014).

ages which is a key quantity for portfolio decisions. The subsequent sections consider specific cases. The basic life-cycle stock-bond allocation problem is discussed in Section 4, after which Section 5 adds housing to the model. Within the model with human capital and housing, Section 6 investigates how households should tilt their stock portfolios towards value or growth stocks. Finally, Section 7 summarizes our findings.

2 A mean-variance model with human capital

This section explains how human capital can be included in the mean-variance framework and thus how this one-period setting can provide a life-cycle perspective on portfolio decisions. Let F denote the financial wealth and L the human capital ("L" for labor income) of the agent so that total wealth is the sum $W = F + L$. The current date is labeled as time t. The agent makes a buy-and-hold investment decision for a period of a given length. The end of the period is labeled as time $t + 1$. We assume that the agent has the traditional mean-variance objective

$$\max \left\{ \mathrm{E}_t \left[\frac{W_{t+1}}{W_t} \right] - \frac{\gamma}{2} \operatorname{Var}_t \left[\frac{W_{t+1}}{W_t} \right] \right\}, \tag{1}$$

where $\gamma > 0$ measures the agent's relative risk aversion. The agent cares about expectation and variance of the return on total wealth, not just on financial wealth.

The agent decides on the portfolio of traded assets to hold over the period. Suppose that the agent can invest in a riskfree asset with rate of return r_f over the period and in a number of risky assets with rates of return given by the vector $\boldsymbol{r}$. The expected rates of return are represented by $\boldsymbol{\mu}$ and the variance-covariance matrix by $\underline{\underline{\Sigma}}$. Let $\boldsymbol{\pi}_t$ denote the vector of fractions of financial wealth invested in the risky assets. The financial wealth not invested in the risky assets, $F_t(1 - \boldsymbol{\pi}_t \cdot \mathbf{1})$, is invested in the riskfree asset. Here, we let $\mathbf{1}$ denote a vector of ones and write vector products using a center dot as in $\boldsymbol{\pi} \cdot \boldsymbol{r}$.

Assumption 1 *The matrix $\underline{\underline{\Sigma}}$ is positive definite and $\boldsymbol{\mu} \neq r_f \mathbf{1}$.*

The end-of-period total wealth is

$$W_{t+1} = F_t \left(1 + r_f + \boldsymbol{\pi}_t \cdot (\boldsymbol{r} - r_f \mathbf{1})\right) + L_t(1 + r_L),$$

where r_L is the rate of return on human capital with expectation μ_L and standard deviation σ_L. Consequently,

$$\begin{aligned}\frac{W_{t+1}}{W_t} &= \frac{F_t}{F_t+L_t}\left(1+r_f+\boldsymbol{\pi}_t\cdot(\boldsymbol{r}-r_f\mathbf{1})\right)+\frac{L_t}{F_t+L_t}(1+r_L)\\ &= \frac{1}{1+\ell_t}\left(1+r_f+\boldsymbol{\pi}_t\cdot(\boldsymbol{r}-r_f\mathbf{1})\right)+\frac{\ell_t}{1+\ell_t}(1+r_L),\end{aligned} \tag{2}$$

where we have introduced the human-to-financial wealth ratio $\ell_t = L_t/F_t$. Hence,

$$\begin{aligned}\mathrm{E}_t\left[\frac{W_{t+1}}{W_t}\right] &= \frac{1}{1+\ell_t}\left(1+r_f+\boldsymbol{\pi}_t\cdot(\boldsymbol{\mu}-r_f\mathbf{1})\right)+\frac{\ell_t}{1+\ell_t}(1+\mu_L),\\ \mathrm{Var}_t\left[\frac{W_{t+1}}{W_t}\right] &= \frac{1}{(1+\ell_t)^2}\,\boldsymbol{\pi}_t\cdot\underline{\underline{\Sigma}}\boldsymbol{\pi}_t+\frac{\ell_t^2}{(1+\ell_t)^2}\,\sigma_L^2+2\frac{\ell_t}{(1+\ell_t)^2}\,\boldsymbol{\pi}_t\cdot\mathrm{Cov}_t[\boldsymbol{r},r_L],\end{aligned}$$

where $\mathrm{Cov}[\boldsymbol{r},r_L]$ is the vector of covariances between the returns on the individual risky assets and the return on human capital. The objective (1) is thus equivalent to

$$\max_{\boldsymbol{\pi}_t}\left\{\boldsymbol{\pi}_t\cdot(\boldsymbol{\mu}-r_f\mathbf{1})-\frac{\gamma}{2}\frac{1}{1+\ell_t}\left[\boldsymbol{\pi}_t\cdot\underline{\underline{\Sigma}}\boldsymbol{\pi}_t+2\ell_t\boldsymbol{\pi}_t\cdot\mathrm{Cov}_t[\boldsymbol{r},r_L]\right]\right\}. \tag{3}$$

The solution of the unconstrained optimization problem (3) is straightforward and stated in the following theorem, which also summarizes some notable less straightforward properties of the solution. First, we define the auxiliary constants

$$\begin{aligned}A &= (\boldsymbol{\mu}-r_f\mathbf{1})\cdot\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu}-r_f\mathbf{1}\right), \quad & B &= (\boldsymbol{\mu}-r_f\mathbf{1})\cdot\underline{\underline{\Sigma}}^{-1}\,\mathrm{Cov}_t[\boldsymbol{r},r_L],\\ C &= \mathrm{Cov}_t[\boldsymbol{r},r_L]\cdot\underline{\underline{\Sigma}}^{-1}\,\mathrm{Cov}_t[\boldsymbol{r},r_L], \quad & D &= AC-B^2.\end{aligned}$$

Assumption 1 implies that $A>0$ and that $\underline{\underline{\Sigma}}^{-1}$ exists and is positive definite. We have $C\geq 0$ with $C>0$ if $\mathrm{Cov}_t[\boldsymbol{r},r_L]\neq\mathbf{0}$. And since

$$AD = \left(B(\boldsymbol{\mu}-r_f\mathbf{1})-A\,\mathrm{Cov}_t[\boldsymbol{r},r_L]\right)\cdot\underline{\underline{\Sigma}}^{-1}\left(B(\boldsymbol{\mu}-r_f\mathbf{1})-A\,\mathrm{Cov}_t[\boldsymbol{r},r_L]\right)\geq 0,$$

we have $D\geq 0$, and provided that $B(\boldsymbol{\mu}-r_f\mathbf{1})\neq A\,\mathrm{Cov}_t[\boldsymbol{r},r_L]$ we even have $D>0$. This condition is violated, and thus $D=0$, if $\boldsymbol{\mu}-r_f\mathbf{1}=k\,\mathrm{Cov}_t[\boldsymbol{r},r_L]$ for some scalar k, which is the case when (i) the agent trades in only one risky asset, or (ii) all risky assets have the same excess expected return, same standard deviation, and same covariance with human capital, and all pairs of risky assets have the same correlation.

Theorem 1 *(a) The optimal unconstrained portfolio is*

$$\boldsymbol{\pi}_t^* = \frac{1}{\gamma}(1+\ell_t)\,\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu}-r_f\mathbf{1}\right)-\ell_t\,\underline{\underline{\Sigma}}^{-1}\,\mathrm{Cov}_t[\boldsymbol{r},r_L]. \tag{4}$$

(b) *The expectation and variance of the rate of return on the optimal portfolio are*

$$\mathrm{E}_t[r_{t+1}] = r_f + \frac{1}{\gamma}(1+\ell_t)A - \ell_t B, \tag{5}$$

$$\mathrm{Var}_t[r_{t+1}] = \frac{1}{\gamma^2}(1+\ell_t)^2 A + \ell_t^2 C - \frac{2}{\gamma}(1+\ell_t)\ell_t B. \tag{6}$$

If $\gamma > A/B$ (resp., $\gamma < A/B$), then $E_t[r_{t+1}]$ is decreasing (increasing) in ℓ_t and the largest (smallest) expected return for a fixed γ over all ℓ_t-values is $r_f + (A/\gamma)$.

(c) *The set of portfolios chosen by agents with the same $\ell_t > 0$, but different levels of risk aversion $\gamma > 0$, satisfy*

$$\mathrm{Var}_t[r_{t+1}] = \frac{(\mathrm{E}_t[r_{t+1}] - r_f)^2}{A} + \ell_t^2 \frac{D}{A}. \tag{7}$$

In the (standard deviation, mean)-diagram, these portfolios form, if $D > 0$ and $\ell_t > 0$, a hyperbola having $(\ell_t\sqrt{C}, r_f - \ell_t B)$ as an end point.

(d) *The set of portfolios chosen by agents with the same level of risk aversion $\gamma > 0$, but different values of the human-financial wealth ratio ℓ_t, satisfy*

$$\begin{aligned}\mathrm{Var}_t[r_{t+1}] = \frac{1}{A}\left(1 + \frac{\gamma^2 D}{(A-\gamma B)^2}\right)\left(\mathrm{E}_t[r_{t+1}] - r_f\right)^2 \\ - \frac{2\gamma D}{(A-\gamma B)^2}\left(\mathrm{E}_t[r_{t+1}] - r_f\right) + \frac{AD}{(A-\gamma B)^2}.\end{aligned} \tag{8}$$

In the (standard deviation, mean)-diagram, these portfolios form, if $D > 0$, a hyperbola having $(\frac{\sqrt{A}}{\gamma}, r_f + \frac{A}{\gamma})$ as an end point.

Appendix A provides a proof. Concerning (c) and (d), note that if $\ell_t = 0$ or $D = 0$, the optimal portfolios for a fixed ℓ_t or a fixed γ trace out a wedge, i.e., a pair of straight lines meeting at $(0, r_f)$ with slopes of $\pm\sqrt{A}$. In Section 4 we consider the case where only one risky asset is traded, which implies $D = 0$. If multiple risky assets are traded, but their expected returns, standard deviations, and correlations with human capital across assets are close, then D will be close to zero, and the hyperbolas are close to straight lines. This turns out to be so in the baseline parametrization of the problem studied in Section 5.

The ratio ℓ_t of human capital to financial wealth is clearly crucial for the optimal portfolio. This ratio is typically very large for young individuals and small for older individuals, and the variations in this ratio over life is arguably the most important generator of age-dependence in portfolio decisions. By applying the simple setting above for different values of ℓ_t, we have effectively introduced a life-cycle perspective on portfolio choice.

We can rewrite the optimal portfolio (4) as

$$\boldsymbol{\pi}_t^* = \frac{1}{\gamma}\left(1+\ell_t\right)\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu}-r_f\mathbf{1}\right)\boldsymbol{\pi}_{\tan} - \ell_t\,\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\operatorname{Cov}_t[\boldsymbol{r}, r_L]\boldsymbol{\pi}_{\mathrm{hdg}},$$

where

$$\boldsymbol{\pi}_{\tan} = \frac{1}{\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu}-r_f\mathbf{1}\right)}\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu}-r_f\mathbf{1}\right), \quad \boldsymbol{\pi}_{\mathrm{hdg}} = \frac{1}{\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\operatorname{Cov}_t[\boldsymbol{r}, r_L]}\underline{\underline{\Sigma}}^{-1}\operatorname{Cov}_t[\boldsymbol{r}, r_L]$$

are the tangency portfolio and income-hedge portfolio, where the latter simply adjusts for the extent to which the human capital replaces investments in the risky assets.

Appendix B shows that (4) is identical to the dynamically optimal portfolio strategy of a power utility investor in a continuous-time setting with constant investment opportunities provided that labor income is either riskfree or spanned by traded assets. With unspanned income risk, the dynamically optimal strategy is unknown for power utility investors. The strategy suggested by our method is then very similar to that derived by the (more complicated) combined analytical-numerical method of Bick, Kraft, and Munk (2013). Their approach approximates both the investment and consumption strategy, and they show that the utility generated by the approximate strategy comes very close to the utility of the unknown optimal strategy as indicated by a tiny certainty equivalent wealth loss. Also note that by applying a log-linearization approach, Viceira (2001) finds an approximate formula for the optimal stock share that has a form similar to (4).

