



An analysis of the Solvency II regulatory framework's Smith-Wilson model for the term structure of risk-free interest rates

Peter Løchte Jørgensen¹

Department of Economics, Aarhus University, Fuglesangs Allé 4, Aarhus V, DK-8210, Denmark

ARTICLE INFO

Article history:

Received 12 July 2017

Accepted 2 October 2018

Available online 12 October 2018

JEL classification:

G12

G22

G28

Keywords:

Solvency II

Financial regulation

Term structure of interest rates

Calibration

Market valuation

ABSTRACT

In the European Union financial regulation requires that life and pension (L&P) companies use the Smith and Wilson (2000) model for the term structure of risk-free interest rates when valuing their liabilities and long term guarantees. Some key features of this model are that it allows for a perfect fit to market observed bond prices, and that its extrapolated long rates converge towards a constant level, the *Ultimate Forward Rate* (UFR). Both this level and the rate at which convergence towards it takes place are directly specified via parameters of the model. Since the Smith–Wilson model is not one of finance theory's standard term structure models, we introduce the model and summarize its most important mathematical properties. We also describe how the European Solvency II regulation came to embrace this particular model.

The paper moves on to document how the regulation also imposes quite detailed and tight restrictions on how the Smith–Wilson model should be parameterized and applied. We argue that many of these implementation instructions – one of which is the regulator's specification of a very high UFR – seem biased in the same direction and that this could indicate a systematic attempt to “lift” the term structure curve up and away from its true location whereby artificially high discount rates are induced. The result of the bias is not only significant undervaluation of L&P liabilities but also a peculiar contradiction of the Solvency II overall objective of enhancing financial stability and of protecting policyholders via the promotion of economic valuation in accordance with market consistent principles.

The paper's analysis is accompanied by valuation illustrations based on data on the liability composition of an actual medium-sized Danish pension fund. The results suggest that the undervaluation of liabilities resulting from use of the regulation compliant Smith–Wilson model can be massive compared to results obtained from some alternative and more freely calibrated models.

© 2018 Elsevier B.V. All rights reserved.

E-mail address: plj@econ.au.dk

¹ I am grateful for comments and useful suggestions from three anonymous referees, from Thomas Sangill, Michael Schmutz, Søren Slipsager and Mikkel Svenstrup as well as from participants at the 31st International Congress of Actuaries in Berlin in June 2018, the 2017 Swiss Risk and Insurance Forum in Rüschlikon, the 21st International Congress on Insurance: Mathematics and Economics in Vienna in July 2017, the 2017 Annual PerCent Conference at Copenhagen Business School, and at the 40th Annual Meeting of the Italian Association for Mathematics Applied to Economics and Social Sciences (AMASES) in Catania in September 2016. Any remaining errors are my own responsibility. Parts of this paper were written while I was visiting Professor Montserrat Guillen at the University of Barcelona (UB) in Spain. I wish to express my gratitude to Professor Guillen for her kind hospitality and to the UB for hosting me and for placing their facilities at my disposal. Financial support from the Pension Research Centre (PerCent) at Copenhagen Business School is gratefully acknowledged.

<https://doi.org/10.1016/j.jbankfin.2018.10.001>

0378-4266/© 2018 Elsevier B.V. All rights reserved.

1. Introduction

After a lengthy development process and many delays the European Union's new regulatory framework for insurance and reinsurance undertakings – “Solvency II” – finally came into effect on January 1, 2016. Solvency II regulates a multitude of areas in relation to, for example, company governance, the duties of auditors, public disclosure, and, of course, solvency and risk management issues. The full text of the Solvency II Directive and its many amendments spans thousands of pages. In the present paper we focus exclusively on the parts of the Solvency II regulation that concern the valuation of life insurance and pension (L&P) companies' assets and liabilities. In particular, we will look into what Solvency II has to say regarding the model for determining the term structure of risk-free interest rates that should be used in valuing L&P companies' liabilities and long term guarantees.

The subject of the term structure of interest rates is a classic one in financial theory and term structure models are some of the finance field's oldest "tools of the trade". It is also a subject of financial economics that is both well-understood and well-developed in terms of research. We know, for example, that accurate estimation of the term structure of interest rates is vital for obtaining precise and market consistent valuation estimates in relation to fixed income streams in general and to L&P companies' liabilities in particular. And it is safe to say that the knowledge that financial theory has produced in this area is brought to daily use by central banks, investment banks, and other financial market participants all over the world. It is therefore all the more interesting that Solvency II requires L&P companies to use a model for the term structure of risk-free interest rates that has not previously received much attention in the financial literature.²

The specific mathematical term structure model in question is referred to as "the Smith-Wilson model". This model originates from a proprietary report from the British actuarial consultancy firm Bacon & Woodrow (Smith and Wilson, 2000). Not only has this model hitherto received very little attention from financial researchers, it also must be applied with several of its key parameters being directly dictated by EIOPA.³ For example, EIOPA has decided that as maturity is increased, the Smith-Wilson model's zero-coupon interest rate should converge towards a level which is currently (2018) fixed at 4.05% p.a. via one of the model's parameters referred to by EIOPA as the "Ultimate Forward Rate".⁴ This decision could appear a bit strange because in many European countries long-term risk-free market interest rates are currently far below this level. It is also somewhat paradoxical because one of the main broad principles of Solvency II states that

"The assessment of the financial position of insurance and reinsurance undertakings should rely on sound economic principles and make optimal use of the information provided by financial markets... In particular, solvency requirements should be based on an economic valuation of the whole balance sheet."

Recital 45 of the Solvency II Directive (2009).

The purpose of this paper is therefore to introduce, analyze, and discuss the Smith-Wilson model that EIOPA, as of January 1, 2016, has required all L&P companies with business in the EU member states to use for determining the fair value of their liabilities.

The remaining part of the paper is structured as follows: In the next section we provide some additional background on the regulatory framework and on how the Smith-Wilson model came to be embraced by the Solvency II regulation. Section 3 introduces the Smith-Wilson model and discusses its main characteristics. We establish the solution of the model and develop the theory behind its calibration to market data. We provide more mathematical de-

² Andersson and Lindholm (2013); Gach (2016); Lagerås and Lindholm (2016) and de Kort and Vellekoop (2016) are recent contributions from the insurance economics literature and/or papers that focus mainly on various technical aspects of the Smith-Wilson model.

³ EIOPA – The European Insurance and Occupational Pensions Authority – is the supervisory authority responsible for overseeing the insurance and pensions markets in the European Union. EIOPA is one of three supervisory authorities that make up the European System of Financial Supervision (ESFS). The two other supervisory institutions are the European Banking Authority (EBA) and the European Securities and Markets Authority (ESMA).

⁴ From at least the year 2010 and until around April 2017 EIOPA had held the UFR for the Euro area fixed at 4.2% and appeared to regard the UFR as a universal constant. However, in a press release dated April 5, 2017, EIOPA announced a methodology for updating the UFR annually. The new methodology (EIOPA, 2017a) implied a preliminary estimate of the (Euro) UFR of 3.65% but since the methodology also specified that annual changes to the UFR could not exceed 15 basis points, the UFR for the year 2018 was set at 4.05%. On March 28, of 2018 EIOPA announced that the UFR for 2019 had been set at 3.90%. More on this in Section 4.2 below.

tail than in the original quite sketchy 7-page consultancy report from Bacon & Woodrow (Smith and Wilson, 2000). Section 4 explains the most significant parts of EIOPA's detailed instructions for use of the Smith-Wilson model including the parameter restrictions alluded to earlier. We also introduce and discuss some additional adjustments and modifications to the term structure curve that EIOPA allows, but which do not all seem well justified by financial theory. In Section 5 we demonstrate the practical application of the Smith-Wilson model using market data for interest rates as well as projected future benefits from an actual and fairly typical Danish occupational pension fund. We show that the term structure curve resulting from calibrating the Smith-Wilson model to market data while observing EIOPA's restrictions can differ significantly from term structure curves obtained by using more well-known models such as the models of Nelson and Siegel (1987) and Svensson (1994). The differences are largest in the "long end" of the maturity spectrum – more specifically in the so-called *extrapolated* part of the curve – and for our sample pension fund this leads to large differences between the technical provision calculated from the different models that we compare. Appendix A provides further discussion of our findings. We propose some adjustments to EIOPA's approach regarding term structure determination, and we suggest some areas for future research. Finally, we present our conclusions.

2. Regulatory background - Term structure models in EU's solvency II regulation

As mentioned in the introduction, the regulation of European insurance and pension companies has now converged to a point where it is required that a specific mathematical model – the Smith-Wilson model – is used to calibrate and extrapolate the term structure of risk-free interest rates. In addition, EIOPA has put forward quite detailed instructions on how the Smith-Wilson model should be implemented. We shall analyze the Smith-Wilson model and the EIOPA implementation instructions imposed on it in due time, but first this section will briefly describe how the Solvency II regulation came to embrace this particular model. Readers who are already familiar with the regulatory background may skip this section and jump directly to our presentation of the Smith-Wilson model in Section 3.

Since the inception of the Lamfalussy Process⁵ in 2001 the EU has worked on the development of a new supervisory framework for pension and insurance companies known as Solvency II. This was arguably long overdue since the last major update of European insurers' solvency standards was from the early 1970es.⁶ After a lengthy process and many delays the primary legislation – the Solvency II Framework Directive – was adopted and published in the Official Journal of the European Union in December 2009 (Directive 2009/138/EC). The objective of the framework directive is to define *broad principles* of the kind cited in the introduction. Another example of a broad principle formulated in the Solvency II Directive is when the directive recognizes that principles of fair valuation should in general be applied:⁷

"liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arms length transaction." Article 75(1b) of the Solvency II Directive (2009).

⁵ The Lamfalussy Process is an approach to the development of regulations of the financial service industry in the European Union.

⁶ See e.g. Lundbergh et al. (2014) for an overview of developments in the regulatory frameworks for pensions in Europe.

⁷ For a more thorough discussion of the application of fair valuation principles in the L&P business, see e.g. Jørgensen (2004).

Of special relevance for the present paper and in direct relation to the concept of “economic valuation” (see the quote from the Solvency II Directive in the Introduction) the directive further states that proper economic valuation involves

“taking account of the time value of money ... using the relevant risk-free interest rate term structure.” Article 77(2) of the [Solvency II Directive \(2009\)](#).

The directive does not go into further detail about the exact way in which the term structure of risk-free interest rates should be determined, but it does state that

“The Commission shall adopt implementing measures⁸ laying down ... the relevant risk-free interest rate term structure to be used to calculate the best estimate (of the value of assets and liabilities and of technical provisions).” Article 86(b) of the [Solvency II Directive \(2009\)](#).

The amending ([Omnibus II Directive, 2014](#)) (Directive 2014/51/EU) clarifies what this means.⁹ This Directive states that

“it is necessary for a central body to derive, publish, and update certain technical information relating to the relevant risk-free interest rate term structure on a regular basis, taking account of observations in the financial market. The manner in which the relevant risk-free interest rate term structure is derived should be transparent. Given the technical and insurance-related nature of those tasks, they should be carried out by EIOPA.” Recital 29 of the [Omnibus II Directive \(2014\)](#).

The 800-page Delegated Regulation of Solvency II from October 2014 ([The European Commission, 2015](#)) provides further detail about the risk-free term structure determination. This document again makes it clear that it is the responsibility of EIOPA to lay down and publish the technical documentation explaining the inputs, assumptions and methodology applied in the calculation of the term structure of risk-free interest rates.

In accordance with the above, EIOPA – which prior to January 2011 was known as CEIOPS¹⁰ – has worked in parallel during the development phase of Solvency II on the task of developing Implementing Technical Standards in relation to the derivation of the term structure of risk-free interest rates. One significant milestone was the fifth Quantitative Impact Study – commonly referred to as QIS 5 – which was initiated immediately after the adoption of the Solvency II Directive and which took place from January 2010 to March 2011. The QIS 5 Technical Specifications contain a document entitled “Risk-free interest rates – Extrapolation method” ([CEIOPS, 2010](#)). This is a key document because this seems to be where the Smith-Wilson model first emerged as CEIOPS/EIOPA’s preferred choice for a model for estimation of the term structure of risk-free interest rates.¹¹ It is also the document where EIOPA’s considered restrictions on the model’s parameterization and implementation were first introduced. Following a consultation process that does not seem to have seriously

challenged the proposed use of the Smith-Wilson model, some of the implementation restrictions were modified, but many remain unchanged to this day. Now that Solvency II has finally come into effect, the reference technical document is EIOPA’s “Technical documentation of the methodology to derive EIOPA’s risk-free interest rate term structures” (the most recent version of which is [EIOPA \(2018b\)](#)). This 131-page document starts out by emphasizing (again) that “The starting point in Solvency II is the economic valuation of the whole balance sheet, where all assets and liabilities are valued according to market consistent principles.” The document introduces the Smith-Wilson model as *the* model for estimating and extrapolating the risk-free term structure curve and it lays out and explains EIOPA’s many specific requirements for and restrictions on the implementation of the Smith-Wilson model. We present the Smith-Wilson model in the next section, and we introduce and discuss the most significant implementation restrictions in [Section 4](#).

3. The Smith–Wilson technique

This section presents the fundamentals of the term structure estimation technique proposed by [Smith and Wilson \(2000\)](#). As for most other term structure estimation techniques the premise of the Smith-Wilson technique is that market prices of a finite number of fixed income instruments are observed. These instruments are assumed to be liquid, completely risk free, and with promised future payments that are known with certainty. As is standard in this string of literature the idea is then to propose a theoretical model by which (theoretical) present values of the fixed income instruments can be determined. Then, it is demonstrated that the theoretical model can be parameterized such that a good fit between theoretical values and observed market prices is obtained.

Before proposing a theoretical model for valuing fixed income instruments it is common to put forward a number of subjective required properties of the approach. Two key properties that [Smith and Wilson \(2000\)](#) require from their approach is that 1) it should be possible to perfectly fit the prices of all market instruments in the estimation sample, and that 2) the (infinitely) long forward interest rate, labeled f_{∞} , should be an exogenous constant, i.e. it should be an input specified by the modeler. As we shall see, the Smith-Wilson approach entails a second (semi-)exogenous constant, α , which is not determined as part of the fitting-to-market procedure and which controls the behavior (i.e. the *smoothness* and *speed of convergence* in a certain sense) of the term structure curve for large maturities. This will become clearer in our later analysis.

Let us now introduce some additional notation and some fundamental relations. As mentioned above, it is assumed that market prices of a number of fixed income instruments are observed without error at time 0. Let there be N of these instruments, and let m_1, m_2, \dots, m_N denote their observed market prices. Furthermore, let J be the number of different maturities where a payment is promised to at least one of the instruments, and let u_1, \dots, u_J (measured in years) denote the payment dates. It will typically be the case that $J \geq N$, i.e. in practice the number of instruments available for determining the term structure of interest rates will be less than or equal to the number of different payment dates from these instruments. (In the opposite case there will be linear dependencies among the instruments). The payment or *cash flow* to instrument i at date u_j is denoted $c_{i,j}$.

