Portfolio Choice with Jumps: A Closed Form Solution*

Yacine Aït-Sahalia
Department of Economics
Princeton University
and NBER

Julio Cacho-Diaz
Department of Economics
Princeton University

T. R. Hurd
Dept. of Mathematics
and Statistics
McMaster University

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Abstract

We analyze the consumption-portfolio selection problem of an investor facing both Brownian and jump risks. By adopting a factor structure for the asset returns and decomposing the two types of risks on a well chosen basis, we provide a new methodology for determining the optimal solution up to an implicitly defined constant, which in some cases can be reduced to a fully explicit closed form, irrespectively of the number of assets available to the investor. We show that the optimal policy is for the investor to focus on controlling his exposure to the jump risk, while exploiting differences in the asset returns diffusive characteristics in the orthogonal space. We also examine the solution to the portfolio problem as the number of assets increases and the impact of the jumps on the diversification of the optimal portfolio.

Keywords: Optimal portfolio; jumps; Merton problem; closed form solution.

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1. Introduction

Economists have long been aware of the potential benefits of international diversification, while at the same time noting that the portfolios held by actual investors typically suffer from a home bias effect (see e.g., Grubel (1968), Levy and Sarnat (1970), Solnik (1974), Grauer and Hakansson (1987)). One possible explanation is due to the risk of contagion across markets in times of crisis, which is a well-documented phenomenon (see e.g., Claessens and Forbes (2001), Longin and Solnik (2001), Ang and Chen (2002), Bae et al. (2003) and Hartmann et al. (2004).)

Jumps of correlated sign will generate the type of asymmetric correlation across markets that is often used to justify the home bias exhibited by investors' portfolios. Namely, when a downward jump occurs, negative returns tend to be experienced simultaneously across most markets, which then results in a high positive correlation in bear markets. When no jump occurs, the only source of correlation is that generated by the driving Brownian motions and will typically be much lower.

Studying the impact of jumps on portfolio choice has a long history, going back to Merton (1971), who first studied a continuous-time consumption-portfolio problem. Many papers have considered the portfolio problem, either in the simple one-period Markowitz setting or in the more complex Merton setting, when asset returns are generated by jump processes, for instance Poisson processes, stable processes or more general Lévy processes. Early papers include Aase (1984), Jeanblanc-Picque and Pontier (1990) and Shirakawa (1990). More recently, see Han and Rachev (2000) and Ortobelli et al. (2003) for a study of the Markowitz one-period mean-variance problem when asset returns follow a stable-Paretian distribution; Kallsen (2000) for a study of the continuous-time utility maximization in a market where risky security prices follow Lévy processes, and a solution (up to integration) for power, logarithmic and exponential utility using the duality or martingale approach; Choulli and Hurd (2001) give solutions up to constants of the primal and dual Merton portfolio optimization problem for the exponential, power and logarithmic utility functions when a risk-free asset and a exponential Lévy stock are the investment assets; Liu et al. (2003) study the implications of jumps in both prices and volatility on investment strategies when a risk-free asset and a stochastic-volatility jump-diffusion stock are the available investment opportunities; Emmer and Klüppelberg (2004) study a continuous-time mean-variance problem with multiple assets; Madan (2004) derives the equilibrium prices in an economy with single period returns driven by exposure to explicit non-Gaussian systematic factors plus Gaussian idiosyncratic components. Cvitanić et al. (2005) propose a model where the asset returns have higher moments due to jumps and study the sensitivity of the investment in the risky asset to the higher moments, as well as the resulting utility loss from ignoring the presence of higher moments.

The potential role of jumps in generating contagion across markets, and hence limiting the benefits of diversification, has been investigated by Das and Uppal (2004), who evaluate the effect on portfolio choice of systemic risk, defined as the risk from infrequent events that are highly correlated across a large number of assets. They find that systemic risk reduces the gains from diversifying across a range of assets, and makes leveraged portfolios more susceptible to large losses. Upon calibrating their model to index returns, they find that the loss from the reduction in diversification is not substantial. Ang and Bekaert (2002) consider a two-regime model in a discrete-time setting, one with low correlations and low volatilities, and one with higher correlations, higher volatilities, and lower conditional means. They find that the existence of a high-volatility bear market regime does not negate the benefits of international diversification for an investor who dynamically rebalances his portfolio in response to regime switches.

In the presence of jumps, the portfolio choice problem has not been amenable to a closed-form solution so far. With n assets, one must solve numerically an $n-$dimensional nonlinear equation. With more efficient global markets, capital flows and a considerably larger number of available assets to invest in, an investor has more investment opportunities than ever before. So we would certainly like to be able to solve models with a large number of assets n. This is difficult to do using existing methodologies.

This paper's main contribution is to show how the solution can be obtained in closed-form, irrespectively of the number of assets. We assume that asset returns follow exponential Lévy processes, which are a natural generalization of the standard geometric Brownian motion model. With n assets, the space of returns is $\mathbb{R}^n$. We show that the jump risk occurs in a well-defined portion of $\mathbb{R}^n$, say $\bar{V}$, and we decompose $\mathbb{R}^n$ into $\bar{V}$ and the orthogonal space, $V^{\perp}$. Brownian risk, by contrast, occurs throughout $\mathbb{R}^n$. By adopting a factor structure on the Brownian variance-covariance of returns, we manage to split the Brownian risk into a portion occurring in $\bar{V}$, where it adds up to the jump risk, and a portion in $V^{\perp}$, where it is the sole source of risk. We are then able to distinguish between the optimal portfolio positions in the space $\bar{V}$ spanned by the jump risk (which the investor will attempt to limit) and those in the orthogonal space $V^{\perp}$ (where the investor will seek to exploit the opportunities arising from the traditional risk-return trade-off.)

In our model, the structure of the Brownian variance-covariance matrix is taken to reflect the existence of one or more economic sectors, each sector comprising a large number of related companies or countries. We start with the case where there is a single economic sector, and then consider the more general case where the economy consists of m sectors or regions of the world, each consisting of k firms or countries. We allow for as many jump terms as there are sectors; those jumps can affect only their sector, or all or some of the sectors.

In general terms, the structure of the optimal solution is as follows. If there is enough cross-sectional variability in the expected excess returns, then the investor will place a linearly increasing amount of wealth in the risky assets as the number of assets n grows. This, in turn, leads to increasing expected return and volatility of the portfolio value, both growing linearly in the number of assets. This happens because of exposure to the risky assets that the investor acquires in the space $V^{\perp}$. But the optimal policy in $\bar{V}$ is to control the exposure to jumps by keeping it bounded as the number of assets grows. As a result, the exposure to jumps is dwarfed by the exposure to diffusive risk asymptotically in n. Indeed, the additional investments in the risky assets are entirely in the direction that is orthogonal to the jump risk; they are all achieved with zero net additional exposure to the jump risk.

In other words, the optimal investment policy is to control the overall exposure to jump risk in $\bar{V}$, and then exploit, in the orthogonal space $V^{\perp}$, any perceived differences in expected returns and diffusive variances and covariances. But in the special case where the expected excess returns have little variability in the orthogonal space, the opportunities for diversification effects are weak since controlling the exposure to jumps trumps other concerns (including the usual diversification policy.) The optimal portfolio in this latter case is not much better protected against those correlated jumps than a nondiversified portfolio.

The rest of the paper is organized as follows. In Section 2, we present our model of asset returns, and examine the investor's portfolio selection problem. In Section 3, we consider a one sector economy where the n risky assets have the same jump size and derive the optimal portfolio weights in closed form. In Section 4, we study the dependence of the optimal portfolio weights on the arrival intensity of the jumps, their magnitude, and the degree of risk aversion of the investor. In Section 5, we extend the model to an m-sector economy where sectors have different jump sizes and show how to solve the optimal portfolio problem in that case, again in closed form. Conclusions are in Section 6.

2. The Portfolio Selection Model

2.1. Asset Return Dynamics

Like most of the above-mentioned literature, our paper focuses on Merton's problem of maximizing the infinite-horizon expected utility of consumption by investing in a set of risky assets. That is, we select the amounts to be held in the n risky assets and the riskless asset at times $t \in [0, \infty)$. The available investment opportunities consist of a riskless asset with price $S_{0,t}$ and n risky assets with prices $\mathbf{S}_t = [S_{1,t}, \ldots, S_{n,t}]'$.

Asset prices follow the exponential Lévy dynamics

$$\frac{dS_{0,t}}{S_{0,t}} = r dt, \tag{2.1}$$

$$\frac{dS_{i,t}}{S_{i,t-}} = (r + R_i)\, dt + \sum\nolimits_{j=1}^{n} \sigma_{i,j} dW_{j,t} + J_i dY_t, \;\; i = 1, \ldots, n \tag{2.2}$$

with a constant rate of interest $r \geq 0$. $\mathbf{W}_t = [W_{1,t}, \ldots, W_{n,t}]'$ is an $n-$dimensional standard Brownian motion. Y_t is a Lévy pure jump process with Lévy measure $\lambda\nu(dz)$, where $\lambda \geq 0$ is a fixed parameter and the measure ν satisfies $\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty$. For any measurable subset A of the real line, $\lambda\nu(A) = \lambda \int_A \nu(dz)$ measures the (possibly infinite) expected number of jumps, per unit of time, whose size belongs to A.

The economy-wide jump amplitude Y_t is scaled on an asset-by-asset basis by the scaling factor J_i. We assume that the random variable $J_i dY_t$ has support on $(-1, \infty)$ which guarantees the positivity (or limited liability) of S_i. We also assume that the jump process Y and the individual Brownian motions in $\mathbf{W}$ are mutually independent.