3 The size of human capital over the life cycle

The previous section highlights the importance of human capital for portfolio decisions. But how large is human capital relative to financial wealth at different stages of life? Data on financial wealth and labor income over life can be found, for example, in the Survey of Consumer Finances (SCF) in the United States. However, to calculate human capital at a certain age, we need to establish how to discount future labor income. For that purpose we set up a discrete-time model in which income is paid out at the end of each year.

Suppose that the year t log returns on the n traded risky assets are of the form

$$\ln R_{it} = \mu_i - \frac{1}{2}\|\boldsymbol{\sigma}_i\|^2 + \boldsymbol{\sigma}_i\cdot\boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\varepsilon}_t$ follows the n-dimensional standard normal distribution. Let $\underline{\sigma}$ be the $n\times n$ matrix with rows $\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_n$ and let $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_n)^\top$. The log riskfree rate is the constant r.

If alive at the end of year t, the agent receives an income of Y_t. We assume that

$$Y_t = Y_{t-1} \exp\left\{\mu_Y(t) - \frac{1}{2}\sigma_Y(t)^2 + \sigma_Y(t)\boldsymbol{\rho}_Y \cdot \boldsymbol{\varepsilon}_t + \sigma_Y(t)\sqrt{1 - \|\boldsymbol{\rho}_Y\|^2}\varepsilon_{Yt}\right\},$$

where ε_{Yt} follows a one-dimensional standard normal distribution independent of $\boldsymbol{\varepsilon}_t$, and where $\mu_Y(t)$ and $\sigma_Y(t)$ are deterministic functions of time. Then

$$\mathrm{E}_{t-1}[Y_t] = Y_{t-1}e^{\mu_Y(t)}, \quad \mathrm{Var}_{t-1}[\ln(Y_t/Y_{t-1})] = \sigma_Y(t)^2, \quad \mathrm{Corr}_{t-1}[\ln(Y_t/Y_{t-1}), \ln \boldsymbol{R}_t] = \boldsymbol{\rho}_Y.$$

Unless $\|\boldsymbol{\rho}_Y\| = 1$ or $\sigma_Y(t) = 0$, the income carries non-spanned risk. Let $\exp(-\nu(t))$ be the probability that the agent survives year t given she was alive at the end of year $t-1$, so that $\nu(t)$ represents the mortality rate, assumed deterministic. We assume that the agent at most lives until the end of year $T = 100$ so that $\nu(T+1) = \infty$ and the final income received, if still alive, is Y_T.

Suppose that the agent associates a price of risk of λ_Y with the unspanned income shock ε_Y and thus evaluates future income using the period-by-period state-price deflator

$$\frac{\zeta_t}{\zeta_{t-1}} = \exp\left\{-r - \frac{1}{2}\left(\|\boldsymbol{\lambda}\|^2 + \lambda_Y^2\right) - \boldsymbol{\lambda} \cdot \boldsymbol{\varepsilon}_t - \lambda_Y \varepsilon_{Yt}\right\}.$$

Here $\boldsymbol{\lambda}$ captures the market price of risk associated with $\boldsymbol{\varepsilon}$ since

$$\mathrm{E}_{t-1}[\ln R_{it}] - r = -\mathrm{Cov}_{t-1}\left[\ln \zeta_t, \ln R_{it}\right] - \frac{1}{2}\mathrm{Var}_{t-1}[\ln R_{it}] = \boldsymbol{\sigma}_i \cdot \boldsymbol{\lambda} - \frac{1}{2}\|\boldsymbol{\sigma}_i\|^2.$$

We assume that $\boldsymbol{\lambda}$ and λ_Y are constant over time. We show in Appendix C that the total human capital at the end of year t, excluding the income just received, is then

$$L_t = Y_t M(t), \tag{9}$$

where

$$M(t) = \sum_{k=1}^{T-t} \exp\left\{-\sum_{s=t+1}^{t+k} r_m(s)\right\},$$
$$r_m(t) = r + \nu(t) - \mu_Y(t) + \sigma_Y(t)\left[\boldsymbol{\rho}_Y \cdot \boldsymbol{\lambda} + \sqrt{1 - \|\boldsymbol{\rho}_Y\|^2}\lambda_Y\right].$$

We can think of $r_m(t)$ as the risk-, growth-, and mortality-adjusted discount rate for future income. The multiplier M is easily calculated by backwards recursion using $M(T) = 0$

and $M(t) = e^{-r_m(t+1)}(M(t+1)+1)$. The expected future human capital is

$$\mathrm{E}_t[L_{t+k}] = M(t+k)\mathrm{E}_t[Y_{t+k}] = M(t+k)Y_t \exp\left\{\sum_{s=t+1}^{t+k} \mu_Y(s)\right\}. \tag{10}$$

Next, we set up an example illustrating how human capital and its share of total wealth vary with age. Labor income is generally found to be hump shaped over working life: rapidly increasing in early years, then flattening out with a peak at an age of 45-55, and then declining somewhat until retirement. Moreover, the life-cycle pattern over the working phase is well approximated by the exponential of a polynomial of order 3 or slightly higher. For example, these facts have been shown for various US data sets by Attanasio and Weber (1995), Cocco et al. (2005), and Guvenen et al. (2019). Consistent with these findings, we model the expected income over working life via

$$\ln\left(\frac{\mathrm{E}_{t_0}[Y_t]}{Y_{t_0}}\right) = a_1(t-t_0) + a_2(t-t_0)^2 + a_3(t-t_0)^3, \quad t = t_0, \dots, t_R - 1,$$

which determines $\mu_Y(t)$ for $t \le T_R - 1$. We set the initial adult age to $t_0 = 24$ and the retirement age to $t_R = 65$. Hence, the agent starts at her 25th birthday, faces a 40-year working period, retires when turning 65, and lives on at most until the day turning 100. The first-year retirement income is expected to be a fraction $\Upsilon = 0.85$ of the income in the preceding year, in line with final-salary pension schemes and a common assumption in the life-cycle literature (Cocco et al., 2005; Lynch and Tan, 2011). Hence, $\mu_Y(t_R) = \ln(\Upsilon)$. In retirement, labor (pension) income is expected to stay the same, i.e., $\mu_Y(t) = 0$, $t = t_R + 1, \dots, T$. We fix a_1, a_2, a_3 by requiring that (i) expected income peaks at the age $t_{\max} = 52$; (ii) expected income at the peak is $K_{\max} = 2.27$ times the initial income; and (iii) expected income just before retirement is $K_{\text{drop}} = 0.85$ times the peak income. Also these values comply with Guvenen et al. (2019) and other references given above. These choices imply that $a_1 = 5.6929 \times 10^{-2}$, $a_2 = -9.2946 \times 10^{-4}$, and $a_3 = -2.0746 \times 10^{-6}$. Measuring income and wealth in thousands of US dollars, we fix $Y_{t_0} = 15$, which seems reasonable for a 24-year old individual given the median before-tax family income of 35.1 for the age group "less than 35" in the 2010 SCF, cf. Bricker, Kennickell, Moore, and Sabelhaus (2012, Table 1). The red curve in the left panel of Figure 1 shows the resulting expected income path.

In order to calculate the discount rate $r_m(t)$, we fix the riskfree rate at $r = 1\%$, roughly the historical average of short-term real interest rates in the US. We derive the mortality intensity $\nu(t)$ from the life tables for the total US population as of 2009 with an

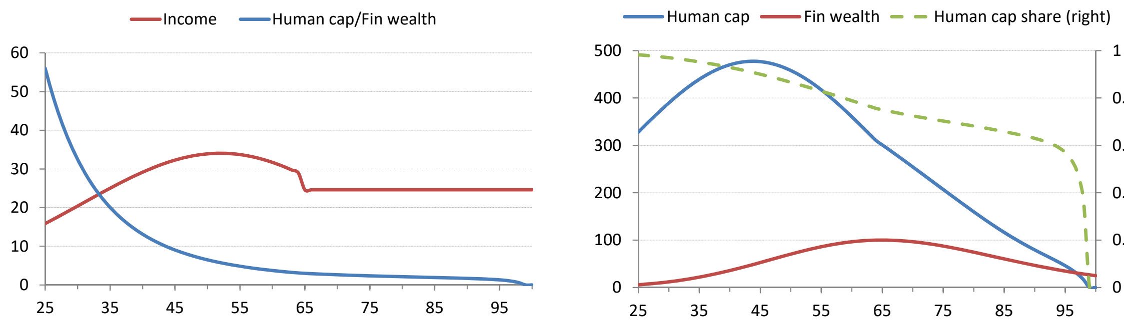


Figure 1: Labor income, human capital, and financial wealth over the life cycle. The left panel shows how expected income in thousand USD (red curve) and the ratio of expected human capital to financial wealth (blue) vary with age measured in years. In the right panel the expected human capital (blue) and the financial wealth (red) as functions of age are measured on the left axis in thousands of USD, whereas the expected human capital's share of total wealth (dashed) is measured on the right axis. The graphs are based on the assumptions and baseline parameter values described in the text.

imposed maximum age of 100.[6] We assume that labor income risk declines linearly from $\sigma_Y(25) = 0.3$ to $\sigma_Y(65) = 0.1$ and stays at that level through retirement. The decline over working life is empirically supported, e.g., by Guvenen et al. (2019, Fig. 3). The retirement income risk is motivated by continued business-related remuneration or uncertain health care costs affecting disposable income (De Nardi, French, and Jones, 2010). Furthermore, we suppose that a single risky asset (the stock index) is traded with a Sharpe ratio of $\lambda = (\mu - r)/\sigma = 0.3$ corresponding, e.g., to $\sigma = 20\%$ and $\mu = 7\%$. The income-stock correlation is assumed to be $\rho = 0.2$; this value is somewhat lower than the estimates 0.29 and 0.34 from Table 1 based on aggregate income changes (using Q1-Q1 changes), in order to reflect idiosyncratic risk in individual income, and the value is also in the range of existing estimates in the literature. Finally, we fix λ_Y by equating the time t present value of a fully unspanned income at time $t+1$ and the agent's certainty equivalent, which leads to $\lambda_Y = \frac{\gamma}{2}\sigma_Y(t+1)$. With $\gamma = 2$ and $\sigma_Y(t+1) = 0.2$, the average income volatility in the above parametrization, we obtain $\lambda_Y = 0.2$. We vary the value of λ_Y and selected other parameters below.

The resulting discount rate over life is seen as the solid black line in both panels of Figure 2. The left panel splits the discount rates into its components. Before retirement, the mortality rate is almost flat, so the shape of the discount rate is determined by how the expected growth rate and the risk adjustment linked to the volatility of income vary with age. In the baseline case, the expected income growth rate decreases more steeply than the risk adjustment, leading to an increasing discount rate. The peak at retirement is

[6] Published at the Centers for Disease Control and Prevention under the US Department of Health and Human Services, see http://www.cdc.gov/nchs/data/nvsr/nvsr62/nvsr62_07.pdf.