Now, if we let $P(u_j)$ denote the model discount factor relating (from time 0) to time u_j , then the requirement that theoretical values and observed market prices match perfectly at time 0 can be

⁸ An implementing measure is the next level of the regulation which spells out the detailed requirements that regulated entities must meet.

⁹ The Omnibus II Directive came into effect in May 2014 following its publication in the Official Journal.

¹⁰ CEIOPS was the acronym for Committee of European Insurance and Occupational Pensions Supervisors.

¹¹ In its “Basis for decision”, point (23), [EIOPA \(2018b\)](#) explains that “The Smith-Wilson method has been applied during the last years of the development of the Solvency II framework, and in particular in the fifth Quantitative Impact Study (QIS 5) and in the Long-term Guarantees Assessment ([EIOPA, 2013](#)) that has underpinned the political agreement of the Omnibus II Directive.”

expressed mathematically as

$$m_i = \sum_{j=1}^J c_{i,j} \cdot P(u_j), \quad i = 1, \dots, N. \quad (1)$$

The discount function, $P(t)$, $t \geq 0$, represents the present (time 0) value of 1 unit of account delivered with certainty at time t . It is natural to require that $P(0) = 1$ and that $\lim_{t \rightarrow \infty} P(t) = 0$, but otherwise the parameterization of $P(\cdot)$ is to be determined via a calibration procedure. By definition the discount function is related to zero-coupon interest rates in the following way

$$P(t) = (1 + R(t))^{-t} = e^{-r(t)t}, \quad t > 0, \quad (2)$$

where $R(\cdot)$ and $r(\cdot)$ denote discretely and continuously compounded zero-coupon interest rates, respectively.

Most previous academic literature on term structure estimation has focused on proposing parameterized functional forms for $R(t)$ (or $r(t)$) (see e.g. Nelson and Siegel, 1987 and Svensson, 1994), but Smith and Wilson (2000) propose a model for the discount function, $P(t)$, directly. To be more specific, Smith and Wilson (2000) propose to model the discount function as

$$P(t) = e^{-f_\infty t} + \sum_{i=1}^N \xi_i \cdot K_i(t), \quad t > 0, \quad (3)$$

where f_∞ is the constant “long forward rate”, the ξ_i ’s are N parameters to be fitted, and the K_i ’s are referred to as kernel functions. These kernel functions are specified as

$$K_i(t) = \sum_{j=1}^J c_{i,j} \cdot W(t, u_j), \quad t > 0, \quad i = 1, \dots, N, \quad (4)$$

where $W(\cdot, \cdot)$ is the (symmetric) “Wilson’s function” which is defined as

$$W(t, u_j) = e^{-f_\infty(t+u_j)} \left\{ \alpha \cdot \min(t, u_j) - 0.5e^{-\alpha \max(t, u_j)} (e^{\alpha \min(t, u_j)} - e^{-\alpha \min(t, u_j)}) \right\}. \quad (5)$$

The parameter α must be positive and exogenously specified. The appendix of Smith and Wilson (2000) shows that the $W(\text{ilson})$ -function in (5) is a result of optimizing in a certain sense the “smoothness” and the “flatness” of the forward rate curve resulting from the proposed form of the discount function. We will not delve further into this issue here, but merely refer the interested reader to the Smith and Wilson (2000)-appendix and to Gach (2016). Instead we shall focus on the properties of the proposed discount function some of which are better understood in light of the following two propositions regarding the Wilson-function, $W(t, u_j)$, and the discount function, $P(t)$, respectively.

Proposition 1 (Wilson’s function). *For any given positive payment date, u_j , Wilson’s function, $W(t, u_j)$, given in (5) is positive as well as continuous and differentiable with respect to t for all $t \in \mathbb{R}_{\geq 0}$. Moreover, we have that*

$$\lim_{t \downarrow 0} W(t, u_j) = 0,$$

$$\lim_{t \rightarrow \infty} W(t, u_j) = 0.$$

Proof of Proposition 1. See Lemmas 1, 2, 3 in the Appendix. \square

In light of Proposition 1 we can interpret the Smith-Wilson discount function (3) as constructed from an exponentially decaying basis function, $e^{-f_\infty t}$, which is then “bumped” by a linear combination of “hump-shaped” perturbation functions (the Wilson-functions) of which there is one for each of the J different calibration instrument payment dates. The influence of each perturbation function declines as the distance between the t - and the u_j -arguments increases and eventually goes to zero as $t \downarrow 0$ or as

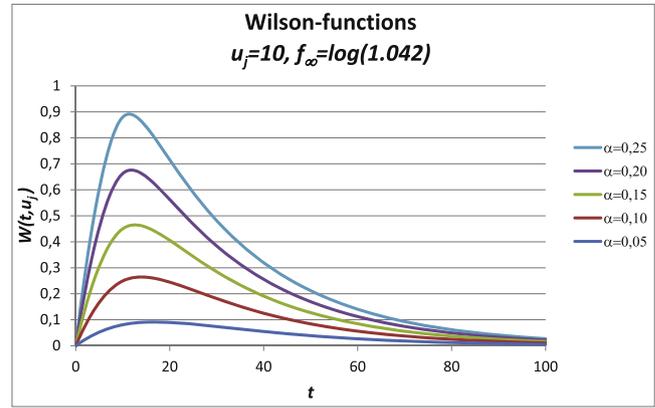


Fig. 1. Wilson-function plots for $u_j = 10$ and with $f_\infty = \log(1.042)$.

$t \rightarrow \infty$. Note that since the ξ_i -parameters in (3) can vary freely, the perturbation elements $\xi_i K_i(t)$ can be both positive and negative. We have plotted some typical Wilson-functions in Fig. 1.

Proposition 2 below highlights some properties of the Smith-Wilson discount function and of the zero-coupon interest rates that it implies. Some of these properties follow quite trivially from Proposition 1, and some require a little more work.

Proposition 2 (The Smith-Wilson discount function and zero-coupon rates). *The Smith-Wilson discount function satisfies the natural requirements*

$$\lim_{t \downarrow 0} P(t) = 1, \quad (6)$$

$$\lim_{t \rightarrow \infty} P(t) = 0. \quad (7)$$

Moreover, with continuously compounded zero-coupon interest rates given by $r(t) = -\frac{1}{t} \log P(t)$, it makes sense to define the initial instantaneous interest rate, r_0 , and the infinitely long zero-coupon rate, r_∞ , as

$$r_0 \equiv \lim_{t \downarrow 0} -\frac{1}{t} \log P(t),$$

$$r_\infty \equiv \lim_{t \rightarrow \infty} -\frac{1}{t} \log P(t).$$

In relation to these rates it can be shown that

$$r_0 = f_\infty - \sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot (\alpha e^{-f_\infty u_j} (1 - e^{-\alpha u_j})), \quad (8)$$

$$r_\infty = f_\infty. \quad (9)$$

Proof of Proposition 2. (6) and (7) follow immediately from the definition in (3) and the results in Proposition 1. The results in (8) and (9) follow from Lemma 4 in the Appendix. \square

Proposition 2 first confirms the two key requirements regarding the limits of $P(t)$ mentioned previously in this section. Secondly, it is established that zero-coupon rates converge towards f_∞ as t goes to infinity and that the initial instantaneous interest rate is given as a function of all the model parameters (and of all calibration instrument payments and payment dates) which is not easy to interpret. We can note that the expression which determines r_0 is generally not guaranteed to be positive.¹²

¹² We do not claim complete originality of the results in Proposition 1 and 2. Some of them can be found in EIOPA (2018b) (see p. 41–46 in particular) and in Smith & Wilson’s original paper, although in most cases only with summaric or indirect proofs. For a more rigorous analysis of some of the properties of the Smith-Wilson model the reader is referred to de Kort and Vellekoop (2016).

Returning now to the model relations (3)–(5) we can conclude that the Smith–Wilson model is basically an $N + 2$ -parameter model, but as the parameters f_∞ and α are supposed to be exogenously specified and thus not to be part of the market calibration procedure, we really have an N -parameter model on our hands. As can also be seen from relations (3)–(5), the resulting model is linear in the ξ 's, so these parameters should be easily obtained provided that none of the equations are linearly dependent on each other. This property makes it possible to match market prices and theoretical values of the calibration instruments perfectly as required by Smith and Wilson (2000).

We will now demonstrate the calibration of the Smith–Wilson model – and thus the estimation of the discount function and term structure curve – in two cases. Following the exposition in the QIS 5 Technical Specifications (CEIOPS, 2010) we first go through the simplified case where the calibration instruments are simple zero-coupon bonds. Second, we go through the more general case where the calibration instruments are fixed income instruments with pay-offs at multiple dates such as coupon bonds and swaps.

3.1. Calibrating to a set of observed zero-coupon bond prices

In this section we illustrate the Smith–Wilson technique in a simplified case which supposes that the market prices of N different zero-coupon bonds with maturities u_1, u_2, \dots, u_N are observed. In this case we have $J = N$ and the number of kernel functions will therefore equal the number of zero-coupon instruments. Since the zero-coupon bonds each have precisely one payment, the kernel functions defined in (4) degenerate to

$$K_i(t) = W(t, u_i),$$

so that the expression for the discount function in (3) simplifies to

$$P(t) = e^{-f_\infty t} + \sum_{i=1}^N \xi_i \cdot W(t, u_i), \quad t > 0. \tag{10}$$

The matching equations (1) become simply

$$m_i = P(u_i), \quad i = 1, \dots, N. \tag{11}$$

Combining (10) and (11) we see that we have the following linear equation system on our hands

$$m_1 = e^{-f_\infty u_1} + \sum_{i=1}^N \xi_i \cdot W(u_1, u_i)$$

$$m_2 = e^{-f_\infty u_2} + \sum_{i=1}^N \xi_i \cdot W(u_2, u_i)$$

⋮

$$m_N = e^{-f_\infty u_N} + \sum_{i=1}^N \xi_i \cdot W(u_N, u_i).$$

Alternatively, we can write the above system in matrix form as follows

$$\mathbf{m} = \boldsymbol{\mu} + \mathbf{W} \cdot \boldsymbol{\xi} \tag{12}$$

where

$$\mathbf{m} = (m_1, m_2, \dots, m_N)^T$$

$$\boldsymbol{\mu} = (e^{-f_\infty u_1}, e^{-f_\infty u_2}, \dots, e^{-f_\infty u_N})^T$$

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_N)^T$$

and where

$$\mathbf{W} = (W(u_i, u_j))_{i=1, \dots, N, j=1, \dots, N}$$

$$= \begin{pmatrix} W(u_1, u_1) & W(u_1, u_2) & \dots & W(u_1, u_N) \\ W(u_2, u_1) & W(u_2, u_2) & \dots & W(u_2, u_N) \\ \vdots & \ddots & \ddots & \vdots \\ W(u_N, u_1) & W(u_N, u_2) & \dots & W(u_N, u_N) \end{pmatrix} \tag{13}$$

Note that the \mathbf{W} -matrix will be symmetric as Wilson's function is symmetric. If \mathbf{W} is invertible, the solution of the system can be found as

$$\boldsymbol{\xi}^* = \mathbf{W}^{-1}(\mathbf{m} - \boldsymbol{\mu}), \tag{14}$$

and the market calibrated discount function takes the form

$$P(t) = e^{-f_\infty t} + \sum_{i=1}^N \xi_i^* \cdot W(t, u_i). \tag{15}$$

The fitted term structure curve can now be determined via relation (2).

3.2. Calibration to coupon bond prices, swap rates, etc.

Consider now the more general case where again prices of N interest rate related instruments are observed, but where these can be instruments with fixed payments at multiple times such as (government) coupon bonds and/or (the fixed leg of) plain vanilla interest rate swaps.

Recall that in this case we use J to denote the number of dates, u_1, \dots, u_J , where a payment is promised to at least one of the instruments. The N instrument prices are denoted m_1, \dots, m_N , and the cash flow to instrument i at time u_j is denoted $c_{i,j}$.

Substituting the Kernel-equation (4) into the basic Smith–Wilson discount function expression (3) we get

$$P(t) = e^{-f_\infty t} + \sum_{i=1}^N \xi_i \left(\sum_{j=1}^J c_{i,j} \cdot W(t, u_j) \right), \quad t > 0, \tag{16}$$

where Wilson's function, $W(\cdot, \cdot)$, is defined as in (5).

Using (16) to substitute for $P(u_j)$ in the system of matching equations (1) gives us the following N equations

$$m_1 = \sum_{j=1}^J c_{1,j} \cdot P(u_j) = \sum_{j=1}^J c_{1,j} \left(e^{-f_\infty u_j} + \sum_{i=1}^N \xi_i \left(\sum_{k=1}^J c_{i,k} \cdot W(u_j, u_k) \right) \right)$$

$$m_2 = \sum_{j=1}^J c_{2,j} \cdot P(u_j) = \sum_{j=1}^J c_{2,j} \left(e^{-f_\infty u_j} + \sum_{i=1}^N \xi_i \left(\sum_{k=1}^J c_{i,k} \cdot W(u_j, u_k) \right) \right)$$

⋮

$$m_N = \sum_{j=1}^J c_{N,j} \cdot P(u_j) = \sum_{j=1}^J c_{N,j} \left(e^{-f_\infty u_j} + \sum_{i=1}^N \xi_i \left(\sum_{k=1}^J c_{i,k} \cdot W(u_j, u_k) \right) \right)$$

or

$$m_1 = \sum_{j=1}^J c_{1,j} \cdot e^{-f_\infty u_j} + \sum_{j=1}^J c_{1,j} \sum_{i=1}^N \xi_i \sum_{k=1}^J c_{i,k} \cdot W(u_j, u_k) = \sum_{j=1}^J c_{1,j} \cdot e^{-f_\infty u_j} + \sum_{i=1}^N \left(\sum_{k=1}^J \left(\sum_{j=1}^J c_{1,j} \cdot W(u_j, u_k) \right) c_{i,k} \right) \xi_i$$

$$\begin{aligned}
m_2 &= \sum_{j=1}^J c_{2,j} \cdot e^{-f_\infty \cdot u_j} + \sum_{i=1}^N \left(\sum_{k=1}^J \left(\sum_{j=1}^J c_{2,j} \cdot W(u_j, u_k) \right) c_{i,k} \right) \xi_i \\
&\vdots \\
m_N &= \sum_{j=1}^J c_{N,j} \cdot e^{-f_\infty \cdot u_j} + \sum_{i=1}^N \left(\sum_{k=1}^J \left(\sum_{j=1}^J c_{N,j} \cdot W(u_j, u_k) \right) c_{i,k} \right) \xi_i.
\end{aligned}$$

In simpler matrix notation the equation system becomes

$$\mathbf{m} = \mathbf{C} \cdot \boldsymbol{\mu} + \mathbf{CWC}^T \cdot \boldsymbol{\xi}, \quad (17)$$

where \mathbf{C} is the $N \times J$ cash flow matrix;

$$\begin{aligned}
\mathbf{C} &= \{c_{i,j}\}_{i=1,\dots,N,j=1,\dots,J} \\
&= \begin{Bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,J} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,J} \\ \vdots & \ddots & \ddots & \vdots \\ c_{N,1} & c_{N,2} & \cdots & c_{N,J} \end{Bmatrix}, \quad (18)
\end{aligned}$$

and where \mathbf{m} , $\boldsymbol{\mu}$, $\boldsymbol{\xi}$ and \mathbf{W} are as before except that \mathbf{W} and $\boldsymbol{\mu}$ are now $J \times J$ and $J \times 1$ -dimensional, respectively.