In the special case where Y is a compound Poisson process, $dY_t = Z_t dN_t$, where N_t is a scalar Poisson process with constant intensity parameter $\lambda > 0$, Z_t is a random jump amplitude with probability measure $\nu(dz)$. Then $\mathbf{S}$ follows a jump-diffusion. In the absence of a jump term altogether, we obtain the usual geometric Brownian motion dynamics for $\mathbf{S}$.

The quantities R_i, σ_{ij} and J_i are constant parameters. We write $\mathbf{R} = [R_1, \ldots, R_n]'$, $\mathbf{J} = [J_1, \ldots, J_n]'$, and $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}'$ where

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix}. \tag{2.3}$$

We assume that Σ is a nonsingular matrix. For now, there is a single jump term in (2.2). In Section 5 below, we will generalize the model to include multiple jumps terms.

2.2. Wealth Dynamics and Expected Utility

Let $\omega_{0,t}$ denote the percentage of wealth (or portfolio weight) invested at time t in the riskless asset and $\boldsymbol{\omega}_t = [\omega_{1,t}, \ldots, \omega_{n,t}]'$ denote the vector of portfolio weights in each of the n risky assets, assumed to be adapted caglad processes. The portfolio weights satisfy

$$\omega_{0,t} + \sum\nolimits_{i=1}^{n} \omega_{i,t} = 1. \tag{2.4}$$

The investor consumes C_t at time t. In the absence of any income derived outside his investments in these assets, the investor's wealth, starting with the initial endowment X_0, follows the dynamics

$$\begin{aligned} dX_t &= -C_t dt + \omega_{0,t} X_t \frac{dS_{0,t}}{S_{0,t-}} + \sum\nolimits_{i=1}^{n} \omega_{i,t} X_t \frac{dS_{i,t}}{S_{i,t-}} \\ &= \left(rX_t + \boldsymbol{\omega}_t'\mathbf{R}X_t - C_t\right) dt + X_t \boldsymbol{\omega}_t' \boldsymbol{\sigma} d\mathbf{W}_t + X_t \boldsymbol{\omega}_t' \mathbf{J} dY_t. \end{aligned} \tag{2.5}$$

The investor's problem at time t is then to pick the consumption and portfolio weights $\{C_s, \boldsymbol{\omega}_s\}_{t \leq s \leq \infty}$ which maximize the infinite horizon, discounted at rate β, expected utility of consumption

$$V(X_t, t) = \max_{\{C_s, \boldsymbol{\omega}_s;\, t \leq s \leq \infty\}} E_t \left[\int_t^{\infty} e^{-\beta s} U(C_s) ds \right] \tag{2.6}$$

subject to the dynamics of his discounted wealth (2.5), and with X_t given. We will consider in detail in the rest of the paper the case where the investor has power utility, and in the Appendix the cases of exponential and log utilities, respectively.

Using stochastic dynamic programming and the appropriate form of Itô's lemma for semi-martingale processes, the Hamilton-Jacobi-Bellman equation characterizing the optimal solution to the investor's problem is:

$$\begin{aligned} 0 = \max_{\{C_t,\boldsymbol{\omega}_t\}} \Bigg\{ & e^{-\beta t}U(C_t) + \frac{\partial V\left(X_t,t\right)}{\partial t} + \frac{\partial V\left(X_t,t\right)}{\partial X}\left(rX_t + \boldsymbol{\omega}_t'\mathbf{R}X_t - C_t\right) \\ & + \frac{1}{2}\frac{\partial^2 V\left(X_t,t\right)}{\partial X^2}X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t \\ & + \lambda\int\left[V\left(X_t + X_t\boldsymbol{\omega}_t'\mathbf{J}z,t\right) - V\left(X_t,t\right)\right]\nu\left(dz\right)\Bigg\} \end{aligned} \tag{2.7}$$

with the transversality condition $\lim_{t\to\infty} E\left[V\left(X_t,t\right)\right] = 0$ (see Merton (1969).)

Using the standard time-homogeneity argument for infinite horizon problems, we have

$$\begin{aligned} e^{\beta t}V\left(X_t,t\right) &= \max_{\{C_s,\boldsymbol{\omega}_s;\; t\le s\le\infty\}} E_t\left[\int_t^\infty e^{-\beta(s-t)}U(C_s)ds\right] \\ &= \max_{\{C_s,\boldsymbol{\omega}_s;\; t\le s\le\infty\}} E_t\left[\int_0^\infty e^{-\beta u}U(C_{t+u})du\right] \\ &= \max_{\{C_s,\boldsymbol{\omega}_s;\; 0\le s\le\infty\}} E_t\left[\int_0^\infty e^{-\beta u}U(C_u)du\right] \\ &\equiv L(X_t) \end{aligned}$$

is independent of time. Thus $V\left(X_t,t\right) = e^{-\beta t}L(X_t)$ and (2.7) reduces to the following equation for the time-homogeneous value function L :

$$\begin{aligned} 0 = \max_{\{C_t,\boldsymbol{\omega}_t\}} \Bigg\{ & U(C_t) - \beta L(X_t) + \frac{\partial L\left(X_t\right)}{\partial X}\left(rX_t + \boldsymbol{\omega}_t'\mathbf{R}X_t - C_t\right) + \frac{1}{2}\frac{\partial^2 L\left(X_t\right)}{\partial X^2}X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t \\ & + \ \lambda\int\left[L\left(X_t + X_t\boldsymbol{\omega}_t'\mathbf{J}z\right) - L\left(X_t\right)\right]\nu\left(dz\right)\Bigg\} \end{aligned} \tag{2.8}$$

with the transversality condition

$$\lim_{t\to\infty} E\left[e^{-\beta t}L\left(X_t\right)\right] = 0. \tag{2.9}$$

The maximization problem in (2.8) separates into one for C_t, with first order condition

$$\frac{\partial U\left(C_t\right)}{\partial C} = \frac{\partial L\left(X_t\right)}{\partial X}$$

and one for $\boldsymbol{\omega}_t$:

$$\max_{\{\boldsymbol{\omega}_t\}}\left\{\frac{\partial L\left(X_t\right)}{\partial X}\boldsymbol{\omega}_t'\mathbf{R}X_t+\frac{1}{2}\frac{\partial^2 L\left(X_t\right)}{\partial X^2}X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t\right.$$
$$\left.+\ \lambda\int\left[L\left(X_t+X_t\boldsymbol{\omega}_t'\mathbf{J}z\right)-L\left(X_t\right)\right]\nu\left(dz\right)\right\} \tag{2.10}$$

Given wealth X_t, the optimal consumption choice is therefore

$$C_t^*=\left[\frac{\partial U}{\partial C}\right]^{-1}\left(\frac{\partial L\left(X_t\right)}{\partial X}\right). \tag{2.11}$$

In order to determine the optimal portfolio weights, wealth and value function, we need to be more specific about the utility function U.

2.3. Power Utility

Consider an investor with power utility, $U\left(c\right)=c^{1-\gamma}/\left(1-\gamma\right)$ for $c>0$ and $U\left(c\right)=-\infty$ for $c\leq 0$ with CRRA coefficient $\gamma\in(0,1)\cup(1,\infty)$. (In the Appendix, we treat the exponential and log utility cases.) We will look for a solution to (2.8) in the form

$$L(x)=K^{-\gamma}x^{1-\gamma}/\left(1-\gamma\right) \tag{2.12}$$

where K is a constant, so that

$$\frac{\partial L\left(x\right)}{\partial x}=\left(1-\gamma\right)L(x)/x,\quad \frac{\partial^2 L\left(x\right)}{\partial x^2}=-\gamma\left(1-\gamma\right)L(x)/x^2. \tag{2.13}$$

Then (2.8) reduces to

$$\begin{aligned}0=\max_{\{C_t,\omega_t\}}\Bigg(&U\left(C_t\right)-\beta L\left(X_t\right)+\left(1-\gamma\right)L(X_t)\left(\boldsymbol{\omega}_t'\mathbf{R}+r\right)-\left(1-\gamma\right)C_t\frac{L\left(X_t\right)}{X_t}\\&-\frac{1}{2}\gamma\left(1-\gamma\right)L(X_t)\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t\\&+\lambda\int\left[\left(1+\boldsymbol{\omega}_t'\mathbf{J}z\right)^{1-\gamma}L\left(X_t\right)-L\left(X_t\right)\right]\nu\left(dz\right)\Bigg)\end{aligned} \tag{2.14}$$

that is

$$\begin{aligned}0=\min_{\{C_t,\omega_t\}}\Bigg(&-\frac{U\left(C_t\right)}{\left(1-\gamma\right)L\left(X_t\right)}+\frac{\beta}{\left(1-\gamma\right)}-\left(r+\boldsymbol{\omega}_t'\mathbf{R}\right)+\frac{C_t}{X_t}\\&+\frac{1}{2}\gamma\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t-\frac{\lambda}{\left(1-\gamma\right)}\int\left[\left(1+\boldsymbol{\omega}_t'\mathbf{J}z\right)^{1-\gamma}-1\right]\nu\left(dz\right)\Bigg)\end{aligned} \tag{2.15}$$

after division by $-\left(1-\gamma\right)L(X_t)<0$, so that max becomes min.

2.4. Optimal Policies

The optimal policy for the portfolio weights $\boldsymbol{\omega}_t$ is

$$\boldsymbol{\omega}_t^* = \arg\min_{\{\boldsymbol{\omega}_t\}} g(\boldsymbol{\omega}_t). \tag{2.16}$$

where the functions

$$g(\boldsymbol{\omega}) = -\boldsymbol{\omega}'\mathbf{R} + \frac{\gamma}{2}\boldsymbol{\omega}'\Sigma\boldsymbol{\omega} + \lambda\psi\left(\boldsymbol{\omega}'\mathbf{J}\right) \tag{2.17}$$

and

$$\psi\left(\boldsymbol{\omega}'\mathbf{J}\right) = -\frac{1}{(1-\gamma)}\int\left[\left(1+\boldsymbol{\omega}'\mathbf{J}z\right)^{1-\gamma} - 1\right]\nu\left(dz\right). \tag{2.18}$$

are both convex.