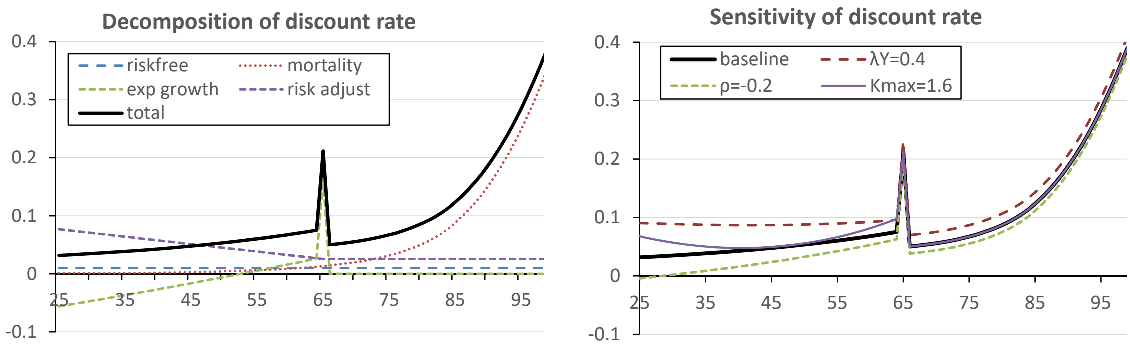


Figure 2: The risk-, growth-, and mortality-adjusted income discount rate. Both panels depict the income discount rate $r_m(t)$ as a function of age. The left panel assumes the baseline parameter values and shows the total discount rate $r_m(t)$ (solid black curve) as well as its four constituents, namely the riskfree rate r (dashed blue), the mortality rate $\nu(t)$ (dotted red), the expected income growth rate $\mu_Y(t)$ (dashed green), and the risk adjustment $\sigma_Y(t)[\rho\lambda + (1-\rho^2)^{1/2}\lambda_Y]$ (dashed purple). The right panel shows the total discount rate with the baseline parameters (solid black) as well as three cases in which the value of a single parameter is changed relative to the baseline value, namely the idiosyncratic income risk premium $\lambda_Y = 0.4$ (red dashed), the income-stock correlation $\rho = -0.2$ (green dotted), and the ratio of peak income to initial income $K_{\max} = 1.6$ (solid purple).

due to the expected income drop. In retirement, the shape of the discount rate is entirely determined by the mortality risk and therefore increases rapidly in the late years.

From the discount rates and the expected income, we compute the expected path of human capital using (10). As shown by the blue curve in the right panel of Figure 1, human capital is expected to grow from an initial level of around 315 (kUSD) to a peak at around 478 at age 44, after which it declines steadily mainly because of the shortening of the income-earning period.[7]

In order to estimate human capital's share of total wealth, we need the financial wealth over the life cycle. We assume a life-cycle pattern in financial wealth of the form

$$\ln\left(\frac{F(t)}{F(t_0)}\right) = b_1(t-t_0) + b_2(t-t_0)^2 + b_3(t-t_0)^3, \quad t = t_0, t_0+1, \dots, T.$$

We fix b_1, b_2, b_3 by requiring that (i) financial wealth peaks at retirement; (ii) financial wealth at the peak is $C_{\max} = 20$ times the initial financial wealth; and (iii) financial wealth at age 100 is $C_{\text{end}} = 5$ times the initial financial wealth. This implies that $b_1 = 0.16065$, $b_2 = -2.4865 \times 10^{-3}$, and $b_3 = 8.5675 \times 10^{-6}$. Fixing the initial level at $F(24) = 5$ (kUSD), we obtain the financial wealth path shown by the red curve in the right panel of Figure 1.

[7] In the spirit of Hall (1978) and others, Guiso and Sodini (2013) discount future income at the riskfree rate and thus ignore the riskiness and expected growth of income as well as mortality risk. Hence, they report a larger human capital that declines monotonically over life. However, the overall life-cycle pattern in the human capital share of total wealth they report is not markedly different from our Figure 1.

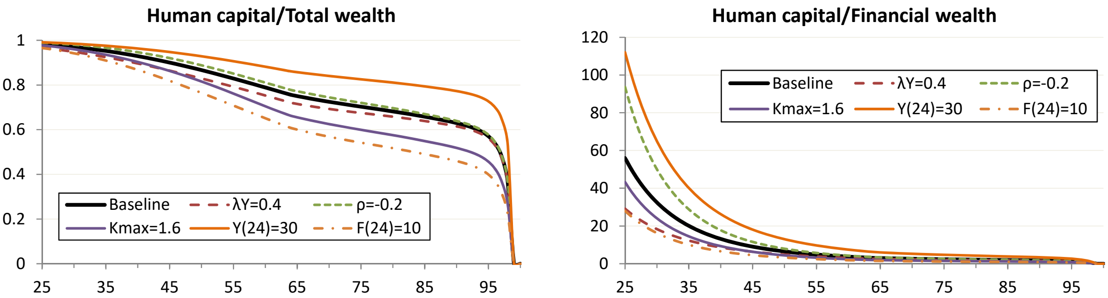


Figure 3: Human capital relative to wealth. The left panel shows how the expected human capital's share of total wealth varies with age, whereas the right panel illustrates the ratio of human capital to financial wealth as a function of age. In both panels the black curve corresponds to the baseline parameter values described in the text, whereas the other curves correspond to cases in which the value of a single parameter is changed relative to the baseline value, namely the idiosyncratic income risk premium $\lambda_Y = 0.4$ (red dashed curve), the income-stock correlation $\rho = -0.2$ (green dotted), the ratio of peak income to initial income $K_{\max} = 1.5$ (purple solid), the initial income $Y_{24} = 30$ thousand USD (orange solid), and the initial financial wealth $F(24) = 10$ thousand USD (dashed-dotted).

The shape and levels are in line with various empirical studies. For example, in the 2010 wave of the SCF, median family net worth is 12.4 kUSD for "less than 35" year old's and reaches around 200 kUSD at retirement (or maybe slightly later), cf. Bricker et al. (2012, Table 4). In retirement, individuals reduce financial wealth to finance consumption.

The dashed curve in the right panel of Figure 1 shows that the human capital's share of total wealth starts at around 98%, remains above 90% until age 46, above 80% until age 59, above 70% until age 76, and above 60% until age 94. The blue curve in the left panel shows the ratio of human capital to financial wealth, corresponding to ℓ in the previous section. This ratio starts at around 55 and declines smoothly over life. When studying specific models below, we report portfolios for $\ell = 1, 2, 5, 10, 20, 50$, which with the above baseline calculation roughly corresponds to ages of 97, 84, 55, 44, 35, and 26, respectively.

Income and wealth paths vary substantially across individuals as indicated, e.g., by the huge difference between means and medians of income and net worth at different ages in the SCF data (Bricker et al., 2012). Of course, if we scale either income or financial wealth up or down and fix the other, the human capital's share of total wealth changes. However, across individuals income and wealth often move together since higher-earning individuals tend to build more wealth. Hence, we expect less cross-sectional variation in the human capital's share of total wealth than seen in income or wealth.

Finally, we briefly consider the sensitivity of the above findings with respect to selected inputs. Figure 3 depicts the human capital over life as a fraction of total wealth (left panel) or financial wealth (right panel) for the baseline set of parameters explained above as well as five cases where one of the key parameter values is varied. First, if initial income is doubled to 30, income at every age and thus the human capital are also doubled, which of course induces a significant increase in the human capital's share of wealth at any age, cf. the solid orange curves. Second, doubling the initial financial wealth to 10 significantly reduces the human capital's share of wealth, cf. the dashed-dotted curve. In both of these cases, the income discount rate $r_m(t)$ is unaffected. The remaining three cases vary either $K_{\max}$, λ_Y, or ρ and thus affect the discount rate and therefore the human capital and its ratio to total or financial wealth. Third, if we keep the initial income level, but assume that the growth until the expected income peak is only half as big as in the baseline case ($K_{\max} = 1.6$ instead of 2.27), the income discount rate increases early in life and just before retirement as indicated in the right panel of Figure 2, and the human capital's share of wealth is obviously reduced as depicted in Figure 3 (see the solid purple curves). Fourth, if we double the idiosyncratic income premium λ_Y to 0.4, the income discount rate increases so that human capital is reduced (dashed red curves). Fifth, if we change the income-stock correlation from 0.2 to -0.2, the discount rate drops as future income is more valuable when negatively correlated with the stock market, so the human capital's share of wealth increases (dotted green curves). Note, however, that the life-cycle pattern in the human capital's share of wealth is little sensitive to these variations in inputs, and in all cases considered the human capital is the dominant component of total wealth up to age 85.

4 Stock-bond asset allocation

We first consider the case with only a single risky asset, the stock market index. From Theorem 1 the unconstrained optimal fraction of financial wealth invested in the stock is

$$\pi_t = \frac{\mu_S - r_f}{\gamma\sigma_S^2}(1+\ell_t) - \ell_t\frac{\rho_{SL}\sigma_L}{\sigma_S} = \frac{\mu_S - r_f}{\gamma\sigma_S^2} + \ell_t\left(\frac{\mu_S - r_f}{\gamma\sigma_S^2} - \frac{\rho_{SL}\sigma_L}{\sigma_S}\right). \tag{11}$$

The term $\frac{\mu_S - r_f}{\gamma\sigma_S^2}$ is the solution in absence of human capital, as is well-known from Markowitz' original analysis, and it is identical to the solution to Merton's intertemporal portfolio choice problem with constant investment opportunities. Human capital affects the optimal stock weight via the scaling term $1+\ell_t$ and through the term $\ell_t\rho_{SL}\sigma_L/\sigma_S$, which adjusts for the extent to which the human capital replaces a stock investment. Since agents combine the riskfree asset and a single risky asset, their choices trace out a wedge in the traditional standard deviation-mean diagram.

The upper panel of Table 2 illustrates the optimal portfolio over a one-year period for frequently used parameter values: $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, $\sigma_L = 10\%$, and $\rho_{SL} = 0.1$. The values of the income volatility σ_L and the income-stock correlation ρ_{SL}

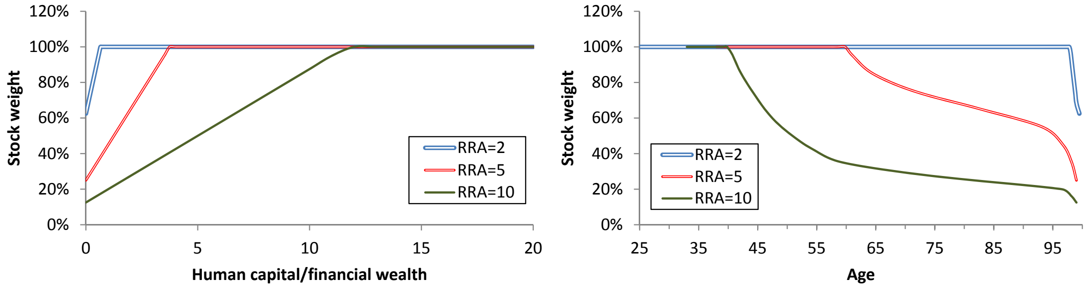


Figure 4: Optimal stock weight with human capital. The figure shows the constrained optimal stock weight as a function of the human capital to financial wealth ratio (left panel) and age (right panel) for three different values of the relative risk aversion coefficient γ. The stock weight is restricted to the interval from 0% to 100%. The baseline parameter values listed in Table 3 are assumed.

are in the range of existing estimates in the literature; due to substantial idiosyncratic income risk, the correlation is lower than the estimates 0.29 and 0.34 from Table 1 based on aggregate income changes (using Q1-Q1 changes).

The numbers in the table are to be read in the following way. For an agent with a relative risk aversion of $\gamma = 5$ and a human/financial wealth ratio of $\ell = 10$, the optimal decision is to invest 225% of current financial wealth in the stock, partly financed by a loan of 125% of current financial wealth. This levered stock investment has an expected rate of return of 12.25% and a standard deviation of 45%. The table reveals that, for a fixed γ, the optimal stock weight is decreasing over life supporting the typical "more stocks when young" advice. Formally, this is because the term in the last bracket in (11) is positive. Intuitively, human capital resembles a riskfree investment more than a stock investment so, to obtain the optimal overall risk profile, young agents (those with large ℓ) short the riskfree asset and invest a lot in stocks. If borrowing constrained so that $\pi_S \leq 1$, then 100% in stocks is optimal for all risk-tolerant and also more risk-averse investors with sufficient human capital relative to financial wealth. Furthermore, the stock weight is decreasing in the degree of risk aversion. Figure 4 illustrates how the constrained optimal stock weight varies with the human-financial wealth ratio for different degrees of risk aversion (left panel) and age (right panel), where the translation to age follows the baseline human capital calculation in Section 3.[8] The patterns in the stock weight are exactly as found in the much more advanced dynamic life-cycle models that have to be solved numerically, cf., e.g., Cocco et al. (2005). Young investors, even quite risk averse, should hold all their wealth in stocks. As they grow older, they should eventually—except the most risk tolerant—start shifting from stocks to bonds.