Again we seek the $N \times 1$ -dimensional $\boldsymbol{\xi}$ -vector that solves the equation system. The system is linear, so solution is easy. If a solution exists it must be given by

$$\boldsymbol{\xi}^* = (\mathbf{CWC}^T)^{-1} (\mathbf{m} - \mathbf{C} \cdot \boldsymbol{\mu}). \quad (19)$$

The crucial requirement for the existence of a solution is thus that the $N \times N$ matrix \mathbf{CWC}^T should be invertible.

4. EIOPA'S adaptation of the Smith–Wilson model

Having presented the mathematical details of the Smith–Wilson model in the previous section we now move on to take a closer look at EIOPA's instructions for the implementation of the model. This will include a description of some of the most significant restrictions that the EU regulation imposes on the parameterization and general estimation of the model. As mentioned earlier, the reference document in this respect is EIOPA's technical documentation of their methodology to derive the term structure of risk-free interest rates, [EIOPA \(2018b\)](#). On the very first page this document emphasizes that

“The starting point in Solvency II is the economic valuation of the whole balance sheet, where all assets and liabilities are valued according to market consistent principles.”

It goes on to recognize that the risk-free interest rate term structure plays a central role for this purpose. The document explains that EIOPA is required to publish the risk-free interest rates monthly for all currencies of relevance for EU's insurance market, and then moves on to explain the methodology applied.

4.1. The data requirement

With regard to the underlying data it is first explained that the basic risk-free interest rate term structure is constructed from market observed risk-free interest rates for a finite number of maturities. It is stipulated that both the *interpolation* within these maturities and the *extrapolation* beyond a *last liquid point* (LLP) (read: the maturity of the last usable data point) will be based on the Smith–Wilson model. As a default approach benchmark swap rates are used for calibrating the model.¹³ For each relevant currency

([EIOPA, 2018b](#)) points out the exact swap instruments to be used and the maturities that are considered sufficiently deep, liquid and transparent (DLT) for this purpose. For the EUR currency, for example, the benchmark instruments are the interest rate swaps that pay fixed rates annually against 6-month EURIBOR floating rates, and the DLT maturities are 1–10, 12, 15 and 20 years. This implies that the EUR's last liquid point is 20 years.¹⁴ Interpolation will take place for interest rates of maturities between 1 year (since swap rates for shorter maturities are rarely available) and the maturity corresponding to the LLP. Extrapolation will take place from the LLP and up to a maturity of 150 years implying that the full EIOPA term structure curve will span all maturities in the interval from 1 to 150 years. The quite long time span of the calibrated term structure curve should of course be seen in light of the fact that obligations of life and pension insurance companies often span beyond policyholders' own lifetime (e.g. for policies that include pensions for widows and children).

4.2. The Ultimate Forward Rate (UFR)

One of the most significant parameters of the Smith–Wilson model is the “long forward rate” which we earlier denoted f_∞ and which in EIOPA's terminology has become the “ultimate forward rate” and is abbreviated as UFR.¹⁵ The parameter value is specified by EIOPA and it is their policy to do so according to a “macroeconomic approach” (see e.g. [CEIOPS, 2010](#)) that basically consists of adding together historical estimates of expected inflation and short-term real interest rates. Based on estimates of 2.0% and 2.2% for the expected inflation and the real interest rates respectively, the QIS 5 report ([CEIOPS \(2010\)](#)) found that 4.2% was an appropriate value for the UFR for most currencies, and this value remained unchanged throughout the development phase of Solvency II.¹⁶ Although some concern was raised over what was arguably a relatively high value of the UFR compared to the level of risk-free market interest rates (see e.g. [Rebel, 2012](#) and [Lundbergh et al., 2014](#)), EIOPA stuck with the 4.2% value even as Solvency II came into effect in January 2016. This could be seen as reflecting the opinion of EIOPA that the “UFR should be stable over time and only change due to fundamental changes in long term expectations” ([CEIOPS, 2010](#), p. 3). In April 2016 EIOPA backed up their decision by publishing a consultation paper on the issue ([EIOPA, 2016](#)). The paper concluded that the 4.2% value of the UFR parameter was still appropriate and that it should be maintained at least throughout the year 2016. The paper also repeatedly emphasized the importance of a stable UFR and recommended that future changes in the numerical value of the UFR should not exceed 20 basis points per year (see p. 51 of [EIOPA, 2016](#)). However, on April 5, 2017, EIOPA announced a methodology for *deriving* the UFR that would entail annual updates of the numerical value of the f_∞ -parameter ([EIOPA, 2017a](#) and [EIOPA, 2017b](#)) from 2018 onwards. The basis of the methodology will continue to be the “macroeconomic approach” referred to above. However, as regards the estimate of the real interest rate newer data will be allowed to influence the estimate (but the data series still is to include historical data all the way back to 1960) and as regards the estimate of the expected inflation rate the country specific official central bank inflation target rates will be used as reference points (see details in [EIOPA, 2017a](#)).

¹⁴ Some currencies have swap markets that EIOPA has found to be deep, liquid and transparent for longer maturities. For example, the LLP of Swiss Francs (CHF) is 25 years, and for British Pounds (GBP) it is 50 years.

¹⁵ In some EU documents the UFR is referred to as the “unconditional ultimate long-term forward rate”.

¹⁶ Prior to January 2018 the UFR was set to 3.2% p.a. for the Swiss (CHF) and Japanese (JPY) currencies. It was 5.2% p.a. for BRL, INR, MXN, TYR, SAF, and it was 4.2% p.a. for all EEA currencies as well as for all non-EEA currencies not mentioned above.

¹³ This actually follows from Article 44 of the Delegated Regulation ([The European Commission, 2015](#)). Government bond prices are the foundation of the secondary approach.

Country specific UFRs will be announced annually by the end of March and will be used for determining the risk-free term structures from January 1, the following year. The new methodology announced in April 2017 implied a preliminary base estimate of the 2018 UFR for the Euro area of 3.65%, but since it has also been decided (in accordance with the above-mentioned “volatility aversion” of EIOPA) that annual changes must not exceed 15 basis points, the final 2018 value of the UFR-parameter was set at 4.05%. In March 2018 EIOPA announced their base estimate of 3.60% for the UFR for the Euro area in 2019 (see [EIOPA, 2018a](#)), but again due to the rule that limits annual changes to a maximum of 15 basis points the UFR to be applied in the Euro area in 2019 will be 3.90%.¹⁷ Should the base estimate remain at 3.60% in the coming years the cap on the magnitude of annual adjustments means of course that a UFR value of 3.60% can be reached in the beginning of 2021 at the earliest.¹⁸

4.3. Convergence parameters

It can be shown (see [Proposition 2](#) and the Appendix) that the Smith-Wilson model’s zero-coupon interest rate, $r(t)$, and the model’s instantaneous forward interest rate for time t , $f(t)$, both converge towards f_∞ as $t \rightarrow \infty$. The starting points for these convergence processes are the interest rates prevailing at the last liquid point, and convergence thus takes place as maturities increase further into the extrapolation period window. The speed of convergence of $r(t)$ and $f(t)$ towards f_∞ depends crucially on the Smith-Wilson model’s α -parameter and EIOPA has decided to regulate this convergence process as follows. First of all the [Omnibus II Directive \(2014\)](#) defines a Convergence Maturity (CM) for each relevant currency.¹⁹ In general this is defined as $\max\{LLP + 40 \text{ years}; 60 \text{ years}\}$, and it is thus 60 years for the Euro which has an LLP equal to 20 years. More precisely, the convergence maturity is the time by which the instantaneous forward interest rate should have converged to the UFR to within an error of maximum 1bp (the Convergence Tolerance).²⁰ Since a large α puts more weight on UFR in the extrapolation period and thus implies faster convergence, EIOPA has dictated that α is determined as the smallest value – although not smaller than 0.05 – which ensures convergence in the sense described above. In mathematical terms the numerical value of EIOPA’s optimal α -parameter, α^* , is determined as

$$\alpha^* \equiv \min\{\alpha \geq 0.05 : |f_\infty - f(CM)| \leq 1\text{bp}\}. \quad (20)$$

There appears to be no clear economic reasoning underlying the above-described convergence requirement apart from the QIS 5 study’s vague reference to [Thomas and Maré \(2007\)](#) who found that a value of the α -parameter of 0.1 was appropriate in their study of the South African market.

¹⁷ The country specific UFRs published by EIOPA in January 2018 varied between 3.05% (for Lichtenstein and Switzerland) and 5.35% (for Brazil, India, South Africa and Turkey). In 2019 they will vary between 2.90% (for Switzerland) and 5.50% (for Brazil, India, South Africa and Turkey), see [EIOPA \(2018a\)](#).

¹⁸ In a recent report the [European Systemic Risk Board \(2017\)](#) comment on EIOPA’s new methodology for adjusting the UFR. The ESRB find it “to be too slow, should a “low-for-long” scenario prevail over the next decade.”

¹⁹ The Convergence Maturity is obtained as the sum of the last liquid point (LLP) and the country-specific Convergence Period (CP). For example, the LLP may be 20 years and the Convergence Period may be 40 years. Then the Convergence Maturity is 60 years.

²⁰ The QIS 5 report, [CEIOPS \(2010\)](#), introduced the convergence point/convergence maturity and suggested that a range between 70 and 120 years was appropriate. The curves provided by CEIOPS for the study were constructed with the convergence point set to 90 years and the convergence tolerance was 3bp.

4.4. Credit risk adjustment (CRA)

Another important aspect of EIOPA’s Smith-Wilson implementation instructions is the Credit Risk Adjustment (CRA) the idea of which is to adjust for any credit spreads in the market (swap) rates that are used as input in the procedure for determining the risk-free term structure curve. The adjustment is applied as a parallel downward shift of the market rates for maturities up to and including the last liquid point. The size of the adjustment is developed in accordance with [Recital 20](#) and [Article 45](#) of the Delegated Regulation ([The European Commission, 2015](#)) as, basically, 50 percent of the average spread between 3-month interbank offered rates (IBOR) and overnight indexed swap (OIS) interest rates of the same maturity where the average is calculated over a time period of one year.²¹ Interestingly, an interval for the value of this parameter is hard-coded into the Delegated Regulation which states that “the adjustment shall not be lower than 10 basis points and not higher than 35 basis points”.

[EIOPA \(2018b\)](#) (p. 31–33) explains the application and the calculation of the CRA in more detail. Most notably it is clarified that when swap rates are used as the basis for the determination of the risk-free term structure curve (as is EIOPA’s preferred method) “the CRA is applied to observed par swap rates before deriving the zero coupon rates”.

4.5. Volatility adjustment (VA)

The final significant element of EIOPA’s Smith-Wilson implementation instructions that we wish to briefly introduce here is the optional Volatility Adjustment (VA) to the term structure curve. According to the introductory remarks in [EIOPA \(2018b\)](#) the VA is introduced “in order to reduce the impact of short term market volatility on the balance sheets of undertakings”. It seems fair to question whether such a purpose could be in conflict with the overall Solvency II principle of market consistency in valuations, but one would have to look more carefully into EIOPA’s methodology for calculating the volatility adjustment before a definitive conclusion can be reached on that issue. However, the description of that quite technical methodology spans almost 40 pages of [EIOPA \(2018b\)](#) placing such an analysis outside the scope of the present paper, and we must relegate that question to future research. For now we will limit the exposition to stating a few basic facts regarding the VA.

First of all, the size of the volatility adjustment to the term structure curve is calculated and published monthly by EIOPA for all relevant countries and currencies. The adjustment is measured in basis points and it is based on 65% of the risk-corrected spread between the yield that can be earned on an average sector portfolio of government and corporate bonds in a given country and the basic risk-free interest rates. A large part of the above-mentioned technical documentation concerns the definition of the country-specific *reference portfolio* (and a similar *representative portfolio*) and how risk-corrections should be done.

With the numerical size of the VA determined it must be applied in the term structure calibration procedure as follows. The Smith-Wilson calibration is first run without the VA in order to determine an intermediary term structure of risk-free interest rates. The VA is then added to the zero-coupon rates obtained from this first step for all maturities up to the last liquid point, and the calibration procedure is then re-run using a zero-coupon bond cash-flow matrix corresponding to these adjusted zero-coupon rates. The effect of the VA is an upwards parallel displacement of a size

²¹ The IBOR–OIS–spread is generally recognized as a good measure of credit risk in the interbank market. See for example ([Filipovic and Trolle, 2013](#)) and references therein.

equal to the VA of the basic risk-free interest rate term structure curve until the last liquid point. The term structure curve is also affected by the VA after the last liquid point – i.e. in the extrapolation interval – but the effect of the adjustment will decrease as maturities increase since the convergence criterion (20) in relation to forward rates must still be observed. For the EUR currency the VA varied between 3 and 33 bps during 2016 and 2017. The VA for the Danish kroner (DKK) has been among the largest. During the years 2016 and 2017 it varied between 30 and 71 bps. The practical application of the volatility adjustment is illustrated in the next section.

5. Illustration and analysis

In this section we illustrate the calibration and application of the Smith–Wilson model on the basis of recent market data and by observing EIOPA's implementation instructions (EIOPA, 2018b) as described in the previous section. Having determined the EIOPA risk-free term structure of interest rates we move on to comparing it with term structure curves obtained from some alternative and arguably more standard methods for term structure estimation. Finally, we illustrate the impact that the choice of term structure model may have on the calculated technical provisions – basically the present value of the liabilities – of a pension company. The illustrations in this section are based on market data of EUR swap interest rates (in accordance with EIOPA's primary recommendations cf. Section 4.1) and on projected future pension benefits payable from an actual Danish pension fund.