Since $(\gamma, \mathbf{R}, \Sigma, \mathbf{J})$ are constant, the objective function g is time independent, so it is clear that any optimal solution will be time independent. Furthermore, the objective function is state independent, so any optimal solution will also be state independent. In other words, any optimal $\boldsymbol{\omega}_t^*$ will be a constant $\boldsymbol{\omega}^*$ independent of time and state. Further, the objective function g is strictly convex, goes to $+\infty$ in all directions, and hence always has a unique minimizer. In the pure diffusive case, $\lambda = 0$ and we obtain of course the familiar Merton solution

$$\boldsymbol{\omega}^* = \frac{1}{\gamma}\Sigma^{-1}\mathbf{R}. \tag{2.19}$$

As to the optimal consumption policy, with $\left[\partial U/\partial C\right]^{-1}(y) = y^{-1/\gamma}$ and $\partial L\left(x\right)/\partial x = K^{-\gamma}x^{-\gamma}$ in equation (2.11), we obtain

$$C_t^* = K\ X_t. \tag{2.20}$$

Next, we evaluate equation (2.15) at the optimal policies $(C_t^*, \boldsymbol{\omega}^*)$ to identify the constant K :

$$K = \frac{\beta}{\gamma} - \frac{(1-\gamma)}{\gamma}\left[\boldsymbol{\omega}^{*\prime}\mathbf{R} + r\right] + \frac{1}{2}\left(1-\gamma\right)\boldsymbol{\omega}^{*\prime}\Sigma\boldsymbol{\omega}^* + \frac{\left(1-\gamma\right)\lambda}{\gamma}\psi\left(\boldsymbol{\omega}^{*\prime}\mathbf{J}\right). \tag{2.21}$$

The constant K will be fully determined once we have solved below for the optimal portfolio weights, $\boldsymbol{\omega}^*$.

Finally, we have to check that the transversality condition is satisfied. By plugging the optimizers X^* and C_t^* into (2.6), and then taking expectations, one finds

$$E\left[V\left(X_t^*, t\right)\right] = E\left[\int_t^\infty e^{-\beta s}U(C_s^*)ds\right].$$

Now $e^{-\beta s}U\left(C_s^*\right) = KV\left(X_s^*, s\right)$ from which it follows that $E\left[V\left(X_t^*, t\right)\right]$ solves $df/dt = -Kf$ and hence decays exponentially to zero as $t \to \infty$, for any $K > 0$.

3. Optimal Portfolio in a One Sector Economy with Homogenous Jumps

3.1. The Orthogonal Decomposition

To begin, we consider the simplest possible case, where the n risky assets have identical jump size characteristics

$$\mathbf{J} = \bar{J}\mathbf{1} \tag{3.1}$$

where $\bar{J}$ is a scalar, and $\mathbf{1}$ is the $n-$vector $\mathbf{1} = [1, ..., 1]'$. The key to finding the solution in closed form is to pick a basis of the space of returns, $\mathbb{R}^n$, that isolates the direction of the jump vector. In this simple case, jumps are parallel to the vector $\mathbf{1}$, so we are led to consider the orthogonal decomposition $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ where $\bar{V}$ is the span of $\mathbf{1}$ and $V^{\perp}$ is the orthogonal hyperplane.

By construction, all jump risk is contained in $\bar{V}$. However, Brownian risk occurs throughout $\mathbb{R}^n$. We will then have to figure out which part of the Brownian risk takes place in $\bar{V}$ (call it $\bar{\Sigma}$) and which part takes place in $V^{\perp}$ (call it $\Sigma^{\perp}$.) This is where assuming a factor structure for the Brownian variance-covariance matrix of returns Σ will be useful, as it will make that decomposition explicit. But throughout the paper, including the multisector case of Section 5 where the jump and factor structures are much richer, we will always decompose the returns space $\mathbb{R}^n$ into the direction of the jump vector(s), where both Brownian and jump risks occur (for now it is simply $\mathbf{1}$) and the orthogonal space (where only Brownian risk occurs). That decomposition is illustrated in Figure 1.

As will become clear below, this decomposition leads to a separation of the problem into two subproblems: one in $\bar{V}$, and one in $V^{\perp}$. Because $V^{\perp}$ contains only Brownian risk, the solution in that space will be the Merton no-jump solution, except that it will be based not on the full Σ, but just on the fraction $\Sigma^{\perp}$ of Brownian risk that lies in that space. That part of the solution will be explicit, as the Merton no-jump solution always is. In $\bar{V}$, we will have to deal with both sources of risk.

But we will already have achieved dimension-reduction: the solution for the portfolio weights in the large-dimensional space, $V^{\perp}$, is known, and the remaining unknowns in $\bar{V}$ are low-dimensional (one-dimensional here in fact). With more structure on the dynamics generating the asset returns, namely specific assumptions on the jump measure ν, we will show how that last unknown (the optimal portfolio policy in $\bar{V}$) can also be obtained in closed form.

So, the key is to work with the orthogonal decomposition of $\mathbb{R}^n$ that is suggested by the jumps. With that in mind, we therefore decompose the vector of excess returns as follows

$$\mathbf{R} = \bar{R}\mathbf{1} + \mathbf{R}^{\perp} \tag{3.2}$$

where $\bar{R}$ is a scalar and $\mathbf{R}^{\perp}$ is an $n-$vector orthogonal to $\mathbf{1}$.

As for Σ, we assume for now the one-factor structure

$$\Sigma = v^2 \begin{pmatrix} 1 & \rho & \cdots \\ \rho & \ddots & \rho \\ \cdots & \rho & 1 \end{pmatrix} \tag{3.3}$$

where $v^2 > 0$ is the variance of the returns generated by the diffusive risk, and $-1/(n-1) < \rho < 1$ is their common correlation coefficient.

The Σ matrix decomposes on $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ as follows:

$$\Sigma \quad = \quad \underbrace{\kappa_1 \frac{1}{n}\mathbf{1}\mathbf{1}'}_{=\ \bar{\Sigma}} \; + \; \underbrace{\kappa_2 \left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)}_{=\ \Sigma^{\perp}} \tag{3.4}$$

where $\mathbf{I}$ denotes the $n \times n$ identity matrix and

$$\kappa_1 = v^2 + v^2 (n-1)\rho \tag{3.5}$$

$$\kappa_2 = v^2 (1-\rho) \tag{3.6}$$

are the two distinct eigenvalues of Σ, κ_1 with multiplicity 1 and eigenvector $\mathbf{1}$ and κ_2 with multiplicity $n-1$.

We then search for the optimal portfolio vector $\boldsymbol{\omega}$ in the form

$$\boldsymbol{\omega} = \bar{\omega}\mathbf{1} + \boldsymbol{\omega}^{\perp}, \tag{3.7}$$

where $\bar{\omega}$ is scalar and $\boldsymbol{\omega}^{\perp}$ is an $n-$vector orthogonal to $\mathbf{1}$.

3.2. Optimal Portfolio Separation

Given the decomposition (3.4), we see from (2.16)-(2.17) that the optimal $\bar{\omega}^*$ and $\boldsymbol{\omega}^{\perp *}$ must satisfy

$$\left(\boldsymbol{\omega}^{\perp *}, \bar{\omega}^*\right) = \arg \min_{\{\boldsymbol{\omega}^{\perp}, \bar{\omega}\}} \left\{ -n\bar{\omega}\bar{R} + \frac{1}{2}\gamma n \bar{\omega}^2 \kappa_1 + \lambda \psi \left(n\bar{\omega}\bar{J}\right) \right. \\ \left. - \boldsymbol{\omega}^{\perp\prime}\mathbf{R}^{\perp} + \frac{1}{2}\gamma \boldsymbol{\omega}^{\perp\prime}\kappa_2 \left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)\boldsymbol{\omega}^{\perp} \right\}. \tag{3.8}$$

And we now see that this separates into two distinct optimization problems: one for $\boldsymbol{\omega}^{\perp}$ and one for the scalar $\bar{\omega}$:

$$\begin{cases} \boldsymbol{\omega}^{\perp *} = \arg\min_{\boldsymbol{\omega}^{\perp}} \left\{ g^{\perp}(\boldsymbol{\omega}^{\perp}) \right\} \\ \bar{\omega}^* = \arg\min_{\bar{\omega}} \left\{ \bar{g}(\bar{\omega}) \right\} \end{cases} \tag{3.9}$$

where

$$g^{\perp}(\boldsymbol{\omega}^{\perp}) = - \underbrace{\boldsymbol{\omega}^{\perp\prime}\mathbf{R}^{\perp}}_{\text{Part of return in } V^{\perp}} + \underbrace{\frac{1}{2}\gamma\kappa_2 \boldsymbol{\omega}^{\perp\prime}\boldsymbol{\omega}^{\perp}}_{\text{Part of Brownian risk in } V^{\perp}} \tag{3.10}$$

$$\bar{g}(\bar{\omega}) = - \underbrace{n\bar{\omega}\bar{R}}_{\text{Part of return in } \bar{V}} + \underbrace{\frac{1}{2}\gamma n \bar{\omega}^2 \kappa_1}_{\text{Part of Brownian risk in } \bar{V}} + \underbrace{\lambda\psi\left(n\bar{\omega}\bar{J}\right)}_{\text{Jump risk}}. \tag{3.11}$$

In $V^{\perp}$, things are simple because there is no jump risk by construction. The first order condition for minimizing (3.10) is

$$-\mathbf{R}^{\perp} + \gamma\kappa_2 \boldsymbol{\omega}^{\perp *} = 0$$

whose solution is

$$\boldsymbol{\omega}^{\perp *} = \frac{1}{\gamma\kappa_2}\mathbf{R}^{\perp} = \frac{1}{\gamma v^2 (1-\rho)}\mathbf{R}^{\perp}. \tag{3.12}$$

This is nothing else than the Merton no-jump solution, but in restriction to the space $V^{\perp}$: that is, we take just the part of the Brownian risk and the vector of excess returns that lie in $V^{\perp}$, namely $\Sigma^{\perp}$ and $\mathbf{R}^{\perp}$, not the full Σ and $\mathbf{R}$ for the purpose of computing the Merton no-jump solution. The investor is going after excess returns , subject to the usual continuous-risk provisions: the higher his risk aversion, the higher the continuous variance of the returns, and the more they are correlated, the less he will invest.