[8] Obviously, we use the parameter values described here in Section 4. We let the value of λ_Y vary with the relative risk aversion as explained in Section 3.

	$\gamma = 1$				$\gamma = 5$				$\gamma = 10$			
ℓ	stock	rf	exp	std	stock	rf	exp	std	stock	rf	exp	std
	Baseline parameter values											
0	125	-25	7.3	25	25	75	2.3	5	13	87	1.6	3
1	245	-145	13.3	49	45	55	3.3	9	20	80	2.0	4
2	365	-265	19.3	73	65	35	4.3	13	28	72	2.4	6
5	725	-625	37.3	145	125	-25	7.3	25	50	50	3.5	10
10	1325	-1225	67.3	265	225	-125	12.3	45	88	12	5.4	18
20	2525	-2425	127.3	505	425	-325	22.3	85	163	-63	9.1	33
50	6125	-6025	307.3	1225	1025	-925	52.3	205	388	-288	20.4	78
	High income volatility or stock-income correlation											
0	125	-25	7.3	25	25	75	2.3	5	13	88	1.6	3
1	230	-130	12.5	46	30	70	2.5	6	5	95	1.3	1
2	335	-235	17.8	67	35	65	2.8	7	-3	103	0.9	1
5	650	-550	33.5	130	50	50	3.5	10	-25	125	-0.3	5
10	1175	-1075	59.8	235	75	25	4.8	15	-63	163	-2.1	13
20	2225	-2125	112.3	445	125	-25	7.3	25	-138	238	-5.9	28
50	5375	-5275	269.8	1075	275	-175	14.8	55	-363	463	-17.1	73

Table 2: Optimal stock-bond allocation with human capital The table shows the percentages of financial wealth optimally invested in stock and riskfree asset, as well as the expectation and standard deviation of the financial return in percent. The upper panel assumes the baseline parameter values $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, $\sigma_L = 10\%$, and $\rho_{SL} = 0.1$. The lower panel also assumes $r_f = 1\%$, $\mu_S = 6\%$, and $\sigma_S = 20\%$, but either (i) $\sigma_L = 10\%$, $\rho_{SL} = 0.4$ or (ii) $\sigma_L = 40\%$, $\rho_{SL} = 0.1$.

The impact of human capital on optimal investments is parameter dependent, however. The lower panel of Table 2 illustrates that results are very different for investors with high risk aversion and either high income-stock correlation or high income uncertainty (or both). For $\gamma = 10$ and either an income-stock correlation of 0.4 (instead of 0.1) or a human capital standard deviation of 40% (instead of 10%), the term in the last bracket in (11) is negative so that the optimal stock weight is now decreasing in the human-financial wealth ratio ℓ. Consequently, very risk-averse agents should hold less stocks when young if their income is sufficiently risky or sufficiently stock-like. If such agents cannot short stocks, the optimal strategy is to have nothing in stocks early in life and only introduce stocks into the portfolio later in life when human capital has declined adequately. Bagliano, Fugazza, and Nicodano (2014) pointed out such effects in the context of the more complex dynamic life-cycle models.

5 Adding housing investments

Residential real estate is a major asset of many households and should therefore be included in household decision problems. Here we consider real estate as a pure financial investment and include it in the mean-variance setting above alongside the stock and the riskfree asset. Let r_H denote the rate of return on real estate or "housing" over the investment period with an expectation of μ_H and a standard deviation of σ_H. At the beginning of the period, the agent now has to choose the portfolio weights π_S and π_H of the stock and of housing, respectively, with the remaining financial wealth invested in the riskfree asset. This fits into our general model specification by choosing

$$\boldsymbol{\pi}_t = \begin{pmatrix} \pi_{St} \\ \pi_{Ht} \end{pmatrix}, \quad \boldsymbol{r} = \begin{pmatrix} r_S \\ r_H \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_S \\ \mu_H \end{pmatrix},$$

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \sigma_S^2 & \rho_{SH}\sigma_S\sigma_H \\ \rho_{SH}\sigma_S\sigma_H & \sigma_H^2 \end{pmatrix}, \quad \mathrm{Cov}_t[\boldsymbol{r}, r_L] = \begin{pmatrix} \rho_{SL}\sigma_S\sigma_L \\ \rho_{HL}\sigma_H\sigma_L \end{pmatrix},$$

with ρ's denoting the various correlations as indicated by the subscripts.

In our illustrations below we assume the baseline parameter values listed in Table 3. Using the US inflation-adjusted home prices over 1948-2018 (data as in Table 1), the geometric average annual return is 0.6% and the volatility is 4.6%. The assumed 4% expected annual return on residential real estate can be justified by adding an annual rent of around 5.4% of home prices (in line with estimates of Flavin and Yamashita (2002) and Fischer and Stamos (2013)) to the 0.6% average price appreciation and subtracting 2% to account for taxes, maintenance, and transaction costs. Individual house prices are more volatile than the nationwide index, so we increase the house price volatility to 10% and lower the correlation estimates based on the aggregate data in Table 1 to levels in line with empirical studies and close to the values used by, e.g., Flavin and Yamashita (2002), Cocco (2005), Yao and Zhang (2005), Davidoff (2006), and Fischer and Stamos (2013).

5.1 Unconstrained solution

In this case we can write the optimal unconstrained portfolio weights in (4) as

$$\pi_{St} = \frac{1}{\gamma(1-\rho_{SH}^2)\sigma_S}(1+\ell_t)\left(\frac{\mu_S - r_f}{\sigma_S} - \rho_{SH}\frac{\mu_H - r_f}{\sigma_H}\right) - \ell_t \frac{\sigma_L}{\sigma_S}\frac{\rho_{SL} - \rho_{SH}\rho_{HL}}{1-\rho_{SH}^2}, \tag{12}$$

$$\pi_{Ht} = \frac{1}{\gamma(1-\rho_{SH}^2)\sigma_H}(1+\ell_t)\left(\frac{\mu_H - r_f}{\sigma_H} - \rho_{SH}\frac{\mu_S - r_f}{\sigma_S}\right) - \ell_t \frac{\sigma_L}{\sigma_H}\frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{1-\rho_{SH}^2}. \tag{13}$$

Again, the speculative demands are scaled due to the presence of human capital and the portfolio weights are subsequently adjusted for the extent to which the human capital resembles a stock and a real estate investment, respectively.

Table 4 shows optimal portfolios for different combinations of the risk aversion coefficient γ and the human-financial wealth ratio ℓ. As in the previous section, agents with either low risk aversion or a combination of medium-to-high risk aversion and a significant human capital want to borrow money to boost their investment in the risky assets. Human capital works like an inherent investment primarily in the riskfree asset due to the low correlations of human capital with risky assets. Hence, the larger the human capital, the more the agent borrows and invests in risky assets. Real estate dominates the risky portfolio. The tangency portfolio has 28% in stocks and 72% in real estate due to real estate having a larger Sharpe ratio. Despite stocks and real estate having identical correlations with human capital, the income hedge portfolio has 1/3 in stocks and 2/3 in real estate

Symbol	Description	Baseline value
r_f	Riskfree rate	0.01
μ_S	Expected stock return	0.06
σ_S	Stock price volatility	0.20
μ_H	Expected housing return	0.04
σ_H	House price volatility	0.10
σ_L	Human capital volatility	0.10
ρ_{SH}	Stock-house correlation	0.20
ρ_{SL}	Stock-human capital correlation	0.10
ρ_{HL}	House-human capital correlation	0.10

Table 3: Parameter values for stock index, real estate, and human capital.

ℓ	$\gamma = 1$ stock	house	rf	$\gamma = 5$ stock	house	rf	$\gamma = 10$ stock	house	rf
0	99	260	-259	20	52	28	10	26	64
1	194	513	-606	35	96	-31	16	44	41
2	289	765	-953	51	140	-91	21	61	17
5	573	1521	-1994	98	271	-269	39	115	-53
10	1047	2781	-3728	176	490	-566	67	203	-170
20	1995	5302	-7197	332	927	-1159	124	380	-405
50	4839	12865	-17603	801	2240	-2941	296	911	-1108

Table 4: Optimal unconstrained portfolios with housing. Percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 3 are assumed.

because of real estate having a standard deviation half the size of stocks. The income hedge portfolio is subtracted from the (magnified) investment in the tangency portfolio and thus causes an increase in the real estate to stock ratio. For example, with a relative risk aversion of 5 the ratio is $52/20 = 2.6$ without human capital and $927/332 \approx 2.8$ with a human-financial wealth ratio of 20. By comparing Table 4 to Table 2, we see that the introduction of real estate reduces the optimal weight in the stock index and in the riskfree asset (for most investors the latter means: increases borrowing).

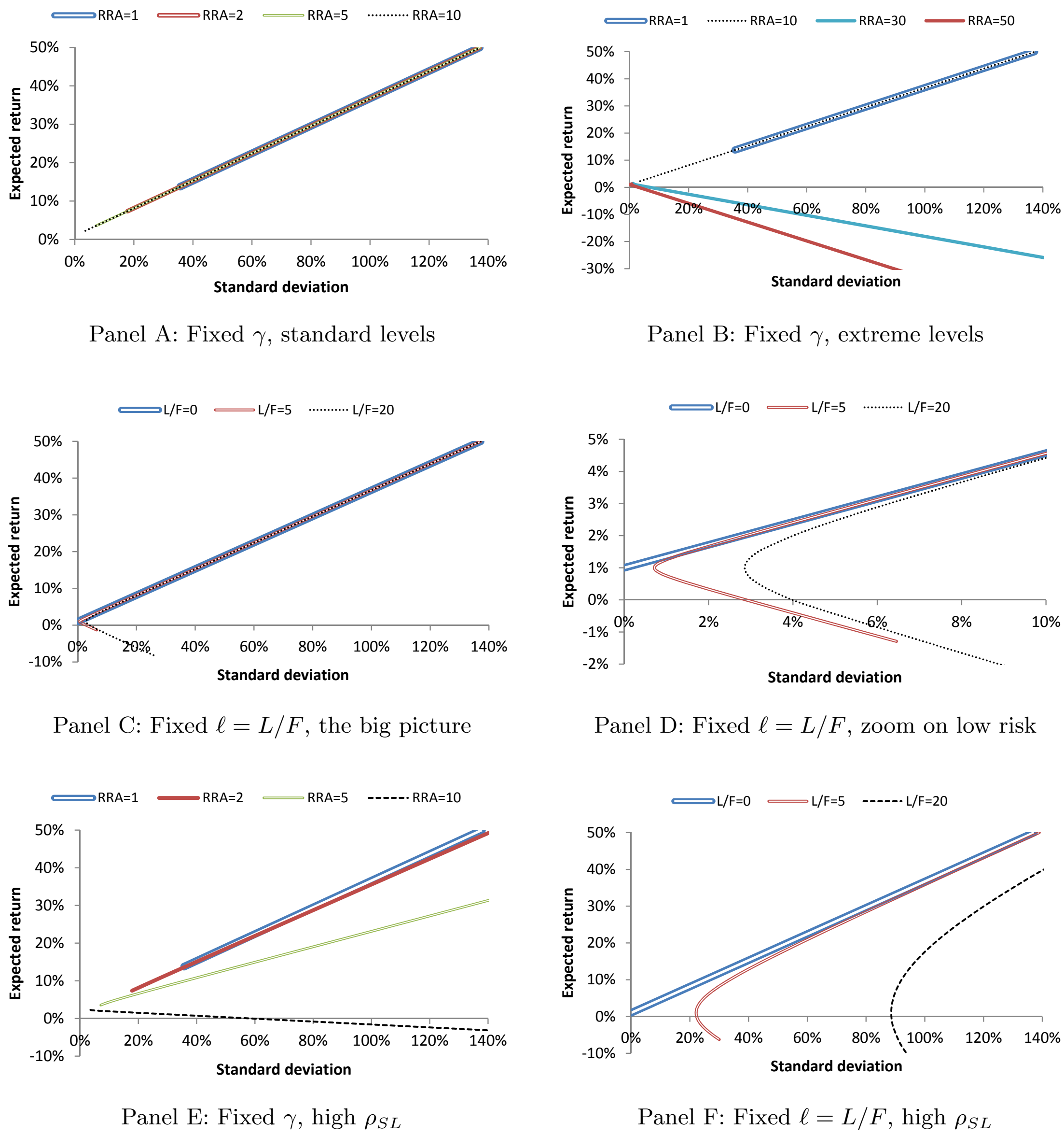
Figure 5 illustrates the optimal portfolios in the typical standard deviation and mean diagram. Panels A and B show efficient frontiers for selected levels of the risk aversion coefficient γ, when the human-financial wealth ratio ℓ is varied from zero to infinity. Panels C and D illustrate efficient frontiers for selected levels of the human-financial wealth ratio ℓ, when the risk aversion coefficient γ is varied from zero to infinity. Panels E and F of Figure 5 show that with a higher stock-income correlation of 0.6, the frontiers vary much more across levels of risk aversion and levels of the human-financial wealth ratio, and the hyperbolic shape of (at least some of) the frontiers become clearer.