5.1. Smith–Wilson model calibration

The swap interest rates used for our illustration were collected from Bloomberg (code “EUSA Curncy”) on July 8, 2016. This particular date was chosen because the data set on projected payable pension benefits that we obtained from a Danish pension fund – and which will be presented in Section 5.3 below – were constructed on precisely this date. July 8, 2016 was a fairly normal day in the market. Interest rates were low, but not record low for the year. Prior to any analysis we subtract a credit adjustment (CRA) of 10 basis points directly from all swap rates, cf. Section 4.4.²² It may be noted that EUR swap interest rates are normally available for annual maturities ranging from 1 to 30 years and hence that risk-free zero-coupon interest rates for corresponding maturities can be obtained by applying a standard bootstrapping procedure to these data.²³ Both types of market rates – i.e. credit risk-adjusted EUR swap rates and the corresponding bootstrapped zero-coupon rates from July 8, 2016 – are plotted in Fig. 2 below. Most notable is the fact that market rates are negative up to and including the 8-year point. For maturities longer than 20 years the plotted rates seem to stabilize at a level of approximately 0.6% and for any given maturity the difference between the two types of rates is seen to be quite small.

Based on the market data presented above we will now determine the full – i.e. interpolated as well as extrapolated – risk-free interest rate term structure curve according to EIOPA's approach. The first step in the procedure is to discard all data points where the swap rates are not deemed by EIOPA to be sufficiently liquid. As explained in the previous section this means that we

should only include swap rates relating to maturities in the set {1–10, 12, 15, 20} years. Having filtered the data in this way we are left with 13 priced swap instruments which offer payments at 20 different future dates. The information in this data set regarding the par-priced fixed legs of the remaining swap instruments enables us to set up a price vector – m in the notation of Section 3 – as a 13-dimensional unit vector, as well as a 13×20 -dimensional cash flow matrix, C (refer again to Section 3), corresponding to the fixed swap rate payments (plus a unitary notional repayment at the swaps' maturity date). The next step is to specify values for the Smith–Wilson model parameters f_∞ and α . As explained in the previous section, the ultimate forward rate specified by EIOPA was 4.2% until January 2018 so this is the value we will use in our illustration of liability valuation in July of 2016. EIOPA's UFR is based on discrete annual compounding. The f_∞ -parameter value for use in Smith–Wilson's continuous compounding formulation is thus $\ln(1.042)$ which is approximately 4.11%.

As regards the convergence parameter, α , the requirement put forward by EIOPA (see Eq. (20)) basically means that this parameter must be determined via a numerical search procedure. In practice one can start out by setting any positive value of α (for example a value of 0.1 as used and recommended by Thomas and Maré (2007)), solve the model and then check if the convergence criterion (20) is met. If the absolute difference between f_∞ and the model's implied forward rate at the convergence maturity (60 years for the EUR currency) is smaller than the convergence tolerance (here, 1 bp) then α can be lowered, and vice versa. In the present illustration and with the given dataset we have determined a value of α equal to 0.1358. It can be noted that in general we have not experienced problems with determining a value of the α -parameter according to the procedure in (20) although (Lagerås and Lindholm, 2016) report that the numerical optimization of α can sometimes be problematic due to singularities in the error function defined in (20). It should also be noted that by depending on a numerical optimization procedure in this way, the solution procedure of the Smith–Wilson model does not differ fundamentally from the way more classic term structure models (like e.g. Nelson and Siegel, 1987 and Svensson, 1994) are implemented. This is contrary to statements made by EIOPA (see e.g. CEIOPS, 2010, p. 14) that highlight the fact that the Smith–Wilson model “is based on solving a linear system of equations analytically” (EIOPA's underlining) and that “this is an advantage compared to other methods that are based on e.g. minimizing sums of least square deviations”.

The calibration of the Smith–Wilson model in the sense described above also involves the determination of a 13-dimensional ξ -parameter vector by solution of the linear equation system (19).²⁴ With this step completed and with the convergence criterion satisfied we can finally say that the full discount function, $P(t)$, and hence the Smith–Wilson zero-coupon interest rates, $r(t)$, as well as instantaneous forward rates, $f(t)$, have been determined for all values of $t \geq 0$. Fig. 3 shows plots of the Smith–Wilson term structure and forward rate curves for the 1- to 100-year range determined from the data on July 8, 2016, following EIOPA's implementation instructions. The fundamental data points in the form of the zero-coupon interest rates bootstrapped from the market swap rates have also been plotted although only a subset of their corresponding swap rates have been used for obtaining the Smith–Wilson curve, cf. above. The parts of the curves that span from 1 to 30 years are re-plotted in Fig. 4 in order to highlight the finer details in the short range.

²² A full-blown empirical analysis of market credit spreads on the chosen date is outside the scope of this paper. We have simply adopted the CRA value used by EIOPA in all of its publications from January 2016 to July 2018, namely 10 bps. It is worth noting, however, that a CRA value of 10 bps is precisely at the lower boundary of the interval for the CRA adjustment specified in The European Commission (2015).

²³ See for example (Veronesi, 2010).

²⁴ We do not report these values as they have no obvious interpretation, but they are available from the author on request.

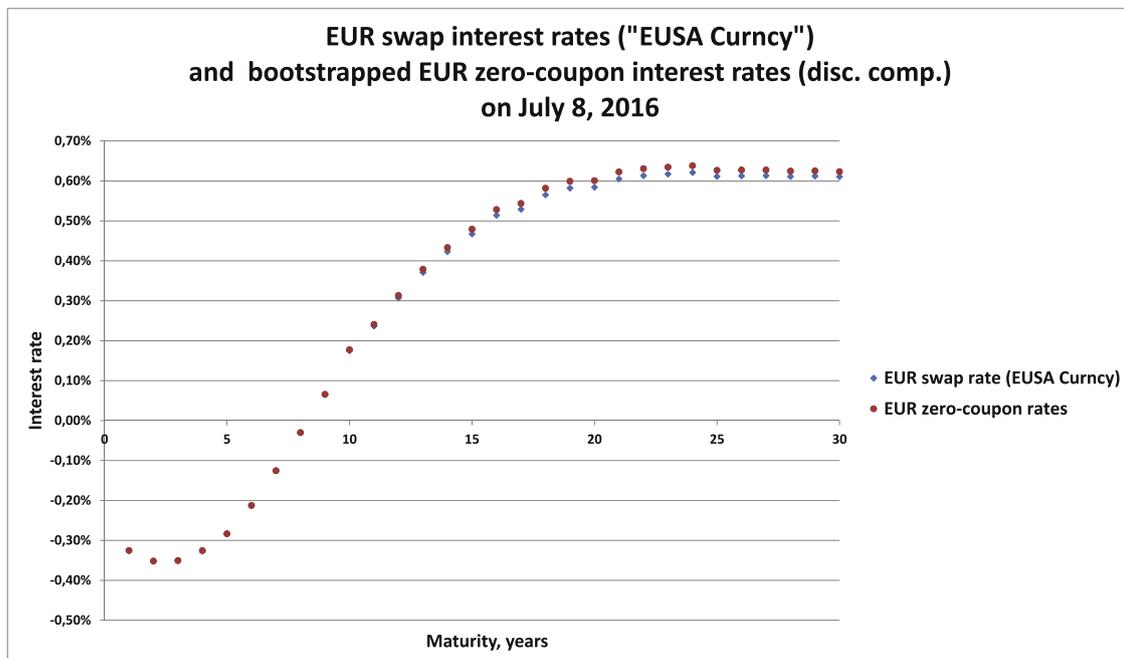


Fig. 2. The basic market data - credit risk-adjusted annual EUR swap interest rates and bootstrapped zero-coupon rates on July 8, 2016.

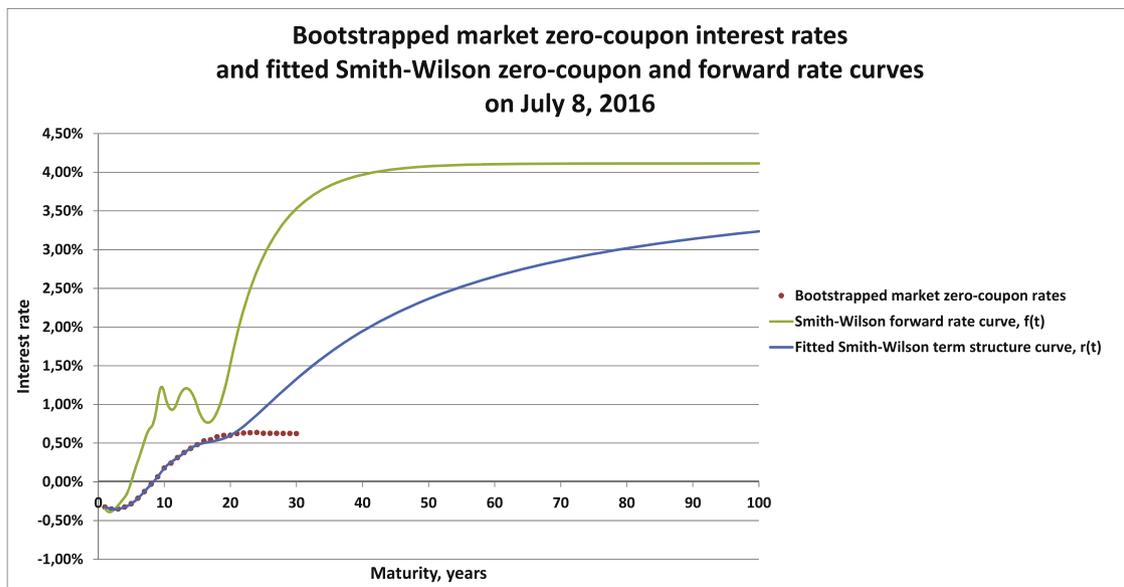


Fig. 3. Bootstrapped market zero-coupon interest rates and fitted Smith-Wilson zero-coupon and forward rate curves on July 8, 2016 (100 year horizon).

It can be seen from Figs. 3 and 4 that the calibrated Smith-Wilson term structure curve follows the data points quite closely until the (last liquid) 20-year point where it takes off steeply upwards towards the given convergence level – the UFR-level – that lies much higher than the market rates in this maturity range. The minor deviations that can be spotted between the Smith-Wilson zero-coupon curve and the market rates in the 15 to 20 year maturity range are due to the discarded data points in this range, cf. above. The plot of the instantaneous forward rate in Fig. 3 confirms that convergence of the model’s forward rate to the UFR-level has happened at the 60-year mark.

We next present a pair of plots that give an indication of the sensitivity of the location of the extrapolated part of the Smith-Wilson curve to the two key parameters f_∞ and α . Fig. 5 illustrates that increasing α speeds up the convergence of the zero-coupon rates towards the level of the UFR. It turns out, however,

that increasing α above the value of 1 has little additional effect.

In Fig. 6 we have again calibrated the Smith-Wilson model to the market data from July 8, 2016, but with alternative values of the exogenously specified UFR. Not surprisingly we see that lower values of f_∞ imply zero-coupon interest rates that are lower in the extrapolation area. As we shall see, it may have a dramatic effect on the present value of long dated guaranteed pension obligations to lower the model’s ultimate forward rate.

Fig. 7 shows the Smith-Wilson curve with the DKK volatility adjustment (red curve) along with the earlier seen standard Smith-Wilson curve (blue curve) and the bootstrapped market zero-coupon rates for the Euro on July 8, 2016. The main effect of the VA is clearly to deliver a significant “lift” to the term structure curve in the short and liquid end. After the application of the VA adjustment there are no longer any rates that are negative.

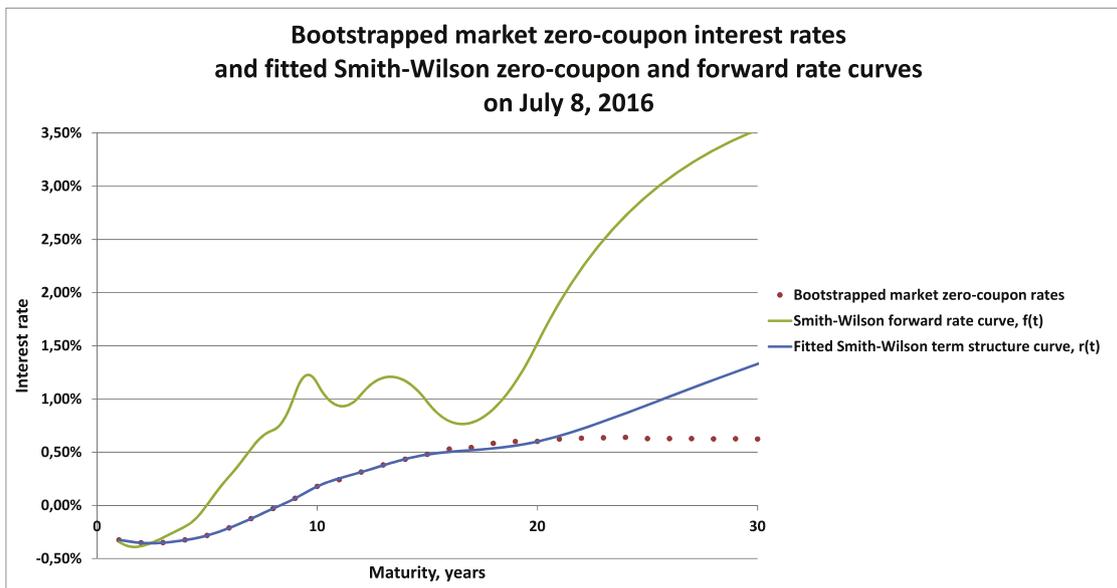


Fig. 4. Bootstrapped market zero-coupon interest rates and fitted Smith-Wilson zero-coupon and forward rate curves on July 8, 2016 (30 year horizon).

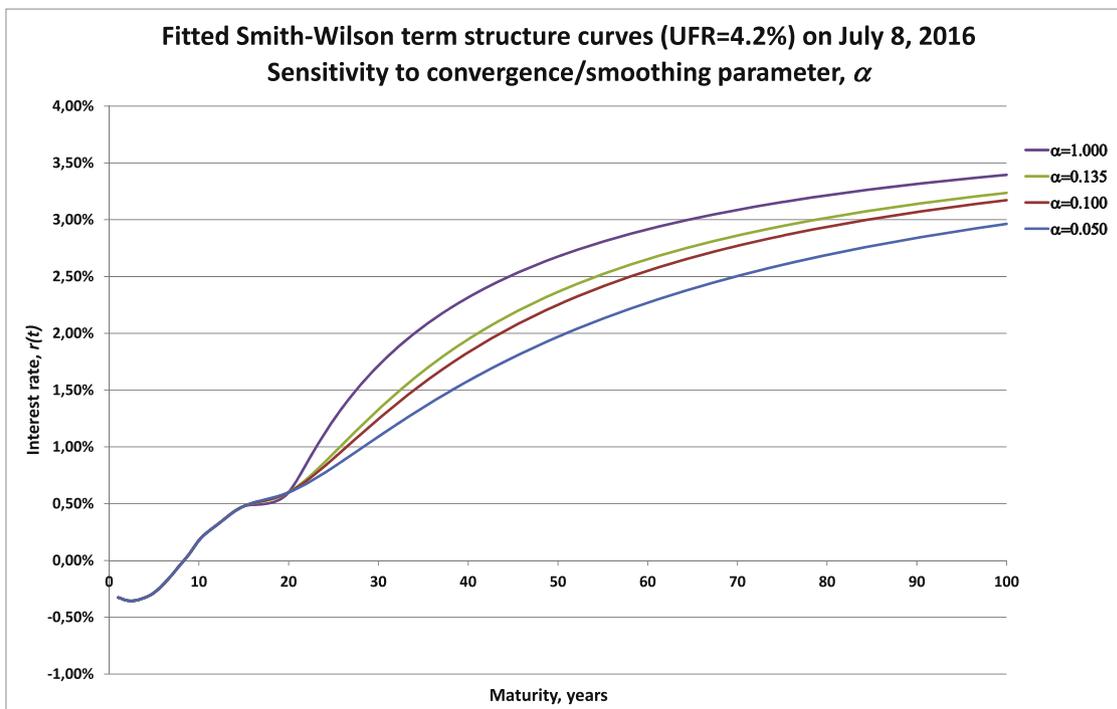


Fig. 5. Fitted Smith-Wilson term structure curves (UFR=4.2%) on July 8, 2016. Sensitivity to convergence/smoothing parameter, α .