In $\bar{V}$, we have to contend with both Brownian and jump risks. As far as the optimal solution for $\bar{\omega}$ is concerned, with the change of variable $\varpi_n = n\bar{\omega}$, we see that

$$\varpi_n^* = \arg\min_{\varpi_n} \left\{ -\varpi_n \bar{R} + \frac{1}{2}\gamma\varpi_n^2\kappa_1/n + \lambda\psi\left(\varpi_n \bar{J}\right) \right\}. \tag{3.13}$$

Letting $n \to \infty$, we have that $\kappa_1/n \to v^2\rho$ and so ϖ_n^* converges to a finite constant ϖ_∞^* given by

$$\varpi_\infty^* = \arg\min_{\varpi_\infty} \left\{ -\varpi_\infty \bar{R} + \frac{1}{2}\gamma\varpi_\infty^2 v^2\rho + \lambda\psi\left(\varpi_\infty \bar{J}\right) \right\}. \tag{3.14}$$

Below, we will show how to determine the constant ϖ_n^* (or, in the asymptotic case, ϖ_∞^*) in closed form under further assumptions on the distribution of the jumps and the investor's utility function. But for now, we see that the optimal portfolio choice is characterized by

$$\begin{aligned} \boldsymbol{\omega}^* &= \bar{\omega}^*\mathbf{1} + \boldsymbol{\omega}^{\perp *} \\ &= \underbrace{\frac{\varpi_n^*}{n}\mathbf{1}}_{\text{optimal policy in } \bar{V}} + \underbrace{\frac{1}{\gamma v^2(1-\rho)}\mathbf{R}^{\perp}}_{\text{optimal policy in } V^{\perp}}. \end{aligned} \tag{3.15}$$

where the last remaining unknown is the constant ϖ_n^* solution of (3.13).

The optimal policy is therefore to invest a common fraction ϖ_n^*/n of wealth into each of the risky assets, and to go long and short in proportions given by $\boldsymbol{\omega}^{\perp *}$ which are independent of the characteristics of the jumps. An investor who selects this optimal portfolio will achieve a wealth process X_t^* which follows a geometric Lévy process with characteristic triple (b, c, f)[1] such that b and c are $O(n)$, as long as there is sufficient cross-sectional dispersion of excess returns, in the sense that $\left\|\mathbf{R}^{\perp}\right\|^2 = \mathbf{R}^{\perp\prime}\mathbf{R}^{\perp} = O(n)$, while the Lévy jump measure f of wealth remains $O(1)$ as $n \to \infty$.

This means that expected excess returns $\mathbf{R}^{\perp}$ lead the investor to place a linearly increasing amount of wealth in the risky assets as n grows, which in turns leads to increasing expected returns b and variance c, both growing linearly in the number of assets. On the other hand, as n grows, the exposure to contagion jumps remains bounded, and is dwarfed by the exposure to diffusive risk. Indeed, the additional investment in the risky assets due to the presence of $\mathbf{R}^{\perp}$ is entirely in the direction of $\boldsymbol{\omega}^{\perp}$, which is orthogonal to $\mathbf{J}$. So these additional amounts invested in the risky assets are all achieved with zero net additional exposure to the jump risk.

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[1] See the Appendix for basic definitions regarding Lévy processes.

3.3. Fully Explicit Portfolio Weights

Some special cases lead to a closed form solution for the last remaining constant, ϖ_n^*, yielding fully closed form solutions for the optimal portfolio weights. For this, we need to specify the utility function and the Lévy measure $\nu(dz)$ driving the common jumps; then we can compute the integral in (3.14).

3.3.1. Negative Jumps

Consider the case of a power utility investor with CRRA coefficient $\gamma = 2$ and Lévy measure satisfying a power law, $\nu(dz) = dz/z$ with support on $[0, 1]$ and $\bar{J} \in (-1, 0]$ (to fix ideas, let us focus on the case $\bar{J} < 0$ in order to capture the downward risk inherent in the types of jumps we are concerned about.) Equation (3.14) specializes to

$$\varpi_\infty^* = \arg\min_{\varpi} \; f_\infty(\varpi) \tag{3.16}$$

where

$$\begin{aligned} f_\infty(\varpi) &= -\varpi\bar{R} + \varpi^2 v^2 \rho + \lambda \int_0^1 \left[\left(1 + \varpi \bar{J} z\right)^{-1} - 1 \right] dz/z \\ &= -\varpi\bar{R} + \varpi^2 v^2 \rho - \lambda \log\left(1 + \varpi\bar{J}\right). \end{aligned} \tag{3.17}$$

The first order condition (FOC) for ϖ is given by

$$-\bar{R} + 2\varpi v^2 \rho - \lambda \bar{J} \left(1 + \varpi\bar{J}\right)^{-1} = 0. \tag{3.18}$$

The optimal solution must satisfy

$$\bar{J}\varpi_\infty^* > -1 \tag{3.19}$$

otherwise, there is a positive probability of wealth X_t becoming negative, which is inadmissible in the power utility case. The asymptotic solution to equation (3.18) is the unique root ϖ_∞^* satisfying the solvency constraint (3.19), and that solution is given by

$$\varpi_\infty^* = \frac{-2\rho v^2 + \bar{J}\bar{R} + \sqrt{\left(2v^2\rho - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R} + \bar{J}\lambda)v^2\rho}}{4\bar{J}v^2\rho}. \tag{3.20}$$

It is also worth nothing that

$$\varpi_\infty^* < \frac{\bar{R}}{2v^2\rho} \tag{3.21}$$

so that the optimal investment in the risky assets is always less than what it would be in the absence of jumps. This is natural since $\bar{J} < 0$. Visual inspection of (3.20) also reveals that $\bar{J}$ and λ do not have a symmetric effect on the optimal portfolio weights (more on that below.) Figure 2 plots ϖ_∞^* as a function of $\bar{J}$ and λ.

In the exact small sample case, the optimal solution to (3.13) under the solvency constraint is

$$\varpi_n^* = \frac{-2\kappa_1/n + \bar{J}\bar{R} + \sqrt{\left(2\kappa_1/n - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R} + \bar{J}\lambda)\kappa_1/n}}{4\bar{J}\kappa_1/n}. \tag{3.22}$$

Figure 3 plots the objective function, $f_n(\varpi) = \bar{g}(\varpi/n)$ and shows its convergence to $f_\infty(\varpi)$ as $n \to \infty$, along with $\arg\min f_n(\varpi) = \varpi_n^*$, converging to ϖ_∞^*.

3.3.2. Negative and Positive Jumps

Another example is given again by the case of a power utility investor with CRRA coefficient $\gamma = 2$ but now the Lévy measure satisfies $\nu(dz) = \lambda dz/|z|$ with support on $[-1, 1]$, so jumps can be either positive or negative, and $\bar{J} \in (-1, 1)$. The jumps have mean zero in this case. The optimal solution must satisfy

$$\left|\bar{J}\varpi\right| < 1 \tag{3.23}$$

otherwise, there is a positive probability of wealth X_t becoming negative, which is inadmissible in the power utility case.

This model yields

$$\begin{aligned} f(\varpi) &= -\varpi\bar{R} + \varpi^2\kappa_1/n + \lambda\int_{-1}^{1}\left[\left(1 + \varpi\bar{J}z\right)^{-1} - 1\right]dz/|z| \\ &= -\varpi\bar{R} + \varpi^2\kappa_1/n - \lambda\log\left(1 - \varpi^2\bar{J}^2\right). \end{aligned} \tag{3.24}$$

The first order condition (FOC) for ϖ is given by

$$-\bar{R} + 2\varpi\left(\kappa_1/n + \lambda\bar{J}^2\left(1 - \varpi^2\bar{J}^2\right)^{-1}\right) = 0 \tag{3.25}$$

which is a cubic equation in ϖ.

For all values of ϖ that satisfy (3.23), the term in parentheses in (3.25) is always positive (recall that $\kappa_1 > 0$). Hence the solution ϖ^* has the same sign as $\bar{R}$, and $\varpi^* = 0$ if $\bar{R} = 0$. This is sensible since jumps have mean zero so in the absence of an expected return compensation from $\bar{R}$ the investor would never want to hold those assets. In this expression, the first term can be either > 0 or < 0, while the second and third are always going to be > 0.