5.2 Solution with collateral constraint

Holding real estate gives easy access to loans through mortgages, while stock investments generally do not, at least not to the same extent. Suppose that you can borrow at most a fraction $1 - \kappa$ of the value of the real estate you own. This corresponds to the constraint

$$\pi_{St} + \kappa \pi_{Ht} \leq 1 \tag{14}$$

on portfolio weights. We take $\kappa = 0.2$ corresponding to an 80% loan-to-value limit as our benchmark. Panel A of Table 5 shows the optimal portfolio for various combinations of the relative risk aversion and the human-financial wealth ratio. Several things are



Panel A: Fixed γ, standard levels

Panel B: Fixed γ, extreme levels

Panel C: Fixed $\ell = L/F$, the big picture

Panel D: Fixed $\ell = L/F$, zoom on low risk

Panel E: Fixed γ, high ρ_{SL}

Panel F: Fixed $\ell = L/F$, high ρ_{SL}

Figure 5: Efficient frontiers with human capital and housing. Each curve shows the combinations of standard deviation and expected return chosen by unconstrained investors either with a certain level of risk aversion but different human-financial wealth ratios (Panels A, B, and E) or with a certain human-financial wealth ratio but different degrees of risk aversion (Panels C, D, and F). The baseline parameter values listed in Table 3 are assumed except that in Panels E and F the stock-income correlation is $\rho_{SL} = 0.6$.

worth noticing. First, a levered house investment is very attractive for investors who are young/middle-aged or relatively risk tolerant. Secondly, non-participation in the stock market is optimal for young investors. Thirdly, the optimal stock weight is increasing or hump-shaped over life. Fourthly, the optimal stock weight can be non-monotonic in risk aversion which with the assumed parameter values is the case for a human-financial wealth ratio of 1, 2, or 5. This phenomenon occurs because the agent compares stocks to a levered house investment and stocks are less risky than a levered house investment. Hence, when increasing the risk aversion, the agent gradually shifts from a levered house investment to stocks and eventually to the riskfree asset.

Let us briefly compare this model output to observed positions over the life cycle. Table 6 indicates the relative importance of various asset and debt categories in different age groups according to the 2017 Panel Study of Income Dynamics (PSID) survey of US families. We include only families reporting that they own a residence with a non-trivial value, as some families choose not to own their residence for reasons excluded from our simple model. The upper part of the panel shows the average across families of the ratio of the value of assets or debt in certain classes relative to the net worth of the family (we exclude families with negative or a trivial positive net worth). For example, across all homeowning families for which the age of the reference person ("head" of family) is between 18 and 30, the value of real estate owned is on average 367.5 percent of the net worth of the family. Clearly, real estate is the major tangible asset of homeowners, and real estate debt (mortgage debt) is the major liability. Note that the portfolio weights of real estate holdings and real estate debt decrease over life as predicted by the simple model. While the main purpose of the model is to build intuition for life-cycle variations in portfolios, and the model obviously excludes several relevant aspects, it is comforting that some of the key predictions are in line with empirical observations. When including retirement accounts, both the stock market participation rate and the stock share for those participating are hump shaped starting low for young households, increasing with age until the peak around retirement, and then decreasing slightly towards the end of life. A similar pattern in the stock share could be generated by our model by pooling investors with different risk attitudes, but then the participation rate of the pool would increase monotonically over life. At least our simple model with housing can explain the observed low stock market participation rate among younger families, in contrast to both the model without housing and to many more advanced life-cycle models.

To understand the impact of the access to collateralized borrowing, Panel B of Table 5 lists optimal portfolios in the case of a 60% loan-to-value limit, whereas Panel C assumes no borrowing at all. Note that a portfolio weight written in blue [red] is larger [smaller] than the corresponding weight in the baseline case. The young investors' appetite for

ℓ	$\gamma = 1$ stock	house	rf	$\gamma = 5$ stock	house	rf	$\gamma = 10$ stock	house	rf
	Panel A: Baseline case with max 80% LTV, $\kappa = 0.2$								
0	52	240	-192	20	52	28	10	26	64
1	13	434	-347	35	96	-31	16	44	41
2	0	500	-400	51	140	-91	21	61	17
5	0	500	-400	50	250	-200	39	115	-53
10	0	500	-400	16	420	-336	60	200	-160
20	0	500	-400	0	500	-400	32	340	-272
50	0	500	-400	0	500	-400	0	500	-400
	Panel B: max 60% LTV, $\kappa = 0.4$								
0	33	167	-100	20	52	28	10	26	64
1	4	241	-145	35	96	-31	16	44	41
2	0	250	-150	47	133	-80	21	61	17
5	0	250	-150	30	174	-105	39	115	-53
10	0	250	-150	3	242	-145	36	159	-95
20	0	250	-150	0	250	-150	12	220	-132
50	0	250	-150	0	250	-150	0	250	-150
	Panel C: no borrowing, $\kappa = 1$								
0	62	38	0	20	52	28	10	26	64
1	100	0	0	31	69	0	16	44	41
2	100	0	0	38	62	0	21	61	17
5	100	0	0	60	40	0	31	69	0
10	100	0	0	95	5	0	43	57	0
20	100	0	0	100	0	0	67	33	0
50	100	0	0	100	0	0	100	0	0

Table 5: Optimal portfolios with housing and borrowing constraints. Percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 3 are assumed. In Panels B and C the numbers in blue are larger than in the baseline case of Panel A, numbers in red are smaller, whereas the remaining numbers are unchanged.

Age group	All	18-30	31-40	41-50	51-60	61-70	71-80	81+
Observations	4373	318	839	912	868	913	351	172
Asset shares in percent								
Real estate	195.7	367.5	264.1	235.1	144.1	141.9	97.7	82.1
Stocks	4.1	1.7	2.7	2.2	3.5	6.4	7.0	9.6
Check/savings	9.9	13.3	11.9	8.9	7.2	9.5	11.8	10.5
IRA/annuities	9.1	3.3	5.1	8.4	10.7	14.1	10.3	6.4
Vehicles	15.0	21.6	19.2	18.0	14.2	11.4	8.0	4.3
Other	5.2	3.6	5.5	5.3	5.4	5.3	5.4	5.3
Debt shares in percent								
Real estate	-138.7	-310.9	-207.3	-177.5	-84.8	-88.6	-40.3	-18.1
Other	0.4	0.1	1.2	0.4	0.3	0.1	0.0	0.1
Direct stocks								
Participation (in pct)	18.8	12.9	15.5	13.7	17.7	24.1	27.1	32.0
Cond. share (in pct)	21.6	13.0	17.5	15.8	19.8	26.4	25.7	30.0
Stocks and retirement accounts								
Participation (in pct)	37.4	22.0	28.8	33.2	38.6	49.1	46.2	44.8
Cond. share (in pct)	35.2	22.4	27.2	31.8	36.9	41.6	37.5	35.7

Table 6: A decomposition of the net worth of homeowners in PSID. The table is based on the 2017 Panel Study of Income Dynamics. Data was retrieved on January 24, 2020 from `https://simba.isr.umich.edu/`. We include only families reporting they own a home worth at least \$10,000 and having a net wealth of at least \$1,000. Furthermore, we remove families for which the recorded age of reference person exceeds 100 or the recorded home value or mortgage value exceeds \$9,999,000. We group PSID entries as follows. Real estate assets: ER66031 plus ER71439. Stocks: ER71445. Check/savings: ER71435. IRA/annuities: ER71455. Vehicles: ER71447 (net worth). Other assets: ER71429 minus ER/1431 (equals business net worth) plus ER71451. Real estate debt: ER66051 plus ER66072 plus ER71441. Other debt: ER71479. Net worth is then the sum of the assets less the sum of the debts. The upper panel shows for each asset or debt class the value in percent of net worth. The lower panel shows the stock market participation rate and the average share of net wealth invested in the stock market conditional on participating, both based only on directly held stocks (including mutual funds) and when retirement accounts (IRA and similar) are included.

financial investments with high risk (and high expected return) implies that if mortgages are available, they prefer housing investments with a mortgage to an unlevered stock investment. However, if borrowing is prohibited, the stock is more attractive than the house because of the stock's higher risk and expected return so young or risk-tolerant households optimally invest their entire wealth in stocks. The credit access embedded in a housing investment is thus essential for the optimal portfolio, especially for young or risk-tolerant households. In line with intuition, a reduction in the loan-to-value limit (i.e., an increase in κ) decreases (or leaves unchanged) the portfolio weight of the house and the borrowed amount, whereas the stock weight can vary non-monotonically.

5.3 Robustness of results

This subsection compares the optimal portfolios for a number of alternative specifications to the optimal portfolios in the baseline case listed in Panel A of Table 5.

Selected parameters. First we illustrate the sensitivity of results to the values of selected parameters that vary across households. Panel A of Table 7 considers an increase in σ_H, the standard deviation of real estate prices, to 0.15 from the baseline value of 0.10. This leads to a higher stock weight, a lower house weight, and a higher weight in the riskfree asset (for most: less borrowing), except for relatively young and risk-tolerant agents who still prefer a fully collateralized investment in real estate and no stocks. The qualitative patterns in how the portfolio weights vary with the risk aversion and the human-financial wealth ratio remain unchanged. Especially for young or fairly risk-tolerant agents, real estate is still the dominant asset in the portfolio. Although real estate now has a lower Sharpe ratio than stocks, real estate is attractive because of its inherent access to loans.

Panel B of Table 7 shows the effect of increasing the standard deviation of human capital from 0.1 to 0.2. The optimal portfolios is unaffected for young or risk-tolerant agents who still opt for a maximally levered house investment and no stocks. In other cases, the larger background risk leads to less borrowing, especially for the most risk-averse households, and a smaller housing portfolio weight, whereas the effect on the stock weight depends on the combination of the risk aversion and the human-to-financial wealth ratio. In the unconstrained case, we know from (12)–(13) that a larger value of σ_L reduces both portfolio weights. The borrowing constraint generally twists weights in the direction of less stocks and more housing, but a larger income risk reduces the credit appetite and thus reduces the magnitude of this twist, which may lead to a larger stock weight.

Alternative assumptions on loan access. In the baseline case the 1% interest rate applied both to lending and borrowing. Now assume that the borrowing rate is 2%, whereas the lending rate is 1%. Panel C of Table 7 lists optimal portfolios for this situation.