The difference between the two Smith-Wilson curves is also plotted (green curve) as this shows the DKK volatility adjustment for the entire maturity spectrum. In accordance with the construction of the curve we observe that the VA curve is horizontal at the given level of 59 bps from year 1 and until the last liquid point.²⁵ After the last liquid point the VA-effect declines, but it is clearly significant far into the extrapolation area. For example, the VA-“lift” is 48 bps at the 30-year point, 30 bps at the 50-year point, and 15 bps at the 100-year point.

²⁵ EIOPA's published VA adjustment for DKK was 59 bps for July, 2016. The VA was 59 bps in June and 56 bps in August. It varied between 51 and 71 bps in 2016. The average over the year was also 59 bps.

5.2. Calibration of some alternative term structure models

Before we turn our attention to the valuation of pension liabilities using the EIOPA/Smith–Wilson term structure model we want to briefly introduce some alternative models for term structure estimation that we will use later for comparison with EIOPA's approach. The models in question can be characterized as follows:

- Bootstrapped market zero-coupon interest rates with horizontal extrapolation
- The standard Nelson–Siegel model (Nelson and Siegel, 1987)
- The extended Nelson–Siegel model (Svensson, 1994)

The first of the models mentioned above is an extremely simple and parameter-free approach where the idea is to use mar-

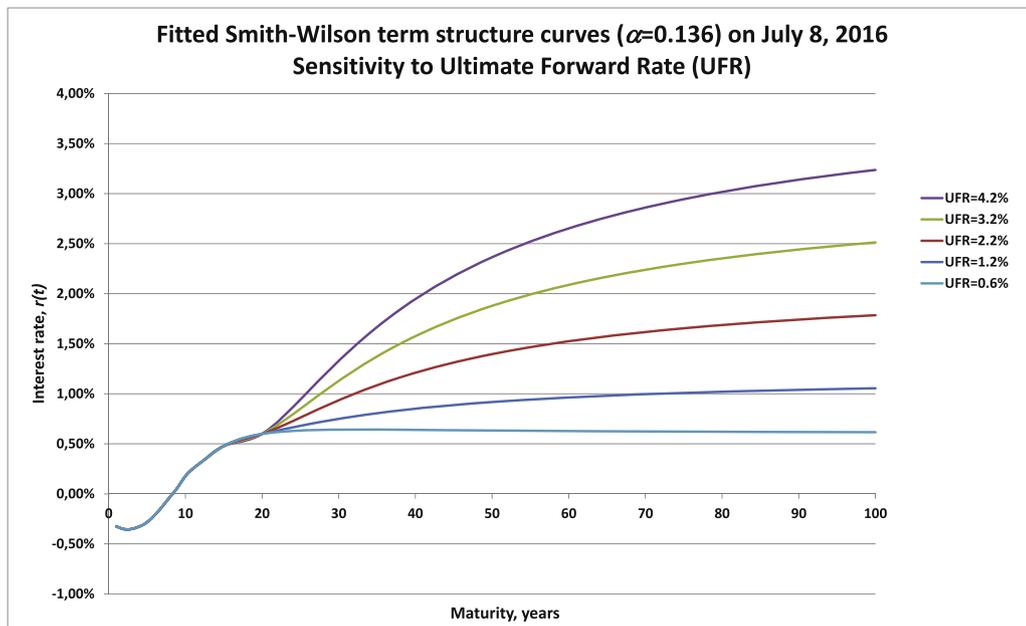


Fig. 6. Fitted Smith-Wilson term structure curves ($\alpha = 0.136$) on July 8, 2016. Sensitivity to Ultimate Forward Rate (UFR).

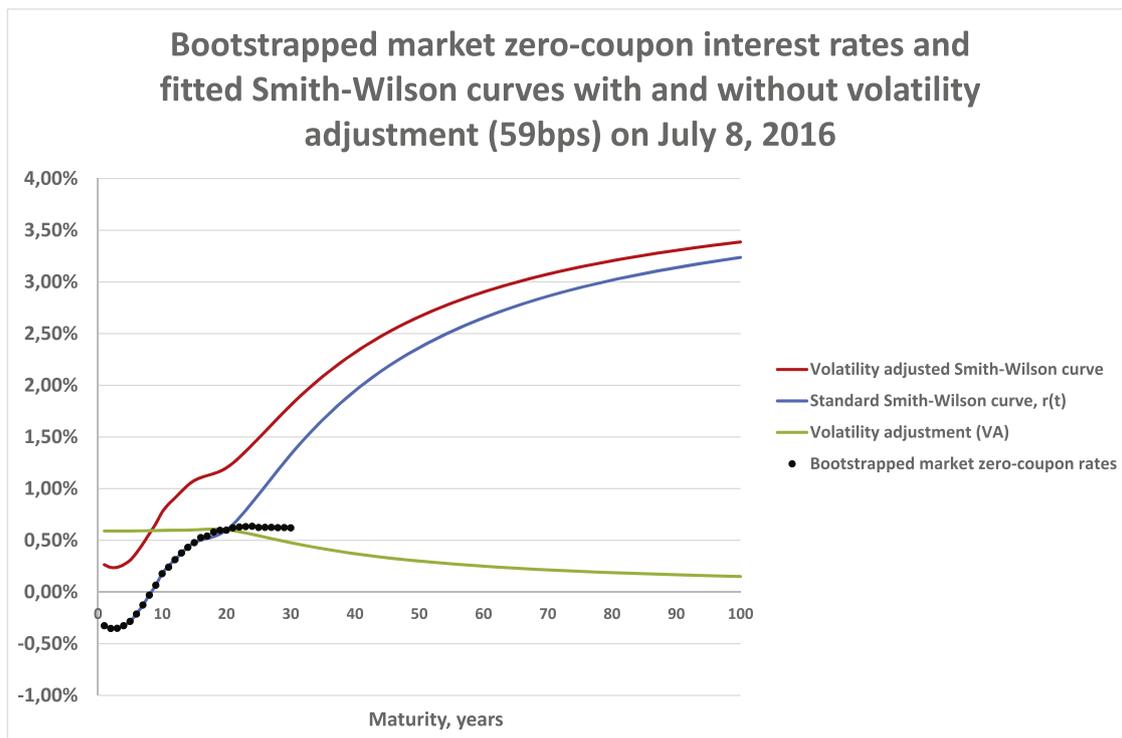


Fig. 7. Bootstrapped market zero-coupon interest rates and fitted Smith-Wilson curves with and without volatility adjustment (59bps) on July 8, 2016.

ket interest rates – possibly interpolated – up to the last available maturity and then to assume that all rates thereafter are identical to this last rate. More formally, if we let $r^{BS}(t^*)$ denote the bootstrapped market zero-coupon interest rate at the longest available maturity, t^* , then this method involves setting $r(t) = r^{BS}(t^*)$ for $t \geq t^*$, where t^* will normally be 30 years when bootstrapping is based on EUR swap rates.²⁶ The term structure curve resulting from this approach will, of course, be flat in the extrapolation in-

terval. In the present illustration we have used the 30-year zero-coupon rate (0.62%) as the reference point for the horizontal extrapolation. Since the market term structure curve is more or less flat already from the 20-year point we note that the results would have been largely unaffected if we had instead used the rate prevailing at EIOPA's last liquid 20-year point (0.60%). Note that the fact that the bootstrapped market curve is already flat in the 20 to 30 year maturity range in the current illustration could be a strong argument for using this particular extrapolation approach. However, we can obviously not rely on this always being the case.

²⁶ A shorter "last available maturity" can of course be chosen subjectively.

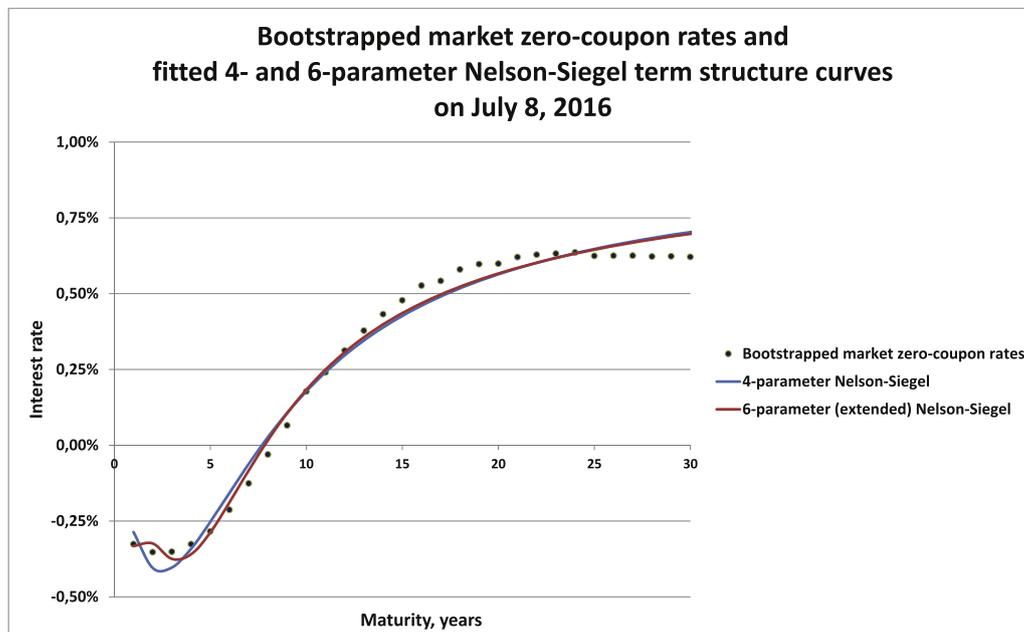


Fig. 8. Bootstrapped market zero-coupon rates and fitted 4- and 6-parameter Nelson-Siegel term structure curves on July 8, 2016 (30 year horizon).

The second and third of the alternative term structure estimation approaches listed above comprise two models from the Nelson-Siegel family – the standard 4-parameter model of (Nelson and Siegel, 1987) as well as its extended version, i.e. the 6-parameter model proposed by Svensson (1994). The main motivation for including these two models in our analysis is that they are widely used and popular in practice. More specifically, in 2005 the Bank for International Settlements (BIS) conducted a study on central banks' approaches to the estimation of the zero-coupon yield curves (BIS, 2005). One conclusion of the study was that "In most cases, ... central banks adopted the so-called Nelson and Siegel approach or the Svensson extension thereof."

In the 6-parameter extended Nelson-Siegel model the term structure of zero-coupon interest rates is parameterized as follows,

$$r^{ENS}(t) = \beta_0 + \beta_1 \left(\frac{1 - \exp\left(-\frac{t}{\tau_1}\right)}{\frac{t}{\tau_1}} \right) + \beta_2 \left(\frac{1 - \exp\left(-\frac{t}{\tau_1}\right)}{\frac{t}{\tau_1}} - \exp\left(-\frac{t}{\tau_1}\right) \right) + \beta_3 \left(\frac{1 - \exp\left(-\frac{t}{\tau_2}\right)}{\frac{t}{\tau_2}} - \exp\left(-\frac{t}{\tau_2}\right) \right), \quad t > 0, \quad (21)$$

where β_0 , β_1 , β_2 , β_3 , τ_1 and τ_2 are the six constant parameters. Clearly it must be required that $\tau_1 > 0$ and $\tau_2 > 0$. The 4-parameter Nelson-Siegel model (Nelson and Siegel, 1987) is the special case of the above obtained by setting $\beta_3 = 0$ (making also the parameter τ_2 redundant).

In Figs. 8 and 9 below we show the term structure curves resulting from fitting the Nelson-Siegel 4- and 6-parameter models (Nelson and Siegel, 1987 and Svensson, 1994) to the market data used above, i.e. zero-coupon interest rates bootstrapped from swap interest rates spanning the full maturity range from 1 to 30 years.

The first of the figures shows the 1- to 30-year range only, whereas the second plot shows the curves resulting from simple

extrapolation of these two models into the maturity range beyond the 30-year point. It is seen that as maturity lengthens, zero-coupon interest rates in both the 4- and the 6-parameter versions of the Nelson-Siegel model converge to a level which is approximately 1%.²⁷ The convergence level is given by the β_1 -parameter in both models, and the exact values of all fitted parameters are given in the figures. It can be seen that the models' fit to the market data is good, although not perfect. This is as expected given the fact that these models are parsimonious with only 4 and 6 parameters compared to the Smith-Wilson model's 15 parameters (in our current illustration of the EIOPA standard implementation). For Figs. 8 and 9 the following parameter values have been applied:

Standard Nelson-Siegel model:

$$\beta_0 = 0.0098, \beta_1 = -0.0093, \beta_2 = -0.0305, \tau = 2.1102.$$

Extended Nelson-Siegel model:

$$\beta_0 = 0.0096, \beta_1 = -0.0201, \beta_2 = -0.3780, \beta_3 = 0.3666, \\ \tau_1 = 1.2854, \tau_2 = 1.1813.$$

5.3. Calculation of technical provisions

In the previous subsection we saw how different models can estimate the term structure of risk-free interest rates very differently even if their calibrations are based on the same market data. The differences between the models were particularly large in the extrapolation area, which is the maturity spectrum beyond 20 years in our basic illustration. In some contexts the long end of the term structure curve may not be particularly important, but for most pension companies it is crucial. We will illustrate this below where we demonstrate how the calculated technical provisions of an actual Danish pension fund can vary enormously depending on the chosen model and its associated parameterization.

²⁷ It can be shown (and plots that we have omitted confirm it) that spot forward rates converge to the same level as zero-coupon rates in the Nelson-Siegel models. In the present case the convergence of forward rates is much faster than in the Smith-Wilson model.