The optimal solution to (3.13) under the solvency constraint (3.23) is

$$\begin{aligned}\varpi_n^* \quad = \quad & \frac{\bar{R}}{6\kappa_1/n} + \sqrt{\frac{4}{3}\left(\frac{\bar{R}^2}{12\left(\kappa_1/n\right)^2} + \frac{\lambda}{\kappa_1/n} + \frac{1}{J^2}\right)} \times \\ & \cos\left(\frac{1}{3}\arccos\left(\frac{\frac{\bar{R}}{6\kappa_1/n}\left(\frac{\bar{R}^2}{36(\kappa_1/n)^2} + \frac{\lambda}{2\kappa_1/n} - \frac{1}{\bar{J}^2}\right)}{\left(\sqrt{\frac{1}{3}\left(\frac{\bar{R}^2}{12(\kappa_1/n)^2} + \frac{\lambda}{\kappa_1/n} + \frac{1}{J^2}\right)}\right)^3}\right) + \frac{4}{3}\pi\right). \qquad (3.26)\end{aligned}$$

The asymptotically optimal solution ϖ_∞^* is obtained by replacing in the equation above κ_1/n with its limit $v^2\rho$ as $n \to \infty$. The optimal solution differs from the negative jump case in that ϖ_n^* is bounded when $\lambda \to \infty$ with limiting value of ϖ_n^* equal to 0. If $R > 0$, the investor faces a trade-off between the excess return, $\bar{R}$, (the investor is pushed up to $\varpi \uparrow \infty$) and the Brownian and jump risks, $\varpi^2\kappa_1/n - \lambda \log\left(1 - \varpi^2\bar{J}^2\right)$, (the investor is pulled down to $\varpi \downarrow 0$). This trade-off causes the investor to have bounded exposure to jump risk, $0 < \varpi_n^* < \bar{R}/\left(2\kappa_1/n\right)$.

3.3.3. Other Examples

Other cases that lead to a closed form solution for ϖ_n^* include power utility with $\gamma = 3$ and either power law jumps $\nu\left(dz\right) = \lambda dz/z$, or uniform jumps $\nu\left(dz\right) = \lambda dz$. In either case, the FOC is then a cubic equation, as (3.25), solvable in closed form using standard methods. Another case is one where the investor has log utility with jumps of a fixed size, $\nu\left(dz\right) = \delta\left(z = \bar{z}\right)dz$, for some $\bar{z} \in [-1, 1]$. Then the FOC leads to a quadratic equation, as (3.18).

4. Comparative Statics

Using the explicit solutions above, we can now investigate how the optimal portfolio responds to different jump intensities, jump sizes and degrees of risk aversion. To save space, we consider only the case of Section 3.3.1, where $\bar{J} < 0$.

4.1. Response to Jumps of Different Arrival Intensity

We have

$$\varpi^*_\infty \to -\infty \text{ as } \lambda \to \infty \tag{4.1}$$

$$\varpi^*_\infty \to \min\left(\frac{\bar{R}}{2v^2\rho}, -\frac{1}{\bar{J}}\right) \text{ as } \lambda \to 0. \tag{4.2}$$

The first limit means that the investor will go short to an unbounded extent on all the risky assets if the arrival rate of the jumps goes to infinity. This is to be expected, since $\bar{J} < 0$ and we impose no short sale constraints. Further, ϖ^*_∞ tends to $-\infty$ when $\lambda \to \infty$ at the following rate

$$\varpi^*_\infty = -\frac{\sqrt{\lambda}}{\sqrt{2v^2\rho}}(1 + o(1)).$$

If, on the other hand, the jumps become less and less frequent, $\lambda \to 0$, then ϖ^*_∞ tends to a finite limit driven by the diffusive characteristics of the assets, $\bar{R}/(2v^2\rho)$, which is the limit in the no jump case, *unless* the jump size $\bar{J}$ is so large that the solvency constraint binds. This gives rise to a kink in the demand function.

If the solvency constraint is not binding, then the higher the variance of the assets and/or the more heavily correlated they are, the smaller the investment in each one of them. And the higher the expected excess return of the assets $\bar{R}$, the higher the amount invested. For a small perceived jump risk (λ small), the optimal solution behaves like

$$\varpi^*_\infty = \min\left(\frac{\bar{R}}{2v^2\rho}, -\frac{1}{\bar{J}}\right) + \frac{\bar{J}\lambda}{|\bar{R} + \bar{J}\lambda|} + o(\lambda).$$

The first correction term is always negative so that the optimal policy is always within (3.19) and (3.21).

4.1.1. Jumps vs. Expected Return Trade-off

The weights ϖ_∞^* are monotonic in λ, with

$$\frac{\partial \varpi_\infty^*}{\partial \lambda} = \frac{\bar{J}}{\sqrt{\left(2v^2\rho - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R} + \bar{J}\lambda)v^2\rho}} < 0.$$

If $\bar{R} > 0$, there exists a critical value $\tilde{\lambda}$ such that

$$\varpi_\infty^* > 0 \text{ for } \lambda < \tilde{\lambda} \tag{4.3}$$

$$\varpi_\infty^* \leq 0 \text{ for } \lambda \geq \tilde{\lambda}. \tag{4.4}$$

That is, as long as jumps do not occur too frequently ($\lambda < \tilde{\lambda}$), the investor will go long on the assets in order to capture their expected return, even though that involves taking on the (negative) risk of the jumps. When the jumps occur frequently enough ($\lambda \geq \tilde{\lambda}$), then the investor decides to forgo the expected return of the assets and focuses on canceling his exposure to the jump risk by going short these assets.

The critical value $\tilde{\lambda}$ takes a particularly simple form. It is given by

$$\tilde{\lambda} = -\frac{\bar{R}}{\bar{J}\int_0^1 z\nu\,(dz)} \tag{4.5}$$

Clearly, the higher $\bar{R}$ relative to $(-\bar{J})$, the higher $\tilde{\lambda}$. And the smaller the expected value of Z, the bigger $\tilde{\lambda}$. With $\nu\,(dz) = dz/z$, we get

$$\tilde{\lambda} = -\frac{\bar{R}}{\bar{J}}. \tag{4.6}$$

This expression can be interpreted as an analogue to the Sharpe ratio, but for jump risk: excess return in the direction of jumps, $\bar{R}$, divided by a measure of the jump magnitude, $|\bar{J}|$.

Now, if $\bar{R} \leq 0$, then $\varpi_\infty^* \leq 0$ for every $\lambda \geq 0$. In that case, there is no point in ever going long those assets since both the expected return and the jump components negatively impact the investor's rate of return.

4.1.2. Flight to Quality

The solution above can capture a well-documented empirical phenomenon. Imagine an investor currently in a normal, low jump risk environment, who receives information suggesting that the jumps become more likely. Starting from a situation where $\lambda < \tilde{\lambda}$, if the perception of the jump risk increases ($\lambda \uparrow \tilde{\lambda}$), then the optimal policy for the investor is to flee-to-quality, by reducing his exposure to the risky assets ($\varpi_\infty^* \downarrow 0$) and investing the proceeds in the riskless asset. If the perception of the jump risk exceeds the critical value $\tilde{\lambda}$ given in (4.5), then the investor should go even further and start short–selling the risky assets. Because the jump risk affects all the assets, the perception of an increase in the intensity of the jumps leads the investor to dump all the risky assets indiscriminately.

4.2. Response to Jumps of Different Magnitudes

If we now concentrate on the effect of an increase in the jump size instead of the jump magnitude, then

$$\frac{\partial \varpi_\infty^*}{\partial \bar{J}} = \frac{1}{2\bar{J}^2}\left(1 - \frac{2\rho v^2 + \bar{J}\bar{R}}{\sqrt{\left(2v^2\rho - \bar{J}\bar{R}\right)^2 + 8\bar{J}(\bar{R}+\bar{J}\lambda)v^2\rho}}\right) > 0$$

for, as usual, $\bar{J} < 0$. The monotonicity implies that as the jump size gets closer to zero ($\bar{J} \uparrow 0$), the investor increases his holdings in the risky assets and conversely as $\bar{J} \downarrow (-1)$.

As to the sign of ϖ_∞^*, we have

$$\varpi_\infty^* > 0 \text{ for } -\bar{R}/\lambda < \bar{J} < 0 \tag{4.7}$$

$$\varpi_\infty^* < 0 \text{ for } -1 < \bar{J} < -\bar{R}/\lambda. \tag{4.8}$$

as long as $\bar{R}/\lambda < 1$. If $\bar{R}/\lambda > 1$, the expected return is high enough relative to the jump intensity that the investor will always maintain a positive investment $\varpi_\infty^* > 0$ in the different assets, no matter how large the jump size (within the constraint $\bar{J} > -1$, of course.)

4.3. Sensitivity to Risk Aversion

Here we consider the effect of the CRRA coefficient γ on ϖ_∞^*. For a CRRA investor, the first order condition of equation (3.14) is given by

$$-\bar{R} + \gamma\varpi_\infty v^2\rho - \lambda\int_0^1 \bar{J}z\left(1+\varpi_\infty\bar{J}z\right)^{-\gamma}\nu\left(dz\right) = 0, \tag{4.9}$$

then, making use of the implicit function theorem, we get

$$\frac{\partial \varpi_\infty^*}{\partial \gamma} = -\frac{\varpi_\infty^* v^2 \rho + \lambda \int_0^1 \bar{J} z \left(1 + \varpi_\infty^* \bar{J} z\right)^{-\gamma} \ln\left(1 + \varpi_\infty^* \bar{J} z\right) \nu\left(dz\right)}{\gamma \left(v^2 \rho + \lambda \int_0^1 \bar{J}^2 z^2 \left(1 + \varpi_\infty^* \bar{J} z\right)^{-\gamma - 1} \nu\left(dz\right)\right)}. \tag{4.10}$$

The denominator is always positive but the numerator could be negative, zero or positive depending on the sign of ϖ_∞^*. That is,

$$\frac{\partial \varpi_\infty^*}{\partial \gamma} = \begin{cases} > 0 & \text{if } \varpi_\infty < 0 \\ = 0 & \text{if } \varpi_\infty^* = 0 \\ < 0 & \text{if } \varpi_\infty^* > 0 \end{cases}. \tag{4.11}$$

This, in turn, implies that the higher the CRRA coefficient of an investor the smaller will be his ϖ_∞^* in absolute value. In the limit where γ increases to ∞, $|\varpi_\infty^*|$ decreases to zero.