	$\gamma = 1$			$\gamma = 5$			$\gamma = 10$		
ℓ	stock	house	rf	stock	house	rf	stock	house	rf
	Panel A: Higher house price risk, $\sigma_H = 0.15$								
0	80	101	-81	22	21	57	11	10	79
1	62	188	-151	40	36	24	18	15	67
2	45	276	-220	57	51	-9	24	20	55
5	0	500	-400	81	94	-76	45	35	20
10	0	500	-400	68	161	-129	79	59	-38
20	0	500	-400	41	295	-236	80	101	-81
50	0	500	-400	0	500	-400	55	226	-181
	Panel B: Higher income risk, $\sigma_L = 0.2$								
0	52	240	-192	20	52	28	10	26	64
1	14	428	-342	31	87	-19	11	35	53
2	0	500	-400	43	123	-66	13	45	42
5	0	500	-400	56	220	-176	18	73	9
10	0	500	-400	28	360	-288	26	120	-45
20	0	500	-400	0	500	-400	41	214	-155
50	0	500	-400	0	500	-400	8	460	-368
	Panel C: Higher borrowing than lending rate, $r_{\text{bor}} = 2\%, r_{\text{len}} = 1\%$								
0	68	160	-128	20	52	28	10	26	64
1	45	274	-219	30	70	0	16	44	41
2	22	388	-310	42	83	-25	21	61	17
5	0	500	-400	69	154	-123	31	69	0
10	0	500	-400	51	244	-195	50	100	-50
20	0	500	-400	15	424	-339	66	172	-138
50	0	500	-400	0	500	-400	30	352	-282
	Panel D: Borrowing against human capital, $\theta = 0.1$								
0	52	240	-192	20	52	28	10	26	64
1	22	438	-360	35	96	-31	16	44	41
2	0	600	-500	51	140	-91	21	61	17
5	0	750	-650	96	270	-266	39	115	-53
10	0	1000	-900	108	460	-468	67	203	-170
20	0	1500	-1400	132	840	-872	124	380	-405
50	0	3000	-2900	204	1980	-2084	296	911	-1108
	Panel E: Buying stocks on margin, $\omega = 0.5$								
0	97	258	-255	20	52	28	10	26	64
1	67	332	-299	35	96	-31	16	44	41
2	38	406	-344	51	140	-91	21	61	17
5	0	500	-400	94	265	-259	39	115	-53
10	0	500	-400	67	333	-300	67	203	-170
20	0	500	-400	12	470	-382	76	311	-286
50	0	500	-400	0	500	-400	3	492	-395

Table 7: Optimal portfolios with housing: robustness. Percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 3 are assumed together with a maximum loan-to-value ratio of 80% ($\kappa = 0.2$), except that $\sigma_H = 0.15$ in Panel A and $\sigma_L = 0.2$ in Panel B. Numbers in blue are larger than in the baseline case, numbers in red are smaller, whereas the remaining numbers are unchanged.

Note that in this case the objective cannot be reduced from (1) to (3), but only to

$$\max_{\boldsymbol{\pi}_t} \Bigg\{ \boldsymbol{\pi}_t \cdot \boldsymbol{\mu} + (1 - \boldsymbol{\pi}_t \cdot \mathbf{1}) \left(r_{\text{len}} 1_{\{\boldsymbol{\pi}_t \cdot \mathbf{1} \leq 1\}} + r_{\text{bor}} 1_{\{\boldsymbol{\pi}_t \cdot \mathbf{1} > 1\}} \right) - \frac{1}{2}\gamma \frac{1}{1+\ell_t} \left[\boldsymbol{\pi}_t \cdot \underline{\underline{\Sigma}} \boldsymbol{\pi}_t + 2\ell_t \, \boldsymbol{\pi}_t \cdot \text{Cov}_t[\boldsymbol{r}, r_L] \right] \Bigg\},$$

where r_{len} is the lending rate and r_{bor} the borrowing rate, and $1_{\{A\}}$ equals 1 if the claim A is true and zero otherwise. Of course, portfolios not involving borrowing are unchanged (high γ, low ℓ_t). Some agents who were borrowing in the baseline case are now neither borrowing nor lending (for $\gamma = 5$, $\ell = 1$ and $\gamma = 10$, $\ell = 5$). Other agents are borrowing less, whereas agents with low risk aversion and high human capital still borrow as much as possible and invest nothing in stocks. Again, the overall qualitative patterns in how the optimal portfolio weights vary with the level of risk aversion and the human-financial wealth ratio remain unchanged.

Next, suppose that households can borrow up to a fraction θ of their human capital in addition to the collateralized mortgage. The constraint (14) is then replaced by $\pi_{St} + \kappa\pi_{Ht} \leq 1 + \theta\ell_t$. Panel D lists optimal portfolios when $\theta = 0.1$. The portfolio is unchanged for the combinations of γ and ℓ_t for which the loan-to-value constraint was not binding. The additional borrowing opportunity is taken by agents with low risk aversion or moderate-to-high human capital. The more risk-averse of these agents increase the weight of both stocks and houses, but the most risk-tolerant agents still prefer the maximal possible position in housing and nothing in stocks.

Finally, suppose that agents can buy stocks on margin and borrow up to a fraction $1 - \omega$ of the value of the stocks owned. The budget constraint (14) is then replaced by $\omega\pi_{St} + \kappa\pi_{Ht} \leq 1$. Panel E shows the optimal portfolios when agents can borrow 50% of the value of their stocks in addition to the mortgage. The agents facing a binding portfolio constraint in the baseline case find stocks relatively more attractive when they give access to loans. Still, young and very risk-tolerant agents choose the maximal possible position in housing and nothing in stocks. The older (smaller human capital) among the very risk-tolerant agents as well as younger and more risk-averse agents do in fact increase their stock share, but all portfolios remain dominated by housing investments.

6 Growth and value tilts in household portfolios

This section studies growth/value investing in the life-cycle mean-variance setting. Numerous exchange-traded funds and mutual funds are devoted to value stocks or growth stocks in specific countries, industries, or with other specific characteristics. Value investing has gained popularity by the success of high-profiled declared value investors (most notably Warren Buffett) and by empirical studies documenting that value stocks offer higher average returns than growth stocks, also after standard market risk adjustments (Rosenberg, Reid, and Lanstein, 1985; Fama and French, 1992). The academic literature on the role of value and growth stocks in household portfolios is sparse, however. Jurek and Viceira (2011) and Larsen and Munk (2012) consider value and growth stocks in theoretical models of multi-period portfolio decisions, but both ignore human capital and housing. This is problematic particularly if value stocks covary with house prices and labor income differently than growth stocks do. Based on the asset holdings of a large number of Swedish households, Betermier et al. (2017) find that, relative to growth investors, value investors are generally older, have lower human capital, lower income risk, lower leverage, and higher financial wealth.

To calculate optimal portfolios we assume the means, variances, and correlations in Table 8. These numbers are based on the 1980-2018 sample of the same data as used in Table 1 (with Q1-Q1 income changes). We adjust the mean and volatility of housing returns and the labor income volatility as explained previously. Average past stock returns are likely to overestimate future expected stock returns because of survivorship biases (Brown, Goetzmann, and Ross, 1995) and the decline in taxes and discount rates and the implied unexpected capital gains over the sample period (Fama and French, 2002). To account for this, we subtract 4 percentage points from the average returns on the three stock portfolios, which leads to expected returns being close to the 6% used in the single-stock settings in earlier sections. We assume a real riskfree rate of 1% as in previous sections. We divide the house-stock and income-stock correlations by two and the house-income correlation by four to account for the idiosyncratic variations in individual house prices and income growth compared to the aggregate time series. Consistent with the prevalent view, value stocks exhibit a higher average return than growth stocks and neutral stocks, but also a slightly higher volatility. Note that, compared to growth stocks, value stock returns are more highly correlated with both labor income and real estate prices. Other things equal, this makes value stocks relatively less attractive to young individuals with a large human capital and individuals with a large position in real estate. However, the optimal portfolio depends on the overall correlation structure of the available assets as well as expected returns and volatilities.

With these inputs, the unconstrained tangency portfolio consists of 18.7% in Growth, -25.3% in Neutral, 46.1% in Value, and 60.5% in housing. The value portfolio has the largest Sharpe ratio, followed by housing, Neutral, and Growth, but due to the correlation structure the neutral portfolio is optimally shorted in absence of human capital. The income-adjustment portfolio (to be shorted) consists of 1.6% in Growth, 81.9% in Neutral,

	Mean	Std dev	Correlations				
			Growth	Neutral	Value	House	Income
Growth stocks	5.5%	17%	1.00	0.78	0.69	0.08	0.14
Neutral stocks	5.4%	15%	0.78	1.00	0.90	0.16	0.18
Value stocks	7.1%	18%	0.69	0.90	1.00	0.19	0.15
House	4.0%	10%	0.08	0.16	0.19	1.00	0.10
Income	N.A.	10%	0.14	0.18	0.15	0.10	1.00

Table 8: Inputs for the growth-value analysis. The table shows the means, standard deviations, and correlations used in the calculations of optimal portfolios with growth and value stocks. The underlying data series are described in the text.

-23.5% in Value, and 40.0% in housing. These values are determined by the correlations of the four assets with income and also by the correlations among the four assets. Neutral stocks have the largest correlations with income, which explains their large weight in the income-adjustment portfolio. From (2) the optimal portfolio of any household is a mix of the speculative portfolio and a short position in the income-adjustment portfolio with the weights being determined by the risk aversion γ and the ratio ℓ of human capital to financial wealth. For any combination of γ and ℓ the position in neutral stocks will be negative and the positions in value stocks positive, and for reasonable values of γ and ℓ growth stocks and housing will also have positive weights.

Panel A of Table 9 shows the optimal unconstrained portfolio for various combinations of γ and ℓ. In the absence of human capital the portfolios have positive weights on housing, value stocks, and growth stocks, and negative weights on neutral stocks and the riskfree asset. When increasing ℓ, the household takes much more risk on its financial portfolio and borrows substantial sums. In the present setting, growth stocks and especially value stocks and housing have very high positive weights, whereas neutral stocks enter with a significant negative weight.

Next we impose short-selling constraints on the assets and a collateral constraint so that up to a fraction $1-\kappa = 0.8$ of the value of the house can be borrowed, in line with (14) but now π_S is replaced by the sum of the weights in the three stock portfolios. Panel B show the optimal portfolios for this case. As in earlier sections, housing (with an associated mortgage) is the dominant asset for young and relatively risk-tolerant households, but value stocks and sometimes growth stocks enter the portfolio for more risk-averse or older households.