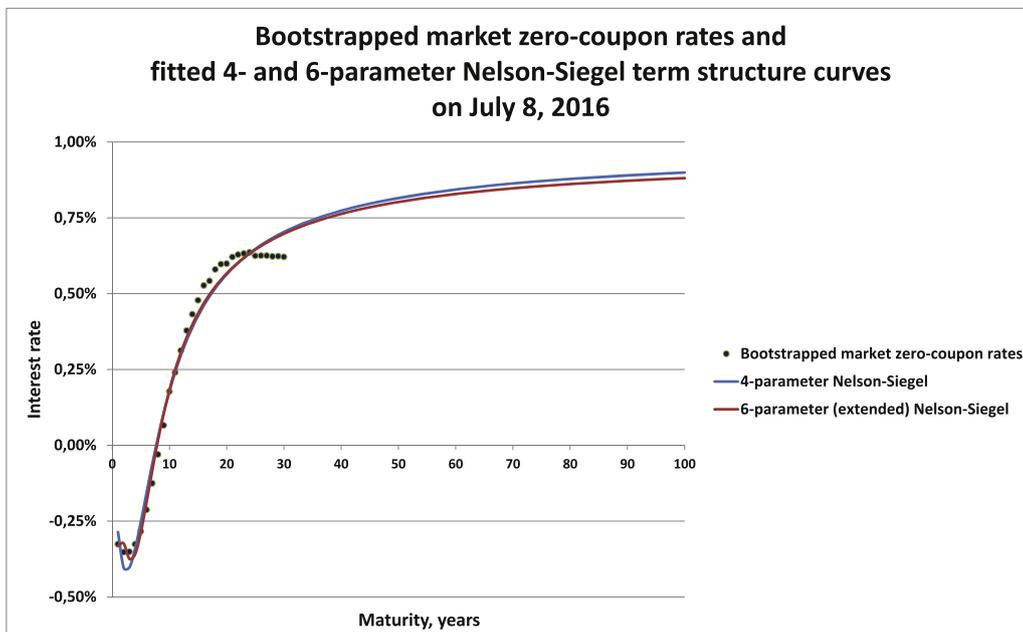


Fig. 9. Bootstrapped market zero-coupon rates and fitted 4- and 6-parameter Nelson-Siegel term structure curves on July 8, 2016 (100 year horizon).

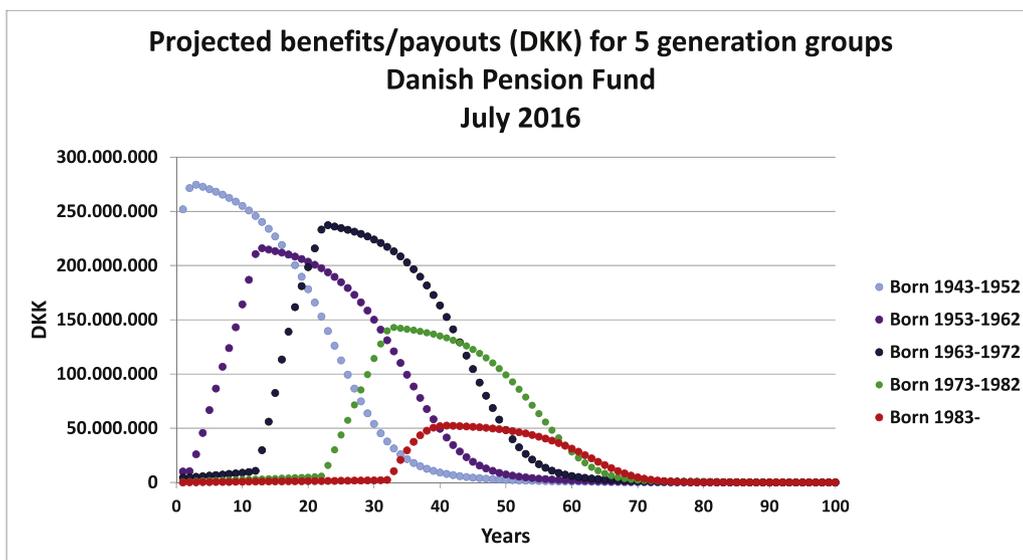


Fig. 10. Projected benefits/payouts (DKK) for 5 generation groups. Danish pension fund, July 2016.

The starting point of our illustration is the projected future benefits from an actual medium-sized Danish pension fund. The nature of the calculated benefits is such that – by the Solvency II regulation – the associated fair valued liability should be determined by a present value calculation using the risk-free term structure of interest rates for discounting. In a Solvency II context this means that the Smith-Wilson term structure curve must be used. The data set was constructed on July 8, 2016, and the structure of the data is as follows. The projected benefits are provided for five different age groups defined by the policyholders’ year of birth. The first group comprises policyholders born from 1943 to 1952. The next three groups also comprise policyholders born within a 10-year interval, i.e. from 1953 to 1962, from 1963 to 1972, and from 1973 to 1982, respectively. Finally there is a “young” age group comprising policyholders born in 1983 or later. Benefits are accumulated within the five age groups and given on an annual basis for time horizons

from 1 to 100 years. The benefits are denoted in nominal Danish kroner (DKK).²⁸

Fig. 10 shows a plot of the projected benefits data and the payout profile is typical for common pension funds: First there is a fairly linear build-up because pension savers enter evenly into the payout phase (a.k.a. the decumulation phase) over a 10-year interval. Then, when all savers in an age group have entered the payout phase, the accumulated payouts can only fall as time passes and pensioners start to die out. This happens slowly in the beginning and then accelerates until only a smaller fraction of the original

²⁸ According to Section 6.A of EIOPA (2018b) provisions in DKK should be calculated using the term structure curve for the EUR with an adjustment for currency risk amounting to 1 bp. The Danish Krone is pegged to the Euro at a central rate of 746.038 DKK per 100 EUR.

Table 1

Present value of liabilities / technical provisions by age group and for different models of the term structure of risk-free interest rates. Values in million DKK. Valuation date, July 8, 2016.

| | Age group by year of birth | | | | | Total |
|---|-------------------------------|-----------|-----------|-----------|-------|--------|
| | 1943–1952 | 1953–1962 | 1963–1972 | 1973–1982 | 1983– | |
| Bootstrap + horiz. extrap. ^a | 5,859 | 5,195 | 5,408 | 3,004 | 1,042 | 20,508 |
| Extended Nelson–Siegel ^b | 5,854 | 5,153 | 5,272 | 2,836 | 960 | 20,075 |
| Standard Smith–Wilson ^c | 5,721 | 4,746 | 4,274 | 1,815 | 494 | 17,051 |
| Smith–Wilson + VA-adj. ^d | 5,331 | 4,279 | 3,747 | 1,572 | 427 | 15,355 |
| | Indexed with Model 1 as basis | | | | | |
| Bootstrap + horiz. extrap. | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| Extended Nelson–Siegel | 99.9 | 99.2 | 97.5 | 94.4 | 92.1 | 97.9 |
| Standard Smith–Wilson | 97.6 | 91.4 | 79.0 | 60.4 | 47.4 | 83.1 |
| Smith–Wilson + VA-adj. | 91.0 | 82.4 | 69.3 | 52.3 | 40.9 | 74.9 |

^a Market rates bootstrapped from annual swap interest rates as explained in main text. Horizontal extrapolation from 30 year point.

^b Estimated parameter values: $\beta_0 = 0.0096$, $\beta_1 = -0.0201$, $\beta_2 = -0.3780$, $\beta_3 = 0.3666$, $\tau_1 = 1.2854$, $\tau_2 = 1.1813$

^c Parameter values: $f_\infty = \ln(1.042)$, $\alpha = 0.1358$

^d EIOPA's volatility adjustment (VA) was 59 bps both at end June and at end July, 2016. This value has been used. See also footnote 25.

members is left in the age groups. Some lifespans are extended so it may take 70 years or more before projected benefits reach zero in a particular age group.

In order to illustrate how the calculated technical provision depends on model assumptions in relation to the risk-free term structure of interest rates we have compiled three tables. In all of the tables we have calculated technical provisions on July 8, 2016 for each of the five age groups as well as a summed total. The technical provisions are given both as an absolute present value in DKK and as an indexed value relative to some specified benchmark.

In Table 1 we provide these calculated technical provisions for four different term structure estimation methods. The first approach is the one that uses bootstrapped risk-free market rates up to and including the 30-year point and horizontal extrapolation thereafter. We use the result from this approach as a benchmark for the indexation in the lower panel of the table. The second model is the extended (6-parameter) Nelson–Siegel model. We do not report results for the standard (4-parameter) Nelson–Siegel model since they are nearly identical to the results obtained using the extended Nelson–Siegel model. The third model for which we report valuation results is the standard EIOPA/Smith–Wilson model, and the fourth model is the volatility adjusted EIOPA/Smith–Wilson model.²⁹

It can be seen from Table 1 that bootstrapping with horizontal extrapolation is the most conservative approach in the sense that it leads to the highest technical provision in all age groups as well as totally. The second largest valuations are obtained from the extended Nelson–Siegel model. Then comes the standard Smith–Wilson approach, and finally the lowest valuations come from the volatility adjusted Smith–Wilson model. Looking at the total technical provision it can be seen that the extended Nelson–Siegel model valuation is only about 2% lower than the valuation which is based on market rates with horizontal extrapolation (the benchmark in Table 1). The total technical provision is 17% lower than the benchmark with the standard Smith–Wilson model and more than 25% lower than the benchmark with the volatility adjusted Smith–Wilson model. It is also interesting to study the differences in valuations across the age groups. In general the valuations

across models differ more the younger the age group. For example, the standard Smith–Wilson approach determines a technical provision for the oldest age group which is merely 2.4% lower than the technical provision found by using market rates with horizontal extrapolation. But if we consider the youngest age group, the Smith–Wilson approach assigns a value of the liability which is less than half of the value found by the benchmark method. If a volatility adjustment is added to the Smith–Wilson curve the technical provision drops to merely 40.9% of the benchmark valuation. The large differences in calculated technical provisions for younger age groups can be explained by the fact that these groups of policyholders have a higher fraction of their projected benefits located in the long end of the maturity spectrum (the extrapolation area) where we have seen that differences between the models' term structure curves are largest. In other words, the way in which the term structure curve is extrapolated beyond the last available market rates is absolutely crucial for the technical provisions that are calculated.

In Table 2 we quantify the influence on technical provisions from the assumed level of the ultimate forward rate (UFR). As explained earlier in the paper the UFR level specified by EIOPA was 4.2% from the time of the QIS 5 study in 2010 and until end 2017. For 2018 a UFR value of 4.05% has been specified. We have shown earlier in the paper that the UFR heavily affects the Smith–Wilson term structure in the extrapolation area, and Table 2 shows how technical provisions would increase in our example if the UFR level was lowered to 4.05% (the UFR prevailing in 2018), 3.2%, 2.2%, 1.2%, and 0.6% (the level of longest liquid market rates at the valuation date in 2016), respectively. Not surprisingly the general picture is that technical provisions increase when the UFR is lowered. The tendency is stronger for the younger age groups. For example, the technical provision for the youngest age group more than doubles when UFR is lowered from 4.2% to 0.6%. For the specific pension fund considered here the overall level of the technical provisions would increase by 20% if a UFR of 0.6% was set instead of 4.2%. Such an increase would undoubtedly wipe out the equity capital of most pension funds and pension benefits would most likely have to be lowered immediately. The effect of lowering the UFR by 15 bps to the 2018 level of 4.05% is seen to be quite marginal.

In Table 3 we illustrate the effect of changing the convergence speed parameter, α . Recall that a higher α makes long rates converge faster to the specified ultimate forward rate which is set at 4.2% in the table. In the current illustration a faster convergence

²⁹ A referee pointed out to us that a somewhat similar analysis to the one in Table 1 was made in Budiono (2012). In a Dutch context this paper compares funding ratios (and their variability) obtained from bootstrapped market curves and various incarnations of the Smith–Wilson model. The paper compares valuations across a 'representative', a 'young', a 'middle', and an 'old' fund.

Table 2

Present value of liabilities / technical provisions by age group obtained by the Smith–Wilson model with different values of the Ultimate Forward Rate (UFR)^a. Values in million DKK. Valuation date, July 8, 2016.

| | Age group by year of birth | | | | | Total |
|-------------|--|-----------|-----------|-----------|-------|--------|
| | 1943–1952 | 1953–1962 | 1963–1972 | 1973–1982 | 1983– | |
| UFR = 4.2% | 5,721 | 4,746 | 4,274 | 1,815 | 494 | 17,051 |
| UFR = 4.05% | 5,726 | 4,762 | 4,312 | 1,850 | 509 | 17,160 |
| UFR = 3.2% | 5,756 | 4,857 | 4,538 | 2,070 | 602 | 17,823 |
| UFR = 2.2% | 5,794 | 4,976 | 4,835 | 2,373 | 738 | 18,715 |
| UFR = 1.2% | 5,833 | 5,105 | 5,168 | 2,735 | 911 | 19,752 |
| UFR = 0.6% | 5,858 | 5,187 | 5,389 | 2,985 | 1,036 | 20,455 |
| | Indexed with UFR=4.2%-results as basis | | | | | |
| UFR = 4.2% | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| UFR = 4.05% | 100.1 | 100.3 | 100.9 | 102.0 | 103.0 | 100.6 |
| UFR = 3.2% | 100.6 | 102.3 | 106.2 | 114.0 | 121.9 | 104.5 |
| UFR = 2.2% | 101.3 | 104.8 | 113.1 | 130.8 | 149.4 | 109.8 |
| UFR = 1.2% | 102.0 | 107.5 | 120.9 | 150.7 | 184.4 | 115.8 |
| UFR = 0.6% | 102.4 | 109.3 | 126.1 | 164.5 | 209.7 | 120.0 |

^a The base estimate of the smoothing parameter has been used, i.e. $\alpha = 0.1358$.

Table 3

Present value of liabilities / technical provisions by age group obtained by the Smith–Wilson model^a with different values of the smoothing parameter, α . Values in million DKK. Valuation date, July 8, 2016.

| | Age group by year of birth | | | | | Total |
|-------------------|--|-----------|-----------|-----------|-------|--------|
| | 1943–1952 | 1953–1962 | 1963–1972 | 1973–1982 | 1983– | |
| $\alpha = 0.0500$ | 5,762 | 4,868 | 4,546 | 2,051 | 583 | 17,809 |
| $\alpha = 0.1000$ | 5,735 | 4,786 | 4,359 | 1,885 | 519 | 17,284 |
| $\alpha = 0.1358$ | 5,721 | 4,746 | 4,274 | 1,815 | 494 | 17,051 |
| $\alpha = 0.5000$ | 5,670 | 4,611 | 4,011 | 1,628 | 437 | 16,357 |
| $\alpha = 1.0000$ | 5,655 | 4,578 | 3,955 | 1,596 | 428 | 16,213 |
| | Indexed with $\alpha = 0.1358$ -results as basis | | | | | |
| $\alpha = 0.0500$ | 100.7 | 102.6 | 106.4 | 113.0 | 118.0 | 104.4 |
| $\alpha = 0.1000$ | 100.2 | 100.8 | 102.0 | 103.9 | 105.0 | 101.4 |
| $\alpha = 0.1358$ | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| $\alpha = 0.5000$ | 99.1 | 97.1 | 93.9 | 89.7 | 88.4 | 95.9 |
| $\alpha = 1.0000$ | 98.9 | 96.5 | 92.5 | 87.9 | 86.6 | 95.1 |

^a The Ultimate Forward Rate is set to 4.2% (discrete compounding), i.e. $f_{\infty} = \ln(1.042)$.

of long rates towards a (relatively high) UFR has the effect of lowering the calculated technical provision. Using as benchmark the values found when α is determined in accordance with the EIOPA instructions – i.e. $\alpha = 0.1358$ – we see for example that technical provisions fall by about 5% at the overall level if α is increased to 1. The effect of such a parameter change is smaller in the oldest age group (1%) and larger in the younger age groups (13.4%), and the explanation is the same as before.