5. Optimal Portfolio in a Multi–Sector Economy with Jumps

We now generalize our previous results by studying the more realistic portfolio selection problem in an economy composed of m sectors (or regions of the world), each containing k firms (or countries). The total number of assets available to the investor is $n = mk$.

To understand the qualitative implications of the solution for portfolio allocation in the presence of jumps, one would presumably be primarily interested in the situation where m is fixed and k goes to infinity with n. But we provide the full solution, including the special case where all the assets are fundamentally different, that is $k = 1$ and $m = n$.

5.1. Sector Jumps

As in the simple one-factor model of Section 3, we will decompose the space of returns as $\mathbb{R}^n = \bar{V} \oplus V^\perp$ where $\bar{V}$ is the span of the jump vector(s) and $V^\perp$ is the orthogonal hyperplane. With m sectors, it is now natural to assume that there is potentially a separate jump term per sector, which can affect not only its own sector but also the other sectors.

We are therefore replacing the dynamics of the risky asset returns in (2.2) with

$$\frac{dS_{i,t}}{S_{i,t-}} = \left(r + R_i\right) dt + \sum\nolimits_{j=1}^{n} \sigma_{i,j} dW_{j,t} + \sum\nolimits_{l=1}^{m} J_{i,l} dY_{l,t}, \quad i = 1, \ldots, n \tag{5.1}$$

where each $Y_{l,t}$ is a Lévy pure jump process with Lévy measure $\lambda_l \nu_l (dz)$, $l = 1, ..., m$. We assume that the Lévy pure jump processes and the Brownian motions are mutually independent and that the sum of the jumps have support on $(-1, \infty)$ to guarantee the positivity (or limited liability) of S_i. The constant $J_{i,l}$ is asset i's scaling of sector l's jump. Let $\mathbf{J}_l = [J_{1,l}, \ldots, J_{n,l}]'$.

Let us define the vector $\mathbf{1}_l$ as the $n-$vector with ones placed in the rows corresponding to the $l-$block and zeros everywhere else:

$$\mathbf{1}_l = [0, \ldots, 0, \underbrace{1, \ldots, 1}_{\text{sector } l}, 0, \ldots, 0]', \tag{5.2}$$

where the first 1 is located in the $k(l-1)+1$ coordinate, and the last one in the $kl+1$ coordinate.

We assume that each jump vector is of the form

$$\mathbf{J}_l = \sum\nolimits_{s=1}^{m} j_{s,l} \mathbf{1}_s = \ [\underbrace{j_{1,l}, \ldots, j_{1,l}}_{\text{sector } 1}, \underbrace{j_{2,l}, \ldots, j_{2,l}}_{\text{sector } 2}, \ldots, \underbrace{j_{m,l}, \ldots, j_{m,l}}_{\text{sector } m}]', \tag{5.3}$$

meaning that firms within a given sector have the same response to the arrival of a jump in a given sector, i.e. to a $dY_{t,l}$, but the proportional response $j_{s,l}$ of firms of different sectors to the arrival of a jump can be different, and also the proportional response of firms in a given sector to the arrival of jumps of different sectors can be different. Since some of the $j_{s,l}$ coefficients can be zero, jumps in one sector can affect only this sector or some or all of the sectors.

So the jump vectors are unrestricted linear combinations of the m vectors $\{\mathbf{1}_l\}_{l=1,..,m}$. Corresponding to the above structure, we now define the orthogonal decomposition $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ where $\bar{V}$ is the m-dimensional span of the vectors $\{\mathbf{1}_l\}_{l=1,..,m}$ that contains the jump vectors, and $V^{\perp}$ is the $(n-m)$-dimensional orthogonal hyperplane. By construction, $V^{\perp}$ contains no jump risk, while $\bar{V}$ contains all the jump risk. As before, there will be Brownian risk in both spaces. So the picture is similar to Figure 2 except that $\bar{V}$ is now of dimension m.

5.2. Multifactor Brownian Risk

To capture the notion of sectors, we assume a block-structure for the variance-covariance matrix of returns,

$$\underset{n\times n}{\Sigma} = \begin{pmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \cdots \\ \Sigma_{2,1} & \ddots & \Sigma_{2,m} \\ \cdots & \Sigma_{m,m-1} & \Sigma_{m,m} \end{pmatrix} \tag{5.4}$$

with within-sector blocks

$$\underset{k\times k}{\Sigma_{l,l}} = v_l^2 \begin{pmatrix} 1 & \rho_{l,l} & \cdots \\ \rho_{l,l} & \ddots & \rho_{l,l} \\ \cdots & \rho_{l,l} & 1 \end{pmatrix} \tag{5.5}$$

and across-sector blocks

$$\underset{k\times k}{\Sigma_{l,s}} = v_l v_s \begin{pmatrix} \rho_{l,s} & \rho_{l,s} & \cdots \\ \rho_{l,s} & \ddots & \rho_{l,s} \\ \cdots & \rho_{l,s} & \rho_{l,s} \end{pmatrix} \tag{5.6}$$

where $1 > \rho_{l,l} > \rho_{l,s}$ and $\rho_{l,l} \geq -1/(k-1)$ and $\rho_{l,s} \geq -1/(n-1)$. In an asset pricing framework, this corresponds to a multifactor model for the returns process with m common Brownian factors and n idiosyncratic Brownian shocks.

We now need to find out which part of the Σ matrix lines up with the jumps (in $\bar{V}$ defined by the specification of the m jump terms), and which part is orthogonal to them (in $V^{\perp}$). That decomposition of Σ is given by

$$\Sigma \;=\; \underbrace{\sum\nolimits_{l,s=1}^{m} \frac{\kappa_{l,s}}{k} \mathbf{1}_l \mathbf{1}_s'}_{=\;\bar{\Sigma}} \;+\; \Sigma^{\perp} \tag{5.7}$$

where

$$\kappa_{l,s} = \begin{cases} v_l^2 \left(1 + (k-1)\,\rho_{l,l}\right) & \text{if } l = s \\ k v_l v_s \rho_{l,s} & \text{if } l \neq s \end{cases} . \tag{5.8}$$

$\mathbf{1}_l \mathbf{1}_s'$ is an $n \times n$ matrix with a $k \times k$ matrix of ones placed in the $(l,s)-$block and zeros everywhere else.

The part of the Σ matrix orthogonal to the jumps is given by the block-diagonal matrix

$$\underset{n\times n}{\Sigma^{\perp}} = \begin{pmatrix} \Sigma^{\perp}_{1,1} & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & \Sigma^{\perp}_{m,m} \end{pmatrix} \tag{5.9}$$

with blocks

$$\underset{k\times k}{\Sigma^{\perp}_{l,l}} = v_l^2\left(1-\rho_{l,l}\right) \begin{pmatrix} \frac{k-1}{k} & -\frac{1}{k} & \cdots \\ -\frac{1}{k} & \ddots & -\frac{1}{k} \\ \cdots & -\frac{1}{k} & \frac{k-1}{k} \end{pmatrix}. \tag{5.10}$$

5.3. Optimal Portfolio Separation in a Multi-Sector Economy

We decompose the vector of expected excess returns on the same basis as above,

$$\mathbf{R} \;=\; \sum\nolimits_{l=1}^{m} r_l \mathbf{1}_l + \mathbf{R}^{\perp} \;=\; \bar{\mathbf{R}} + \mathbf{R}^{\perp} \tag{5.11}$$

where $\mathbf{R}^{\perp}$ is orthogonal to each $\mathbf{1}_l$ and has the form

$$\mathbf{R}^{\perp} = [\mathbf{R}_1^{\perp\prime}, ..., \mathbf{R}_m^{\perp\prime}]'.$$

Each of the k-vectors $\mathbf{R}_l^{\perp}$ is orthogonal to the k-vector of ones.

We will be looking for a vector of optimal portfolio weights in the form

$$\boldsymbol{\omega} \;=\; \sum\nolimits_{l=1}^{m} \bar{\omega}_l \mathbf{1}_l + \boldsymbol{\omega}^{\perp} = \bar{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\perp}. \tag{5.12}$$

By construction, $\Sigma^{\perp}$ is orthogonal to $\bar{\Sigma}$ and to the space generated by $\{\mathbf{1}_l\}_{l=1,..,m}$. Thus, the minimization problem again separates as

$$\begin{cases} \boldsymbol{\omega}^{\perp *} = \arg\min_{\boldsymbol{\omega}^{\perp}} \left\{ g^{\perp}(\boldsymbol{\omega}^{\perp}) \right\} \\ \bar{\omega}^{*} = \arg\min_{\bar{\omega}} \left\{ \bar{g}(\bar{\omega}) \right\} \end{cases} \tag{5.13}$$

where

$$g^{\perp}(\boldsymbol{\omega}^{\perp}) = -\boldsymbol{\omega}^{\perp\prime}\mathbf{R}^{\perp} + \frac{1}{2}\gamma \boldsymbol{\omega}^{\perp\prime}\Sigma^{\perp}\boldsymbol{\omega}^{\perp} \tag{5.14}$$

$$\begin{aligned} \bar{g}(\bar{\boldsymbol{\omega}}) &= -k\sum\nolimits_{l=1}^{m} \bar{\omega}_l r_l + \frac{\gamma}{2} k \sum\nolimits_{l,s=1}^{m} \kappa_{l,s}\bar{\omega}_l\bar{\omega}_s \\ &\quad + \sum\nolimits_{l=1}^{m} \lambda_l \psi_l \left(k \sum\nolimits_{s=1}^{m} \bar{\omega}_s j_{s,l} \right) \end{aligned} \tag{5.15}$$

with

$$\psi_l\left(x\right) = -\frac{1}{\left(1-\gamma\right)} \int \left[\left(1+xz\right)^{1-\gamma} - 1\right] \nu_l\left(dz\right). \tag{5.16}$$