With the baseline parameters, growth stocks play only a small role in the optimal portfolios, but only small adjustments of the inputs lead to significantly different outcomes. As an example, suppose that the correlation between income and growth stocks is lowered from 0.14 to 0.07. In this case, growth stocks are less like human capital and thus have a

	γ = 1					γ = 5					γ = 10				
ℓ	Gro	Neu	Val	Hou	Rf	Gro	Neu	Val	Hou	Rf	Gro	Neu	Val	Hou	Rf
	Panel A: Unconstrained portfolios, baseline parameters														
0	77	-104	190	250	-411	15	-21	38	50	-81	8	-10	19	25	-40
1	153	-224	385	491	-804	30	-57	81	92	-145	15	-37	43	42	-62
2	230	-344	579	733	-1197	46	-94	123	134	-208	22	-63	66	59	-84
5	460	-704	1163	1459	-2376	91	-204	251	261	-397	45	-141	137	111	-150
10	843	-1304	2135	2668	-4341	166	-387	463	472	-713	81	-273	254	197	-260
20	1609	-2505	4080	5087	-8270	317	-754	889	894	-1345	155	-535	490	370	-479
50	3906	-6105	9916	12343	-20058	769	-1853	2164	2160	-3239	377	-1321	1195	887	-1137
	Panel B: Constrained portfolios, baseline parameters														
0	0	0	58	210	-168	10	0	26	50	14	5	0	13	25	57
1	0	0	23	385	-308	15	0	48	92	-54	5	0	21	42	31
2	0	0	0	500	-400	11	0	63	129	-103	6	0	30	59	5
5	0	0	0	500	-400	0	0	55	223	-178	7	0	55	110	-72
10	0	0	0	500	-400	0	0	24	379	-303	0	0	64	182	-145
20	0	0	0	500	-400	0	0	0	500	-400	0	0	37	315	-252
50	0	0	0	500	-400	0	0	0	500	-400	0	0	0	500	-400
	Panel C: Constrained portfolios, growth-income correlation 0.07														
0	0	0	58	210	-168	10	0	26	50	14	5	0	13	25	57
1	0	0	23	385	-308	23	0	42	92	-58	13	0	16	43	28
2	0	0	0	500	-400	23	0	51	129	-103	21	0	19	60	-1
5	0	0	0	500	-400	0	0	55	223	-178	46	0	29	114	-89
10	0	0	0	500	-400	0	0	24	379	-303	44	0	20	181	-145
20	0	0	0	500	-400	0	0	0	500	-400	37	0	0	314	-251
50	0	0	0	500	-400	0	0	0	500	-400	0	0	0	500	-400
	Panel D: Constrained portfolios, growth-income correlation 0.07, higher income risk $\sigma_L = 20\%$														
0	0	0	58	210	-168	10	0	26	50	14	5	0	13	25	57
1	0	0	24	380	-304	26	0	33	85	-44	16	0	6	36	41
2	0	0	0	500	-400	39	0	37	119	-95	28	0	0	46	26
5	0	0	0	500	-400	36	0	25	199	-159	49	0	0	73	-22
10	0	0	0	500	-400	30	0	4	332	-266	77	0	0	116	-93
20	0	0	0	500	-400	0	0	0	500	-400	66	0	0	170	-136
50	0	0	0	500	-400	0	0	0	500	-400	33	0	0	333	-266
	Panel E: Constrained portfolios, alternative income-stock correlations, income risk $\sigma_L = 20\%$														
0	9	0	46	222	-177	20	0	16	52	12	10	0	8	26	56
1	0	0	19	403	-322	34	18	10	88	-50	13	13	0	35	39
2	0	0	0	500	-400	42	29	5	121	-97	16	17	0	44	23
5	0	0	0	500	-400	46	6	7	205	-164	24	30	0	70	-24
10	0	0	0	500	-400	32	0	0	341	-272	33	45	0	110	-88
20	0	0	0	500	-400	0	0	0	500	-400	27	37	0	176	-141
50	0	0	0	500	-400	0	0	0	500	-400	12	13	0	375	-300

Table 9: Optimal portfolios with growth and value stocks. Percentages of financial wealth optimally invested in growth stocks, neutral stocks, value stocks, real estate, and the riskfree asset. See text for explanations.

larger weight in the optimal portfolios of more risk-averse individuals, at the expense of value stocks, as shown in Panel C. Among the more risk-averse households, the portfolio of younger investors includes growth stocks, but not value stocks. As the household matures, the weight of growth stocks declines and the weight of value stocks increases. This fits well with the empirical findings of Betermier et al. (2017) that value investors tend to be older, have lower human capital and higher financial wealth, as well as lower leverage. Furthermore, they find that value investors have lower income risk. Panel D lists optimal portfolios when the income risk is doubled to 20%. The increased income risk does indeed lead to lower portfolio weights of value stocks, whereas the weight of growth stocks increase for individuals with $\gamma = 5$ and $\ell = 5, 10$.

Finally, in Panel E we use the income-stock correlations based on the 1947-2018 sample with Q4-Q4 income growth. After dividing by two to reflect idiosyncratic income risk, the correlation with income is 0.09 for both growth and neutral stocks, and 0.13 for value stocks. In this case, the optimal portfolios for individuals with moderate or high risk aversion are more balanced. In particular, neutral stocks are now included in the optimal portfolio for individuals with high risk aversion or with a combination of moderate risk aversion and moderate human capital.

To sum up, the above numerical examples highlight that, for a given correlation structure, the optimal growth/value tilts vary substantial with risk aversion and the ratio of human capital to financial wealth, and that the optimal portfolios are very sensitive to the assumed correlations of income with the different types of stock. Future empirical studies may provide better estimates of these correlations based on survey or registry income data. Our results demosntrate that even small cross-sectional differences in the income-asset correlations can lead to substantial differences in households' optimal portfolios.

7 Conclusion

Human capital is one of the most valuable assets held by households. We have shown how Markowitz' basic mean-variance portfolio choice model can be extended to include human capital. By solving the extended mean-variance problem (with relevant constraints) for different ratios of human capital to financial wealth, the method effectively delivers portfolio decisions over the life cycle of a household. We have argued that the life-cycle investment strategy generated in this way comes close to the strategy that can be derived using a much more involved, formal dynamic optimization approach.

Two of our three applications address settings that have been solved in the literature by numerical dynamic optimization routines. These applications confirm that our approach generates theoretically correct life-cycle portfolio patterns. The first application considers the classical stock-bond asset allocation with human capital, but no housing. With standard parameter values our results corroborates the findings of Cocco et al. (2005) that 100% in stocks are optimal for young households, but we also show that results are markedly different for certain changes in parameter values. The second application adds housing as an investment object to the problem. Here our approach provides justification and transparent arguments for the findings of Cocco (2005) that housing tends to crowd out stock investments especially for young households. We provide additional results highlighting the importance of the access to borrowing offered by housing investments.

Our final application generalizes the setting further by allowing investments in three stock portfolios, representing growth stocks, value stocks, and neutral stocks. This is, to the best of our knowledge, the first theoretical model of the role of growth and value stocks in households' portfolio decisions. We show that the optimal portfolios to some extent agree with the growth/value tilts found in Swedish household portfolios by Betermier et al. (2017), but results are highly sensitive to the assumed correlation values.

A Proof of Theorem 1

(a) follows by direct optimization of the objective function.

(b) The expected rate of return on the optimal portfolio is

$$\begin{aligned}
\mathrm{E}_t[r] &= r_f + \boldsymbol{\pi}_t^* \cdot (\boldsymbol{\mu} - r_f \mathbf{1}) \\
&= r_f + \frac{1}{\gamma}(1+\ell_t)(\boldsymbol{\mu} - r_f\mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) - \ell_t(\boldsymbol{\mu} - r_f\mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1}\operatorname{Cov}_t[\boldsymbol{r}, r_L] \\
&= r_f + \frac{1}{\gamma}(1+\ell_t)A - \ell_t B,
\end{aligned}$$

which shows (5). The variance of the optimal portfolio is

$$\begin{aligned}
\operatorname{Var}_t[r] &= \boldsymbol{\pi}_t^* \cdot \underline{\underline{\Sigma}} \boldsymbol{\pi}_t^* \\
&= \frac{1}{\gamma^2}(1+\ell_t)^2(\boldsymbol{\mu} - r_f\mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) + \ell_t^2 \operatorname{Cov}_t[\boldsymbol{r}, r_L] \cdot \underline{\underline{\Sigma}}^{-1}\operatorname{Cov}_t[\boldsymbol{r}, r_L] \\
&\quad - \frac{2}{\gamma}(1+\ell_t)\ell_t(\boldsymbol{\mu} - r_f\mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1}\operatorname{Cov}_t[\boldsymbol{r}, r_L] \\
&= \frac{1}{\gamma^2}(1+\ell_t)^2 A + \ell_t^2 C - \frac{2}{\gamma}(1+\ell_t)\ell_t B,
\end{aligned}$$

which confirms (6).

(c) It follows from (5) that

$$\frac{1}{\gamma}(1+\ell_t) = \frac{\mathrm{E}_t[r] - r_f}{A} + \ell_t \frac{B}{A}, \tag{15}$$

and by substituting this into (6), we obtain

$$\operatorname{Var}_t[r] = \left(\frac{\mathrm{E}_t[r] - r_f}{A} + \ell_t \frac{B}{A}\right)^2 A + \ell_t^2 C - 2\left(\frac{\mathrm{E}_t[r] - r_f}{A} + \ell_t \frac{B}{A}\right)\ell_t B,$$

which can be rewritten as (7). The minimum standard deviation equals $\frac{L_t}{F_t}\sqrt{D/A}$ and is obtained for $\mathrm{E}_t[r] = r_f$. From (15), we see that this combination is chosen by an agent with a risk aversion coefficient of $\gamma = (1+\ell_t^{-1})\frac{A}{B}$. For $\gamma \to \infty$, the expected rate of return drops towards $r_f - \ell_t^{-1}B$ so one branch of the hyperbola is cut off at that level. This is the downward-sloping branch if $B > 0$, and the upward-sloping branch if $B < 0$.

(d) Eq. (15) implies that

$$\ell_t = \frac{\gamma(\mathrm{E}_t[r] - r_f) - A}{A - \gamma B}, \tag{16}$$

and by substituting that into (6), we find (8). For $\ell_t = 0$, the expected return is $r_f + \frac{A}{\gamma}$ and the variance is $\frac{A}{\gamma^2}$, which defines the endpoint of one of the branches of the hyperbola.

B Optimal investments in a continuous-time model

Let $\boldsymbol{S}_t$ denote the n-vector of traded risky asset prices at time t, and assume that

$$d\boldsymbol{S}_t = \operatorname{diag}(\boldsymbol{S}_t)\left[\boldsymbol{\mu}\, dt + \underline{\underline{\sigma}}\, d\boldsymbol{z}_t\right], \tag{17}$$

where $\operatorname{diag}(\boldsymbol{S}_t)$ is the $n \times n$ matrix with $\boldsymbol{S}_t$ along the diagonal and zeros off the diagonal, $\boldsymbol{z} = (\boldsymbol{z}_t)$ is an n-dimensional standard Brownian motion representing shocks to prices, $\boldsymbol{\mu}$ is the n-vector of expected returns, and $\underline{\underline{\sigma}}$ is the $n \times n$ matrix of asset price sensitivities towards the shocks. In addition a riskfree asset with a constant rate of return of r (continuously compounded) is traded. We assume that $\boldsymbol{\mu} \neq r\boldsymbol{1}$ and that $\underline{\underline{\sigma}}$ is non-singular.

The investor receives a labor income stream given by the income rate Y_t with dynamics

$$dY_t = Y_t\left[\mu_Y(t)\, dt + \sigma_Y(t)\boldsymbol{\rho}_Y \cdot\, d\boldsymbol{z}_t + \sigma_Y(t)\sqrt{1 - \|\boldsymbol{\rho}_Y\|^2}\, dz_{Yt}\right], \tag{18}$$

where $z_Y = (z_{Yt})$ is a one-dimensional standard Brownian motion independent of $\boldsymbol{z}$, μ_Y is the expected income growth rate, $\sigma_Y \geq 0$ is the income volatility, and $\boldsymbol{\rho}_Y$ is the n-vector of instantaneous correlations of the income rate with the risky asset prices. The income stream contains unspanned risk if $\sigma_Y > 0$ or $\|\boldsymbol{\rho}_Y\| \neq 1$ or both. We assume that the agent lives until a known terminal date T. At the known retirement date $T_R < T$, there is a one-time drop in the income rate,

$$Y_{T_R+} = \Upsilon Y_{T_R-},$$

where $\Upsilon > 0$ can be interpreted as the replacement rate in final-salary pension scheme.

We consider an investor maximizing the expected life-time power utility depending on consumption or terminal wealth or both. The indirect utility is thus defined as

$$J(F, Y, t) = \sup_{c,\boldsymbol{\pi}} \mathrm{E}_t\left[\varepsilon_c \int_t^T e^{-\delta(\tau - t)} u(c_\tau)\, d\tau + \varepsilon_F e^{-\delta(T-t)} u(F_T)\right], \tag{19}$$

where F is current financial wealth, $\delta \geq 0$ is the subjective time preference rate, and $\varepsilon_c, \varepsilon_F \geq 0$ are indicators with $\varepsilon_c \varepsilon_F > 0$. We assume power utility $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$, where $\gamma > 1$ is the constant coefficient of relative risk aversion.