6. Conclusion

The financial literature that deals with models for the term structure of interest rates is extensive and well-developed, but when the European supervisory authority for insurance and pensions markets, EIOPA, recently decided on a term structure model for Solvency II compliant valuations in European L&P companies, EIOPA chose a model from outside the set of classic and well-known term structure models. EIOPA chose the Smith–Wilson model which was first put forward in a proprietary report by the British actuarial consultancy firm Bacon & Woodrow in the year 2000. It has been stressed by EIOPA that the Smith–Wilson model is a method in the “open domain” (CEIOPS, 2010, p. 13), but so are the finance literature’s classic models and the claim is curious in light of the fact that Bacon & Woodrow prohibits the disclosure and distribution of their report to third parties without their prior written permission (Smith and Wilson, 2000). The latter fact may at least partly explain why the Smith–Wilson model has hitherto

not received much attention in the academic literature, and one purpose of the present paper has been to contribute to filling this gap in the literature.

Having first briefly described the regulatory background, we provided a thorough introduction to the Smith–Wilson model and its main characteristics, and we demonstrated the model’s analytical calibration to a given set of fixed income market data. We then moved on to present and discuss the most significant of the very detailed implementation instructions that EIOPA (2018b) has put forward in relation to the Smith–Wilson model’s practical application. Some of these are quite puzzling. First of all there is the requirement that the Smith–Wilson model’s UFR-parameter should be fixed at 4.2% (4.05% in 2018 and 3.90% in 2019) for most relevant currencies despite the fact that long market rates are currently much lower than this. The dictate that the convergence of long rates towards the relatively high UFR-level should begin already from the 20-year point also seems strange given that fairly reliable market interest rates are available up to at least 30 years for most currencies. And it appears paradoxical that so much emphasis is put on the Smith–Wilson model’s ability to fit market data *exactly* in the short end of the curve when market data after the last liquid point is completely ignored. It is also quite peculiar that EIOPA (2018b) has decided on a convergence point that lies at 60 years (for main currencies such as the Euro) when the most significant preparatory report – the QIS 5 study (e.g. CEIOPS, 2010) – specifically recommended fixing the convergence point at

somewhere between the 70- and the 120-year point, and when the same study provided numerical illustrations that used a convergence maturity of 90 years. In relation to the credit risk adjustment (CRA) that the Solvency II regulation specifies it is noteworthy, firstly, that the adjustment – which must be subtracted from market observed rates – is restricted to the interval between 10 and 35 bps, and, secondly, that in the full sample (at the time of writing this paper) of monthly EIOPA publications of the term structure of risk-free interest rates between December 2015 and July 2018 the applied CRA-value for the Euro has never deviated from its minimum value of 10 bps. Finally, the allowance of a volatility adjustment, the description of which is very opaque and which does not appear well justified by financial theory, is a decision that seems highly questionable.

The string of peculiar implementation requirements mentioned above all seem biased in the same direction and could thus indicate a systematic attempt to “lift” the risk-free term structure curve up and away from its true location. The consequence of such an effort could of course be a systematic and massive undervaluation of L&P liabilities across Europe – as also clearly illustrated by our numerical illustrations – and this would appear in clear conflict with Solvency II’s stated objectives of promoting financial stability, of protecting policyholders and of ensuring “market consistency, prudent assessment of technical provisions and optimal use of market information” (EIOPA, 2018b, p. 12).

Another potential problem with EIOPAs term structure model is the hedging incentives that it may generate for life and pension companies in need of managing the risk of their long term liabilities. When considering their regulatory balance sheets these companies may find an enormous amount of exposure to interest rate risk at the last liquid point. This is so since by construction of the Smith–Wilson model the value of all liability cash flows with maturities equal to or larger than the LLP will be highly sensitive to the interest rate prevailing at this particular point. This may again create an artificially high demand for fixed income hedge instruments with this maturity which may depress corresponding interest rates and thus create further obstacles in the way of correct economic and market consistent valuation of liabilities.³⁰ While there may thus be a particular problem with L&P companies influencing the market pricing of instruments with near-LLP maturities, the point that the L&P sector’s transactions can “move the market” is likely to apply more generally to the entire long end of the term structure. This issue is analyzed in a recent and important report from the [European Systemic Risk Board \(2017\)](#) and in a fine paper by [Domanski et al. \(2015\)](#). The ESRB report considers the broad macroprudential consequences of EIOPA’s risk-free yield curve as analyzed in this paper, and in a more general setting the paper by [Domanski et al. \(2015\)](#) shows that L&P companies’ “hunt for duration” may amplify declines in long term interest rates and that negative feedback loops may develop as a consequence.³¹ Such findings raise the question of whether it is really justified to assume L&P companies to be simple price takers as the market consistency concept of Solvency II does. It can thus be argued that Solvency II provides a microprudential framework that to some extent neglects the broader

macro-economic role the L&P industry plays in the capital markets, and future research could look into whether this can be improved upon.

In light of what seems to be a number of significant conflicts between, on the one hand, EIOPA’s detailed and specific implementation instructions for determining the term structure of risk-free interest rates and, on the other hand, both well-established financial theory and some of the fundamental and very sensible overall principles in the Solvency II Directive, it seems as if EIOPA’s key document, the “Technical documentation of the methodology to derive EIOPA’s risk-free interest rate term structures” (EIOPA, 2018b) could benefit from another round of revision. A first priority here must be to get the part about market consistent liability valuation right. As we have argued in this paper there can be little doubt that the way 0- to 30-year liabilities are handled today is not market consistent. EIOPA’s artificially early last liquid point and the more recent invention – the VA adjustment – constitute simple proof of that. The effective consequence of these manipulations is that life and pension companies are allowed to use two different term structure curves when setting up their balance sheets – one for assets (the true risk-free market curve) and another, higher, curve for valuing liabilities. That cannot be right.

While it is relatively easy to pinpoint things that are wrong with the 0- to 30-year span of the EIOPA term structure curve it is harder to devise how to correctly construct a term structure curve for very long maturities where markets for risk-free assets are illiquid or non-existent.³² But what we have documented in this paper is that EIOPA’s current approach implies a very aggressive “lift” of rates beyond 20 years towards a level which is much higher than the level of the longest interest rates that we can observe. Despite attempts to justify this convergence level – the UFR – with macroeconomic data it remains that no one can invest without risk at these rates and thus that using them for discounting hard, long-term pension guarantees conflicts severely not only with basic financial theory but also with the fundamental insurance principle of *prudence*, see e.g. [Welzel \(1996\)](#).

There are some obvious directions for future research. First of all we suggest that more work is done on how the term structure curve can be extrapolated in a way that takes proper account of price information from the “long end” of the market. This could result in recommendations for maintaining the use of the Smith–Wilson model but with a variable UFR that gives recognition to information contained in long market interest rates – a point that has also been made by others including ([Budiono, 2012](#); [European Systemic Risk Board, 2017](#)), and ([Rebel, 2012](#)). Another interesting and important subject for academic research would be an analysis of EIOPA’s various adjustments to the risk-free interest rate curve. In particular, the volatility adjustment that we have briefly introduced in this paper seems worthy of closer scrutiny. In addition to what has already been mentioned it could seem as if there is a conflict between the way in which the volatility adjustment is defined and the Solvency II Directive’s fundamental and very sensible principle that there should be made “no adjustment to take account of the own credit standing of the insurance or reinsurance undertaking” when valuing its liabilities (see Article 75(1b) of the [Solvency II Directive \(2009\)](#)). Future research should look closer into the details of EIOPA’s method for determining the VA and into whether it is really empirically and theoretically sound.

³⁰ [Lagerås and Lindholm \(2016\)](#) point out other problematic issues with the SW model in relation to hedging.

³¹ The ESRB report also agrees with many of the critical points raised here and the overall recommendation of the report is that “the parameters for deriving the regulatory euro risk-free yield curve should be reset”, (p. 29). More specifically, the report directly states that “The current level of the UFR, set at 4.2%, is too high”, (p. 25), it proposes to increase the LLP to 30 years, and to extend the convergence period from the current 40 to 100 years. Interestingly and in relation to the critique of the current 20-year LLP for the EUR, the report observes that “on three out of four measures of liquidity ... 30-year swaps appear, if anything, more liquid than 20-year swaps”, (p. 18).

³² See [Balter et al. \(2016\)](#) for an empirical investigation of the extrapolation of yield curves based on continuous time dynamic term structure models such as the famous Vasicek-model ([Vasicek, 1977](#)).

Appendix A. Some properties of Wilson’s function and their implications

Recall first that in the general case, the Smith–Wilson discount function is given as

$$P(t) = e^{-f_\infty t} + \sum_{i=1}^N \xi_i \left(\sum_{j=1}^J c_{i,j} \cdot W(t, u_j) \right), \quad t \geq 0, \quad (A.1)$$

with Wilson’s function defined as

$$\begin{aligned} W(t, u_j) &= e^{-f_\infty \cdot (t+u_j)} \left\{ \alpha \cdot \min(t, u_j) - 0.5e^{-\alpha \cdot \max(t, u_j)} \right. \\ &\quad \left. (e^{\alpha \cdot \min(t, u_j)} - e^{-\alpha \cdot \min(t, u_j)}) \right\} \\ &= \begin{cases} e^{-f_\infty \cdot (t+u_j)} \left\{ \alpha \cdot t - 0.5e^{-\alpha u_j} (e^{\alpha t} - e^{-\alpha t}) \right\} & t < u_j \\ e^{-f_\infty \cdot (t+u_j)} \left\{ \alpha \cdot u_j - 0.5e^{-\alpha t} (e^{\alpha u_j} - e^{-\alpha u_j}) \right\} & t \geq u_j. \end{cases} \end{aligned} \quad (A.2)$$

The parameters f_∞ and α are exogenously given and must be strictly positive. The ξ_i ’s are (unrestricted) calibration parameters. The payment dates, $\{u_j\}_{j=1, \dots, J}$, must be strictly positive and it is natural to also assume that the payments, $\{c_{i,j}\}_{i=1, \dots, N, j=1, \dots, J}$, are positive although this is not strictly necessary. Below we present a string of results relating to the properties of Wilson’s function.

Lemma 1. *Continuity and limits of the $W(t, u_j)$ ’s for $t \downarrow 0$ and $t \rightarrow \infty$*
 It follows directly from (A.2) that the $W(t, u_j)$ ’s are continuous in $t, t \in \mathbb{R}_{\geq 0}$, and that

$$\lim_{t \downarrow 0} W(t, u_j) = \lim_{t \downarrow 0} e^{-f_\infty \cdot (t+u_j)} \left\{ \alpha \cdot t - 0.5e^{-\alpha u_j} (e^{\alpha t} - e^{-\alpha t}) \right\} = 0,$$

and

$$\lim_{t \rightarrow \infty} W(t, u_j) = \lim_{t \rightarrow \infty} e^{-f_\infty \cdot (t+u_j)} \left\{ \alpha \cdot u_j - 0.5e^{-\alpha t} (e^{\alpha u_j} - e^{-\alpha u_j}) \right\} = 0.$$

Lemma 2. *Partial derivatives of the $W(t, u_j)$ ’s w.r.t. t and their limits*
 From (A.2) it follows that for $t < u_j$ we have

$$\frac{\partial W(t, u_j)}{\partial t} = -f_\infty \cdot W(t, u_j) + \alpha e^{-f_\infty \cdot (t+u_j)} \left\{ 1 - 0.5e^{-\alpha u_j} (e^{\alpha t} + e^{-\alpha t}) \right\}.$$

Similarly, for $t > u_j$ it holds that

$$\begin{aligned} \frac{\partial W(t, u_j)}{\partial t} &= -f_\infty \cdot W(t, u_j) + e^{-f_\infty \cdot (t+u_j)} \left\{ \alpha 0.5e^{-\alpha t} (e^{\alpha u_j} - e^{-\alpha u_j}) \right\} \\ &= -f_\infty \cdot W(t, u_j) + \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-(\alpha + f_\infty)t}. \end{aligned} \quad (A.3)$$

Using in part Lemma 1 it may now be noted that

$$\lim_{t \downarrow 0} \frac{\partial W(t, u_j)}{\partial t} = \alpha e^{-f_\infty u_j} (1 - e^{-\alpha u_j}) > 0, \quad (A.4)$$

and that

$$\lim_{t \rightarrow \infty} \frac{\partial W(t, u_j)}{\partial t} = 0. \quad (A.5)$$

It is finally of interest to note that

$$\begin{aligned} \lim_{t \uparrow u_j} \frac{\partial W(t, u_j)}{\partial t} &= -f_\infty \cdot W(u_j, u_j) + \alpha e^{-2f_\infty u_j} (1 - 0.5(1 + e^{-2\alpha u_j})) \\ &= -f_\infty \cdot W(u_j, u_j) + \frac{\alpha}{2} e^{-2f_\infty u_j} (1 - e^{-2\alpha u_j}), \end{aligned} \quad (A.6)$$

and that

$$\begin{aligned} \lim_{t \downarrow u_j} \frac{\partial W(t, u_j)}{\partial t} &= -f_\infty \cdot W(u_j, u_j) + \frac{\alpha}{2} (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-(\alpha + f_\infty)u_j} \\ &= -f_\infty \cdot W(u_j, u_j) + \frac{\alpha}{2} e^{-2f_\infty u_j} (1 - e^{-2\alpha u_j}) \\ &= \lim_{t \uparrow u_j} \frac{\partial W(t, u_j)}{\partial t}, \end{aligned} \quad (A.7)$$

which demonstrates that Wilson’s function is indeed differentiable w.r.t. t everywhere in its domain.