The key consequence of this separation is that in the space $V^{\perp}$ where only Brownian risk occurs, we again get back the Merton no-jump solution (with $\Sigma^{\perp}$ and $\mathbf{R}^{\perp}$ instead of the full Σ and $\mathbf{R}$). Indeed, the first order condition for minimizing $g^{\perp}(\boldsymbol{\omega}^{\perp})$ is

$$\Sigma^{\perp}\boldsymbol{\omega}^{\perp *} = \frac{1}{\gamma}\mathbf{R}^{\perp} \tag{5.17}$$

which, by the block diagonal form of $\Sigma^{\perp}$ and by construction $\mathbf{1}_l'\boldsymbol{\omega}^{\perp} = 0$ for all l, leads to the optimal solution $\boldsymbol{\omega}^{\perp *} = [\boldsymbol{\omega}_1^{\perp *\prime}, ..., \boldsymbol{\omega}_m^{\perp *\prime}]'$ with

$$\boldsymbol{\omega}_l^{\perp *} = \frac{1}{\gamma v_l^2\left(1-\rho_{l,l}\right)}\mathbf{R}_l^{\perp} \tag{5.18}$$

for $l = 1, \ldots, m$. Note that in general $\Sigma^{\perp}$ is not invertible, and $\text{rank}(\Sigma^{\perp}) = n - m$. However, the null space of $\Sigma^{\perp}$ is $\bar{V}$ and so (5.17) has a unique solution in $V^{\perp}$.

In the space $\bar{V}$, both jump and Brownian risks coexist. Minimizing $\bar{g}(\bar{\boldsymbol{\omega}})$ leads to the analogue of what happens with one sector, as in Section 3, but in dimension m. Similarly its solution has a limit as k goes to infinity with n (the number of sectors m being fixed). With the change of variable $\varpi_n = k\bar{\omega}$ we see that

$$\begin{aligned} \varpi_n^* \quad = \quad & \arg\min_{\{\varpi_n\}} \Bigg\{ - \underbrace{\sum\nolimits_{l=1}^{m} \varpi_{ln} r_l}_{\text{Return contribution}} + \underbrace{\frac{\gamma}{2} \sum\nolimits_{l,s=1}^{m} \frac{\kappa_{l,s}}{k} \varpi_{ln}\varpi_{sn}}_{\text{Brownian risk contribution}} \\ & + \underbrace{\sum\nolimits_{l=1}^{m} \lambda_l \psi_l \left(\sum\nolimits_{s=1}^{m} \varpi_{sn} j_{s,l}\right)}_{\text{Jump risk contribution}} \Bigg\} \end{aligned} \tag{5.19}$$

which, compared to (3.14), is an $m-$dimensional minimization problem, instead of a one-dimensional one. The convexity of the objective function implies the existence of a unique minimizer. Letting $k \to \infty$, we see that ϖ_n^* converges to a finite limit, ϖ_∞^*, with the limit given by the equation above with $\kappa_{l,s}/k$ replaced by $v_l v_s \rho_{l,s}$. As in the one-factor case, we can determine below ϖ_n^* in closed form under some specific jump distributions.

The wealth process X_t^* of the optimizing investor will have geometric Lévy dynamics. Under the natural condition that $\mathbf{R}_l^{\perp\prime}\mathbf{R}_l^{\perp} = O(k)$, as k, the number of stocks per sector, increases, the optimal portfolio can achieve expected gains at the expense of variance which both grow approximately linearly with k, while keeping the exposure to the sector jumps bounded. This result is achieved by the investor apportioning an increasing fraction of assets in the subspace $V^{\perp}$ orthogonal to the vectors $\mathbf{1}_l$.

5.4. Examples of Fully Closed-Form Portfolio Weights

For the model where $\nu_l(dz) = dz/z$ and $\mathbf{J}_l = \mathbf{J}$ for all $l = 1, ..., m$, similar to Section 3.3.1, we get the following objective functions for a CRRA investor with CRRA coefficient $\gamma = 2$ in the m-sector case:

$$\begin{aligned} f_n(\varpi) &= -\textstyle\sum_{l=1}^{m} \varpi_{ln} r_l + \frac{\gamma}{2} \sum_{l=1}^{m} \sum_{s=1}^{m} \frac{\kappa_{l,s}}{k} \varpi_{ln} \varpi_{sn} \\ &\quad - m\lambda \log\left(1 + \textstyle\sum_{s=1}^{m} \varpi_{sn} j_s\right) \end{aligned} \tag{5.20}$$

The first order conditions for ϖ_n are given by

$$-r_l + \gamma \sum_{s=1}^{m} \frac{\kappa_{l,s}}{k} \varpi_{sn} - m\lambda j_l \left(1 + \textstyle\sum_{s=1}^{m} \varpi_{sn} j_s\right)^{-1} = 0 \tag{5.21}$$

for $l = 1, \ldots, m$. These first order conditions form a system of m quadratic equations which admit a unique solution ϖ_n satisfying the solvency constraint $\sum_{l=1}^{m} \varpi_{ln} j_l > -1$.

These are solvable in closed form:

$$\varpi_n^* = \frac{k}{2}\mathbf{K}^{-1}\mathbf{r} + \mathbf{K}^{-1}\mathbf{j}\left(\frac{-1 - \frac{k}{2}\mathbf{j}'\mathbf{K}^{-1}\mathbf{r} + \sqrt{\left(1 + \frac{k}{2}\mathbf{j}'\mathbf{K}^{-1}\mathbf{r}\right)^2 + 2m\lambda k \mathbf{j}'\mathbf{K}^{-1}\mathbf{j}}}{2\mathbf{j}'\mathbf{K}^{-1}\mathbf{j}}\right) \tag{5.22}$$

where $\mathbf{r} = [r_1, ..., r_m]'$, $\mathbf{j} = [j_1, ..., j_m]'$ and

$$\mathbf{K} = \begin{pmatrix} \kappa_{11} & \cdots & \kappa_{1m} \\ \vdots & \ddots & \vdots \\ \kappa_{m1} & \cdots & \kappa_{mm} \end{pmatrix}$$

In this example we can have perfectly negatively correlated jumps. In particular, some sectors can have positive jumps while other sectors have negative jumps. One might imagine a situation where $\mathbf{1}'\mathbf{K}^{-1}\mathbf{j} = 0$, i.e. the vector $\mathbf{j}$ is orthogonal to the vector $\mathbf{1}'\mathbf{K}^{-1}$. In this case, the total exposure to risky assets $\mathbf{1}'\varpi_n^* = k\mathbf{1}'\mathbf{K}^{-1}\mathbf{r}/\mathbf{2}$ is independent of the value of λ and the proportion of wealth invested in the risky assets, $\mathbf{1}'\varpi_n^*$, is identical to the proportion of wealth invested in the risky assets when assets are affected by Brownian risk alone. An investor who receives information suggesting that the jumps become more likely will increase his exposure to sectors with positive jumps and reduce his exposure to sectors with negative jumps but the total proportion of wealth invested in the risky assets will not change. This is in contrast to section 4.1.2 where the exposure to risky assets is reduced as λ increases.

Similar to section 4, if $\mathbf{r} = -\lambda\mathbf{j}$ then the investor has no exposure to jump risk. The reason is that jumps in one sector are use to offset jump risk in another sector.

In this Figure 4 plots the objective function, $f_n(\varpi) = \bar{g}(\varpi/k)$ that we obtain in a two-sector economy, that is the function (5.20) with $m = 2$.

For the example where $\nu_l(dz) = dz/|z|$ and $\mathbf{J}_l = \mathbf{J}$, similar to that of Section 3.3.2, we get the following objective functions for a power utility investor with CRRA coefficient $\gamma = 2$ in the m-sector case:

$$\begin{aligned} f_n(\varpi) &= -\textstyle\sum_{l=1}^{m} \varpi_{ln} r_l + \dfrac{\gamma}{2}\displaystyle\sum_{l=1}^{m}\sum_{s=1}^{m} \frac{\kappa_{l,s}}{k}\varpi_{ln}\varpi_{sn} \\ &\quad -m\lambda \log\left(1 - \left(\textstyle\sum_{l=1}^{m} \varpi_{ln} j_l\right)^2\right) \end{aligned} \tag{5.23}$$

The first order conditions for ϖ_n are given by

$$-r_l + \gamma \sum_{s=1}^{m} \frac{\kappa_{l,s}}{k}\varpi_{sn} + 2m\lambda j_l \left(\textstyle\sum_{s=1}^{m} \varpi_{sn} j_s\right)\left(1 - \left(\textstyle\sum_{s=1}^{m} \varpi_{sn} j_s\right)^2\right)^{-1} = 0 \tag{5.24}$$

for $l = 1, \ldots, m$. These first order conditions form a system of m cubic equations. Equations (5.24) admit a unique solution ϖ_n satisfying the solvency constraint $|\sum_{l=1}^{m} \varpi_{ln} j_l| < 1$. These again are solvable in closed form:

$$\begin{aligned} \varpi_n^* = \tfrac{k}{2}\mathbf{K}^{-1}\mathbf{r} - \tfrac{k}{3}\tfrac{\mathbf{j}'\mathbf{K}^{-1}\mathbf{r}}{\mathbf{j}'\mathbf{K}^{-1}\mathbf{j}}\mathbf{K}^{-1}\mathbf{j} + \sqrt{\tfrac{4}{3(\mathbf{j}'\mathbf{K}^{-1}\mathbf{j})^2}\left(1 + \tfrac{(k\mathbf{j}'\mathbf{K}^{-1}\mathbf{r})^2}{12} + m\lambda k\mathbf{j}'\mathbf{K}^{-1}\mathbf{j}\right)} \times \\ \cos\left(\tfrac{1}{3}\arccos\left(\frac{\frac{k\mathbf{j}'\mathbf{K}^{-1}\mathbf{r}}{6}\left(-1+\frac{1}{36}\left(k\mathbf{j}'\mathbf{K}^{-1}\mathbf{r}\right)^2+\frac{m\lambda}{2}k\mathbf{j}'\mathbf{K}^{-1}\mathbf{j}\right)}{\left(\sqrt{\frac{1}{3}\left(1+\frac{(k\mathbf{j}'\mathbf{K}^{-1}\mathbf{r})^2}{12}+m\lambda k\mathbf{j}'\mathbf{K}^{-1}\mathbf{j}\right)}\right)^3}\right) + \tfrac{4}{3}\pi\right)\mathbf{K}^{-1}\mathbf{j}. \end{aligned} \tag{5.25}$$