The investor must choose a portfolio strategy $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)$, where $\boldsymbol{\pi}_t$ is the n-vector of fractions of financial wealth invested in the n risky assets at time t. The remaining financial wealth $F_t(1 - \boldsymbol{\pi}_t \cdot \boldsymbol{1})$ is invested in the riskfree asset. If $\varepsilon_c > 0$, the investor must also choose a consumption strategy $c = (c_t)$, where c_t is the consumption rate at time t.

If there is no unspanned income risk, and the investor is not facing any portfolio constraints, we can find the optimal portfolio in closed form.

Theorem 2 *Suppose the investor is unconstrained and that either* $\sigma_Y = 0$ *or* $\|\boldsymbol{\rho}_Y\| = 1$. *Then the indirect utility is*

$$J(F, y, t) = \frac{1}{1-\gamma} G(t)^{\gamma} \left(F + yM(t)\right)^{1-\gamma}, \tag{20}$$

where

$$M(t) = \begin{cases} \int_t^T e^{-\int_t^u r_M(s)\,ds}\, du, & \text{if } t \in [T_R, T], \\ \int_t^{T_R} e^{-\int_t^u r_M(s)\,ds}\, du + \Upsilon \int_{T_R}^T e^{-\int_t^u r_M(s)\,ds}\, du, & \text{if } t < T_R, \end{cases} \tag{21}$$

$$G(t) = \varepsilon_c^{1/\gamma} \frac{1}{r_G} \left(1 - e^{-r_G(T-t)}\right) + \varepsilon_F^{1/\gamma} e^{-r_G(T-t)}, \tag{22}$$

$$r_M(t) = r - \mu_Y(t) + \sigma_Y(t) \boldsymbol{\lambda} \cdot \boldsymbol{\rho}_Y, \tag{23}$$

$$r_G = \frac{\delta}{\gamma} + \frac{\gamma - 1}{\gamma} \left(r + \frac{\|\boldsymbol{\lambda}\|^2}{2\gamma}\right). \tag{24}$$

The optimal portfolio at any time t *is*

$$\boldsymbol{\pi}_t = \frac{1}{\gamma} \left(1 + \frac{Y_t M(t)}{F_t}\right) \left(\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}^{\top}\right)^{-1} (\boldsymbol{\mu} - r\mathbf{1}) - \frac{Y_t M(t)}{F_t} \sigma_Y(t) (\underline{\underline{\sigma}}^{\top})^{-1} \boldsymbol{\rho}_Y, \tag{25}$$

and the optimal consumption rate is

$$c_t = \varepsilon_c^{1/\gamma} \frac{F_t + Y_t M(t)}{G(t)}. \tag{26}$$

Proof: The financial wealth dynamics are

$$dF_t = (Y_t - c_t)\, dt + F_t \left[(r + \boldsymbol{\pi}_t \cdot (\boldsymbol{\mu} - r\mathbf{1}))\, dt + \boldsymbol{\pi}_t \cdot \underline{\underline{\sigma}}\, d\boldsymbol{z}_t\right].$$

If we let subscripts on J indicate partial derivatives, the HJB equation is

$$\delta J(F, y, t) = \mathcal{L}_1 J(F, y, t) + \mathcal{L}_2 J(F, y, t) + \mathcal{L}_3 J(F, y, t),$$

where

$$\begin{aligned}
\mathcal{L}_1 J &= \sup_c \left\{\varepsilon_c u(c) - c J_F\right\}, \\
\mathcal{L}_2 J &= \sup_{\boldsymbol{\pi}} \left\{F J_F \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r\mathbf{1}) + \frac{1}{2} F^2 J_{FF} \boldsymbol{\pi} \cdot \underline{\underline{\sigma}}\,\underline{\underline{\sigma}}^{\top} \boldsymbol{\pi} + Y F J_{YF} \sigma_Y \boldsymbol{\pi} \cdot \underline{\underline{\sigma}} \boldsymbol{\rho}_Y\right\}, \\
\mathcal{L}_3 J &= J_t + Y J_Y \mu_Y + r F J_F + Y J_F + \frac{1}{2} Y^2 J_{YY} \sigma_Y^2.
\end{aligned}$$

The first-order condition for c leads to

$$c = \varepsilon_c^{1/\gamma} J_F^{-1/\gamma}$$

and

$$\mathcal{L}_1 J = \varepsilon_c^{1/\gamma} \frac{\gamma}{1-\gamma} J_F^{\frac{\gamma-1}{\gamma}}.$$

The first-order condition for $\boldsymbol{\pi}$ leads to

$$\boldsymbol{\pi} = -\frac{J_F}{F J_{FF}} \left(\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}^\top\right)^{-1} (\boldsymbol{\mu} - r\mathbf{1}) - \frac{Y J_{YF}}{F J_{FF}} \sigma_Y \left(\underline{\underline{\sigma}}^\top\right)^{-1} \boldsymbol{\rho}_Y,$$

which implies that

$$\mathcal{L}_2 J = -\frac{1}{2} \frac{J_F^2}{J_{FF}} \|\boldsymbol{\lambda}\|^2 - \frac{1}{2} \frac{Y^2 J_{YF}^2}{J_{FF}} \sigma_Y^2 \|\boldsymbol{\rho}_Y\|^2 - \frac{Y J_F J_{YF}}{J_{FF}} \sigma_Y \boldsymbol{\lambda} \cdot \boldsymbol{\rho}_Y.$$

where

$$\boldsymbol{\lambda} = \underline{\underline{\sigma}}^{-1}(\boldsymbol{\mu} - r\mathbf{1}).$$

With the conjecture (20), we obtain (26) and

$$\mathcal{L}_1 J = \varepsilon_c^{1/\gamma} \frac{\gamma}{1-\gamma} G^{\gamma-1} (F + yM)^{1-\gamma}.$$

Since

$$-\frac{J_F}{F J_{FF}} = \frac{1}{\gamma}\left(1 + \frac{yM(t)}{F}\right), \qquad \frac{Y J_{YF}}{F J_{FF}} = \frac{yM(t)}{F}, \qquad \frac{J_F^2}{J_{FF}} = -\frac{1}{\gamma} G^\gamma (F + yM)^{1-\gamma},$$
$$\frac{Y^2 J_{YF}^2}{J_{FF}} = -\gamma G^\gamma y^2 M^2 (F + yM)^{-1-\gamma}, \qquad \frac{Y J_F J_{YF}}{J_{FF}} = G^\gamma y M (F + yM)^{-\gamma},$$

Eq. (25) follows and

$$\mathcal{L}_2 J = G^\gamma (F+yM)^{-1-\gamma} \left\{ \frac{1}{2\gamma} (F + yM)^2 \|\boldsymbol{\lambda}\|^2 - yM(F + yM)\sigma_Y \boldsymbol{\lambda} \cdot \boldsymbol{\rho}_Y + \frac{\gamma}{2} y^2 M^2 \sigma_Y^2 \|\boldsymbol{\rho}_Y\|^2 \right\}.$$

Furthermore,

$$\mathcal{L}_3 J = G^\gamma (F + yM)^{-1-\gamma} \Bigg\{ \left[\frac{\gamma}{1-\gamma} \frac{G'}{G} + r \right] (F + yM)^2 + \left[M' - (r - \mu_Y)M + 1\right] y(F + yM) - \frac{\gamma}{2} y^2 M^2 \sigma_Y^2 \Bigg\}.$$

If either $\sigma_Y = 0$ or $\|\boldsymbol{\rho}_Y\| = 1$, then the final terms of $\mathcal{L}_2 J$ and $\mathcal{L}_3 J$ cancel, and the HJB equation is satisfied provided that

$$\begin{aligned} M'(t) - (r - \mu_Y(t) + \sigma_Y(t)\boldsymbol{\lambda}\cdot\boldsymbol{\rho}_Y)\, M(t) + 1 = 0, \\ G'(t) - \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\left[r + \frac{\|\boldsymbol{\lambda}\|^2}{2\gamma}\right]\right) G(t) + \varepsilon_c^{1/\gamma} = 0. \end{aligned}$$

To ensure the terminal condition $J(F,y,T) = \frac{\varepsilon_F}{1-\gamma}F^{1-\gamma}$, we need $M(T)=0$ and $G(T) = \varepsilon_F^{1/\gamma}$. The solutions are given by (21) and (22). □

C Proof of Equation (9)

The value at the end of period t of the income in period $t+k$ is

$$V_{t,t+k} = \mathrm{E}_t\left[e^{-[\nu(t+1)+\cdots+\nu(t+k)]}Y_{t+k}\frac{\zeta_{t+k}}{\zeta_t}\right] = \exp\left\{-\sum_{s=t+1}^{t+k}\nu(s)\right\}\mathrm{E}_t\left[Y_{t+k}\frac{\zeta_{t+k}}{\zeta_t}\right].$$

For any t,

$$\begin{aligned} \mathrm{E}_t\left[Y_{t+1}\frac{\zeta_{t+1}}{\zeta_t}\right] &= Y_t\mathrm{E}_t\Bigg[\exp\Bigg\{\mu_Y(t+1) - \frac{1}{2}\sigma_Y(t+1)^2 - r - \frac{1}{2}\left(\|\boldsymbol{\lambda}\|^2 + \lambda_Y^2\right) \\ &\qquad + (\sigma_Y(t+1)\boldsymbol{\rho}_Y - \boldsymbol{\lambda})\cdot\boldsymbol{\varepsilon}_{t+1} + \left(\sigma_Y(t+1)\sqrt{1-\|\boldsymbol{\rho}_Y\|^2} - \lambda_Y\right)\varepsilon_{Y,t+1}\Bigg\}\Bigg] \\ &= Y_t\exp\left\{-\hat{r}_m(t+1)\right\}, \end{aligned}$$

where

$$\hat{r}_m(s) = r - \mu_Y(s) + \sigma_Y(s)\left[\boldsymbol{\rho}_Y\cdot\boldsymbol{\lambda} + \sqrt{1-\|\boldsymbol{\rho}_Y\|^2}\lambda_Y\right].$$

By recursion and the law of iterated expectations we then get

$$\begin{aligned} \mathrm{E}_t\left[Y_{t+k}\frac{\zeta_{t+k}}{\zeta_t}\right] &= \mathrm{E}_t\left[\frac{\zeta_{t+k-1}}{\zeta_t}\mathrm{E}_{t+k-1}\left[Y_{t+k}\frac{\zeta_{t+k}}{\zeta_{t+k-1}}\right]\right] \\ &= \mathrm{E}_t\left[\frac{\zeta_{t+k-1}}{\zeta_t}Y_{t+k-1}e^{-r_m(t+k)}\right] \\ &= e^{-r_m(t+k)}\mathrm{E}_t\left[\frac{\zeta_{t+k-1}}{\zeta_t}Y_{t+k-1}\right] \\ &= \cdots = Y_t\exp\left\{-\sum_{s=t+1}^{t+k}r_m(s)\right\} \end{aligned}$$

so that

$$V_{t,t+k} = Y_t \exp\left\{-\sum_{s=t+1}^{t+k} (\hat{r}_m(s) + \nu(s))\right\} = Y_t \exp\left\{-\sum_{s=t+1}^{t+k} r_m(s)\right\},$$

where $r_m(s) = \hat{r}_m(s) + \nu(s)$ is a risk-, mortality, and growth-adjusted discount rate. The total human capital at the end of period t, excluding the income just received, is therefore

$$L_t = \sum_{k=1}^{T-t} V_{t,t+k} = Y_t \sum_{k=1}^{T-t} \exp\left\{-\sum_{s=t+1}^{t+k} r_m(s)\right\} = Y_t M(t),$$

which was to be shown.

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