Lemma 3. *Positivity of the $W(t, u_j)$ ’s*

To establish that Wilson’s function is positive everywhere it will suffice to establish positiveness of the function

$$\begin{aligned} I(t, u_j) &\equiv e^{f_\infty \cdot (t+u_j)} W(t, u_j) \\ &= \alpha \cdot \min(t, u_j) - 0.5e^{-\alpha \cdot \max(t, u_j)} (e^{\alpha \cdot \min(t, u_j)} - e^{-\alpha \cdot \min(t, u_j)}) \\ &= \begin{cases} \alpha \cdot t - 0.5e^{-\alpha u_j} (e^{\alpha t} - e^{-\alpha t}) & t < u_j \\ \alpha \cdot u_j - 0.5e^{-\alpha t} (e^{\alpha u_j} - e^{-\alpha u_j}) & t \geq u_j. \end{cases} \end{aligned} \quad (A.8)$$

We first see that $I(\cdot, \cdot)$ is a continuous function and that

$$\lim_{t \downarrow 0} I(t, u_j) = 0, \quad (A.9)$$

and

$$\lim_{t \rightarrow \infty} I(t, u_j) = \alpha u_j > 0. \quad (A.10)$$

As regards the partial derivative of $I(\cdot, \cdot)$ w.r.t. t we conclude that

$$\frac{\partial I(t, u_j)}{\partial t} = \begin{cases} \alpha - 0.5e^{-\alpha u_j} (\alpha e^{\alpha t} + \alpha e^{-\alpha t}) & t < u_j \\ 0.5\alpha e^{-\alpha t} (e^{\alpha u_j} - e^{-\alpha u_j}) & t \geq u_j. \end{cases} \quad (A.11)$$

The differentiability of $I(t, u_j)$ in the point $t = u_j$ is established in the same manner as we did for $W(t, u_j)$ in Lemma 2.

Let us now consider the sign of $\frac{\partial I(t, u_j)}{\partial t}$. For $t < u_j$ we have

$$\begin{aligned} \frac{\partial I(t, u_j)}{\partial t} &= \alpha - 0.5e^{-\alpha u_j} (\alpha e^{\alpha t} + \alpha e^{-\alpha t}) \\ &= \alpha (1 - 0.5e^{-\alpha u_j} (e^{\alpha t} + e^{-\alpha t})) \\ &> \alpha (1 - 0.5e^{-\alpha t} (e^{\alpha t} + e^{-\alpha t})) \\ &= \alpha (1 - 0.5(1 + e^{-2\alpha t})) \\ &> \alpha (1 - 0.5(1 + 1)) \\ &= 0. \end{aligned} \quad (A.12)$$

For $t \geq u_j$ we see that

$$\begin{aligned} \frac{\partial I(t, u_j)}{\partial t} &= 0.5\alpha e^{-\alpha t} (e^{\alpha u_j} - e^{-\alpha u_j}) \\ &= 0.5\alpha e^{-\alpha t} e^{\alpha u_j} (1 - e^{-2\alpha u_j}) \\ &> 0, \end{aligned} \quad (A.13)$$

and we can conclude that $I(t, u_j)$ is a positive function that increases from 0 towards a constant, αu_j , as $t \rightarrow \infty$. Along with Lemma 1 this implies that Wilson’s function is positive, continuous, and differentiable function with limits as established above.

Lemma 4. *The initial instantaneous interest rate and the infinitely long zero-coupon interest rate*

As the relation between the discount function, $P(t)$, and continuously compounded zero-coupon interest rates, $r(t)$, is given as

$$P(t) = e^{-r(t) \cdot t}$$

or equivalently as

$$r(t) = -\frac{1}{t} \log P(t),$$

then in order for us to establish the initial instantaneous interest rate (r_0) and the infinitely long zero-coupon interest rate (r_∞) implied by the Smith–Wilson discount function we should consider the limits

$$r_0 \equiv \lim_{t \downarrow 0} -\frac{1}{t} \log P(t), \tag{A.14}$$

and

$$r_\infty \equiv \lim_{t \rightarrow \infty} -\frac{1}{t} \log P(t), \tag{A.15}$$

where $P(t)$ is the Smith–Wilson discount function given in (A.1).

Considering first (A.14) we see that both the numerator and the denominator of the expression tend to zero, so we invoke l'Hôpital's rule to first conclude that if the limits exist

$$\begin{aligned} r_0 &= \lim_{t \downarrow 0} -\frac{1}{t} \log P(t) \\ &= \lim_{t \downarrow 0} -\frac{\partial}{\partial t} \log P(t) \\ &= \lim_{t \downarrow 0} -\frac{\frac{\partial P(t)}{\partial t}}{P(t)}. \end{aligned} \tag{A.16}$$

At this point we note that we have established in Lemma 4 that

$$\begin{aligned} &= \lim_{t \downarrow 0} \left(-f_\infty \cdot e^{-f_\infty t} + \sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot \frac{\partial W(t, u_j)}{\partial t} \right) \\ &= -f_\infty + \sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot (\alpha e^{-f_\infty u_j} (1 - e^{-\alpha u_j})), \end{aligned} \tag{A.17}$$

where we have used the appropriate limiting result for the derivative of $W(\cdot, u_j)$ from Lemma 2. We conclude that the Smith–Wilson model's initial instantaneous interest rate, r_0 , is given by

$$r_0 = f_\infty - \sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot (\alpha e^{-f_\infty u_j} (1 - e^{-\alpha u_j})). \tag{A.18}$$

It can be noted that this is not in general guaranteed to be positive.

As for the infinitely long zero-coupon interest rate we must evaluate (A.15). Again both numerator and denominator are indeterminate and we thus apply l'Hôpital's rule once more to conclude that (if the limit exists):

$$\begin{aligned} r_\infty &= \lim_{t \rightarrow \infty} -\frac{\partial}{\partial t} \log P(t) \\ &= -\lim_{t \rightarrow \infty} \frac{\frac{\partial P(t)}{\partial t}}{P(t)}. \end{aligned} \tag{A.19}$$

Since the instantaneous forward rate at time t , f_t , is defined as $-\frac{\partial P(t)}{P(t) \partial t}$ we see that if we can establish this limit then we will at the same time have determined the instantaneous forward rate at the infinite horizon.

Let us now work on (A.19). Calculating and inserting the time-derivative of the Smith–Wilson discount function for large t we get

$$\begin{aligned} &= -\lim_{t \rightarrow \infty} \frac{\frac{\partial P(t)}{\partial t}}{P(t)} \\ &= \lim_{t \rightarrow \infty} \left(\frac{f_\infty e^{-f_\infty t} - \sum_{i=1}^N \xi_i \left(\sum_{j=1}^J c_{i,j} (-f_\infty W(t, u_j) + \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-(\alpha + f_\infty)t}) \right)}{P(t)} \right) \\ &= \lim_{t \rightarrow \infty} \left(\frac{f_\infty \cdot P(t) - \sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-(\alpha + f_\infty)t}}{P(t)} \right) \\ &= f_\infty - \lim_{t \rightarrow \infty} \left(\frac{\sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-(\alpha + f_\infty)t}}{P(t)} \right) \\ &= f_\infty - \lim_{t \rightarrow \infty} \left(\frac{\sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-(\alpha + f_\infty)t}}{e^{-f_\infty t} + \sum_{i=1}^N \xi_i \left(\sum_{j=1}^J c_{i,j} \cdot W(t, u_j) \right)} \right) \\ &= f_\infty - \lim_{t \rightarrow \infty} \left(\frac{\sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-\alpha t}}{1 + \sum_{i=1}^N \xi_i \left(\sum_{j=1}^J c_{i,j} \cdot e^{f_\infty t} W(t, u_j) \right)} \right) \\ &= f_\infty - \lim_{t \rightarrow \infty} \left(\frac{\sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-\alpha t}}{1 + \sum_{i=1}^N \xi_i \left(\sum_{j=1}^J c_{i,j} \cdot e^{f_\infty(t+u_j-u_j)} W(t, u_j) \right)} \right) \\ &= f_\infty - \lim_{t \rightarrow \infty} \left(\frac{\sum_{i=1}^N \xi_i \sum_{j=1}^J c_{i,j} \cdot \alpha 0.5 (e^{\alpha u_j} - e^{-\alpha u_j}) e^{-f_\infty u_j} e^{-\alpha t}}{1 + \sum_{i=1}^N \xi_i \left(\sum_{j=1}^J c_{i,j} \cdot e^{-f_\infty u_j} I(t, u_j) \right)} \right) \\ &= f_\infty. \end{aligned} \tag{A.20}$$

The final equality is obtained by noting that the numerator tends to zero and that – by the limiting result regarding the functions $I(\cdot, u_j)$ from Lemma 3 – the denominator converges to some strictly positive constant.

To reiterate we have proven in this second half of Lemma 4 that the infinitely long zero-coupon interest rate, r_∞ , and the Smith–

$\lim_{t \downarrow 0} P(t) = 1$ so it remains only to consider

$$\lim_{t \downarrow 0} \frac{\partial P(t)}{\partial t}$$

Wilson model's ultimate forward rate, f_∞ , coincide. And along the way we have established that the instantaneous forward rate (implied at time zero) for the infinitely long horizon is also equal to f_∞ , i.e. $\lim_{t \rightarrow \infty} f_t = f_\infty$. This enables us to give some interpretation to the parameter, f_∞ , that needs to be exogenously specified before estimating/calibrating the Smith–Wilson model to market prices.

References

- Andersson, H., Lindholm, M., 2013. On the Relation between the Smith-Wilson method and integrated Ornstein-Uhlenbeck processes. Research Report 2013:1. Mathematical Statistics, Stockholm University.
- Balter, A., Pelsser, A., Schotman, P., 2016. What does a term structure model imply about very long-term interest rates? Netspar Discussion Paper No. 02/2014-065. Available at <http://ssrn.com/abstract=2564487>.
- BIS, 2005. Zero-coupon yield curves: Technical Documentation. BIS Papers No. 25, Monetary and Economics Department. Bank for International Settlements.
- Budiono, D., 2012. Variable Ultimate Forward Rate Curve. Working paper, Syntus Achmea Asset Management, August 23, 2012. Available at <http://ssrn.com/abstract=2126717>.
- CEIOPS, 2010. Risk-free interest rates – Extrapolation method. QIS 5 Background Document. Available at <https://eiopa.europa.eu/>
- Domanski, D., Shin, H.S., Sushko, V., 2015. The Hunt for Duration: Not wading but drowning. BIS Working paper No. 519. Monetary and Economic Department, Bank for International Settlements.
- EIOPA, 2013. Technical Findings on the Long-Term Guarantees Assessment. Document EIOPA/13/296, 14 June, 2013. European Insurance and Occupational Pensions Authority.
- EIOPA, 2016. Consultation Paper on the methodology to derive the UFR and its implementation. Document EIOPA-CP-16/03, 6 April, 2016. European Insurance and Occupational Pensions Authority.
- EIOPA, 2017a. EIOPA sets out the methodology to derive the Ultimate Forward Rate (EIOPA Press Release), April 5, 2017.
- EIOPA, 2017b. Risk-free interest rate term structures: Specification of the methodology to derive the UFR. Technical Report. European Insurance and Occupational Pensions Authority, 30 March, 2017, Document EIOPA-BoS-17/072.
- EIOPA, 2018a. Risk-free Interest Rate Term Structures: Report on the Calculation of the UFR for 2019. Technical Report. European Insurance and Occupational Pensions Authority, EIOPA-BoS-18/141, March 21, 2018.
- EIOPA, 2018b. Technical documentation of the methodology to derive EIOPA's risk-free interest rate term structures. Technical Report. European Insurance and Occupational Pensions Authority, Version 31 January, 2018, Document EIOPA-BoS-15/035.
- European Systemic Risk Board, 2017. Regulatory risk-free yield curve properties and macroprudential consequences. Technical Report. European Systemic Risk Board, The ATC Expert Group on Insurance, August 2017.
- Filipovic, D., Trolle, A.B., 2013. The term structure of interbank risk. J. Financ. Econ. 109 (3), 707–733.
- Gach, F., 2016. Note on the Smith-Wilson interest rate curve. Int. J. Theoretical Appl. Finance 19 (7), 1–16.
- Jørgensen, P.L., 2004. On accounting standards and fair valuation of life insurance and pension liabilities. Scand. Actuar. J. 104 (5), 372–394.
- de Kort, J., Vellekoop, M.H., 2016. Term structure extrapolation and asymptotic forward rates. Insurance: Mathematics and Economics 67, 107–119.
- Lagerås, A., Lindholm, M., 2016. Issues with the Smith-Wilson method. Insurance: Mathematics and Economics 71, 93–102.
- Lundbergh, S., Laros, R., Rebel, L., 2014. Developments in European regulation: Risks and challenges. In: Mitchell, O.S., Maurer, R., Hammond, P.B. (Eds.), *Recreating Sustainable Retirement: Resilience, Solvency, and Tail Risk*. Oxford University Press, pp. 186–214.
- Nelson, C., Siegel, A.F., 1987. Parsimonious modelling of yield curves. J. Bus. 60, 473–489.
- Omnibus II Directive, 2014. Directive 2014/51/EU of the European Parliament and of the Council amending Directives 2003/71/EC and 2009/138/EC and Regulations (EC) No 1060/2009, (EU) No 1094/2010 and (EU) No 1095/2010 in respect of the powers of the European Supervisory Authority (European Insurance and Occupational Pensions Authority) and the European Supervisory Authority (European Securities and Markets Authority). Official Journal of the European Union L153/1–61.
- Rebel, L., 2012. The Ultimate forward rate methodology to value pension liabilities: A review and an alternative methodology. Working paper, Cardano Rotterdam and VU Amsterdam, September 5, 2012.
- Smith, A., Wilson, T., 2000. Fitting Yield Curves with Long Term Constraints. Technical Report. Bacon and Woodrow, August 2000.
- Solvency II Directive, 2009. Directive 2009/138/EC of the European Parliament and of the Council on the taking-up and pursuit of the business of insurance and reinsurance (Solvency II). Official Journal of the European Union L335/1–155.
- Svensson, L.E.O., 1994. Estimating and Interpreting Forward Interest Rates: Sweden 1992–1994. Working Paper No. 4871. National Bureau of Economic Research.
- The European Commission, 2015. Delegated regulation EU 2015/35 of 10 October 2014 supplementing directive 2009/138/EC of the European Parliament and of the Council on the taking-up and pursuit of the business of insurance and reinsurance (Solvency II). Official Journal of the European Union L12/1–797.
- Thomas, M., Maré, E., 2007. Long Term Forecasting and Hedging of the South African Yield Curve. Working Paper. Presentation at the 2007 Convention of the Actuarial Society of South Africa.
- Vasicek, O., 1977. An equilibrium characterization of the term structure. J. Financ. Econ. 5, 177–188.
- Veronesi, P., 2010. *Fixed Income Securities: Valuation, Risk, and Risk Management*. Wiley, New Jersey, USA.
- Welzel, H.-J., 1996. The prudence concept in EC insurance accounts law. The Geneva Papers on Risk and Insurance – Issues and Practice 21 (78), 36–49.