6. Conclusions

We have proposed a new approach to characterize in closed form the portfolio selection problem for an investor concerned with the possibility of jumps in asset returns, and who seeks to control this risk by diversification or other means. We extended the standard multi–asset geometric Brownian motion models to exponential Lévy models through the inclusion of correlation effects due to jumps. For the general exponential Lévy model, the portfolio selection problem for n assets reduces to the minimization of a convex function in n dimensions.

However, our key decomposition of the space of returns into a low-dimensional space containing both jump and Brownian risks, and a large dimensional one containing only Brownian risk, enables a further reduction of the problem to a convex optimization in the dimension of the number of sectors m, which we can typically take to be small while the total number of assets n is large. In all cases, the problem in the $(n-m)$-dimensional space containing only Brownian risk is solvable in closed form. And we proposed specific models for the jump distribution which allow for the orthogonal problem, in the m-dimensional space where the jumps take place, to be also solvable in closed form.

With explicit optimal portfolios policies, one can address important practical questions. How exactly does increasing the number of available assets improve the investor's exposure to both diffusive and jump risk? How does the portfolio of an investor who fears jumps differ from the portfolio of one who does not? Is there a simple form for the optimal portfolio which is achieved asymptotically as the number of assets grows to infinity? etc.

One can wonder if the approach remains valid when extensions and generalizations of this work are considered:

1. One could consider modelling the spectral decomposition of the Σ matrix directly, instead of parametrizing the matrix itself and then determining its decomposition across the two spaces $\bar{V}$ and $V^{\perp}$. As an empirical strategy, one could determine the number of sectors through factor analysis or similar techniques, in order to determine in a data-driven manner the shape of the Σ matrix.
2. Stochastic volatility of the type considered in Liu et al. (2003) requires solving our

nonlinear equations for weight vectors stepwise in time, in parallel with ordinary differential equations (which themselves depend on the current portfolio weights). This does not appear doable in closed form.

3. Portfolio restrictions such as short-selling constraints are relevant in practice but, when generic constraints are imposed on the optimal portfolio, we cannot expect the dimensional reduction to be preserved or our conclusions to hold. However, a utility function such as power utility which becomes $-\infty$ for wealth below a finite threshold, sometimes automatically implies certain constraints: it appears that in this case, much of our analysis remains intact.

4. Finally, we would like to be able to better capture contagion, in the form not just of simultaneous jumps within or across-sectors, as we are currently able to model, but rather in the form of a jump in one sector causing an increase in the likelihood that a different jump will occur in another sector. To capture this effect, self-exciting jump processes seem a promising approach which we intend to investigate in future work.

Appendix

A. Lévy Processes

We give here a definition of Lévy processes and our notation. An n dimensional Lévy process $\mathbf{L}_t$ is specified by its "characteristic triple" $(\mathbf{b}, \mathbf{c}, f)$ where $\mathbf{b} \in \mathbb{R}^n$ is the drift or mean return vector, $\mathbf{c} \in \mathbb{R}^{n\times n}$ is the diffusion matrix, or local variance of the continuous part of $\mathbf{L}_t$, and f is the Lévy measure on $\mathbb{R}^n$, which satisfies

$$\int_{\mathbb{R}^n} \left(1 \wedge \|\mathbf{x}\|^2\right) f(d\mathbf{x}) < \infty.$$

The characteristic function of $\mathbf{L}_t$ is given by the Lévy-Khintchine formula

$$E(e^{i\mathbf{u}'\mathbf{L}_t}) = \exp\left(t\left(i\mathbf{u}'\mathbf{b} - \frac{\mathbf{1}}{2}\mathbf{u}'\mathbf{c}\mathbf{u} + \int_{\mathbb{R}^n\setminus\{0\}} f(d\mathbf{x})\left(e^{i\mathbf{u}'\mathbf{x}} - 1 - i\mathbf{u}'\mathbf{h}\left(\mathbf{x}\right)\right)\right)\right) \tag{A.1}$$

for $\mathbf{u} \in \mathbb{R}^n$and $\mathbf{h}\left(\mathbf{x}\right)$ is a truncation function (see e.g., Chapter II.2 in Jacod and Shiryaev (2003).) The stochastic differential equation for $\mathbf{L}_t$ written in terms of its characteristics is

$$d\mathbf{L}_t = \mathbf{b}dt \quad + \mathbf{c}^{1/2}d\mathbf{W}_t + \int_{\mathbb{R}^n} \mathbf{x}\; N^{(\mu)}(d\mathbf{x}, dt) \tag{A.2}$$

where $\mathbf{c}^{1/2}$ is a matrix square root satisfying $\mathbf{c}^{1/2}(\mathbf{c}^{1/2})' = \mathbf{c}$, and $N^{(\mu)}$ is called the Poisson random measure associated with the intensity measure $\mu\left(dt, dz\right) = f\left(dz\right)dt$.

We can identify the right hand side of equation (2.2) as the dynamics of a Lévy process with triple $(r + \mathbf{R}, \boldsymbol{\sigma\sigma}', f)$ where $f(d\mathbf{l}) = f(dz)$ with $\mathbf{l} = \mathbf{J}z$ a measure on a line segment in the direction of $\mathbf{J}$ in $\mathbb{R}^n$. We say that $\mathbf{S}_t$ itself has geometric Lévy dynamics, meaning that each component satisfies $dS_{i,t}/S_{i,t-} = dL_{i,t}$ where $\mathbf{L}_t = [L_{1,t}, \ldots, L_{n,t}]'$ follows an SDE of the type (A.2).

B. Exponential Utility

For Merton's problem with CRRA utility function the optimal wealth process X_t^* achieved by picking the constant portfolio fractions $\boldsymbol{\omega}^*$ is itself a one dimensional geometric Lévy process whose characteristic triple is $\left(X_{t-}^*\left(\boldsymbol{\omega}^{*\prime}\mathbf{R} + r - K\right), X_{t-}^{*2}\boldsymbol{\omega}^{*\prime}\boldsymbol{\sigma\sigma}'\boldsymbol{\omega}^*, f\right)$ where $f(dy) =$

$f(dz)$ with $y = X^*_{t-}\boldsymbol{\omega}'\mathbf{J}z$. In this case, the investor keeps constant fractions of wealth in each risky asset, and the constant remaining fraction $1 - \sum_j \omega_j$ in the riskless asset.

Now, consider an investor with exponential utility, $U(C) = -\frac{1}{q}\exp(-qC)$ with CARA coefficient $q > 0$. We can look for a solution to (2.8) in the form

$$L(x) = -\frac{K}{q}e^{-rqx} \tag{B.1}$$

so that

$$\frac{\partial L(x)}{\partial x} = -rqL(x), \quad \frac{\partial^2 L(x)}{\partial x^2} = r^2q^2L(x). \tag{B.2}$$

Then (2.8) reduces to

$$\begin{aligned} 0 = \max_{\{C_t,\boldsymbol{\omega}_t\}} & \big\{U(C_t) - \beta L(X_t) - rqL(X_t)\left(X_t r + X_t\boldsymbol{\omega}_t'\mathbf{R} - C_t\right) \\ & + \frac{1}{2}r^2q^2L(X_t)X_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t + \lambda\int\left[e^{-rqX_t\boldsymbol{\omega}_t'\mathbf{J}z}L(X_t) - L(X_t)\right]\nu(dz)\Big\} \end{aligned} \tag{B.3}$$

that is

$$\begin{aligned} 0 = \min_{\{C_t,\boldsymbol{\omega}_t\}} & \left\{\frac{U(C_t)}{rqL(X_t)} - \frac{\beta}{qr} - \left(X_t r + X_t\boldsymbol{\omega}_t'\mathbf{R} - C_t\right)\right. \\ & \left. + \frac{1}{2}rqX_t^2\boldsymbol{\omega}_t'\Sigma\boldsymbol{\omega}_t + \frac{\lambda}{rq}\int\left[e^{-qX_t\boldsymbol{\omega}_t'\mathbf{J}z} - 1\right]\nu(dz)\right\} \end{aligned} \tag{B.4}$$

after division by $qL(X_t)$ (note that max becomes min as a result of $qL(X_t) < 0$).

The optimal policy of $\boldsymbol{\omega} = X_t\boldsymbol{\omega}_t$ is given by the objective function

$$\min_{\{\omega\}}\left(-\boldsymbol{\omega}'\mathbf{R} + \frac{1}{2}rq\boldsymbol{\omega}'\Sigma\boldsymbol{\omega} + \frac{\lambda}{rq}\int\left[e^{-rq\boldsymbol{\omega}'\mathbf{J}z} - 1\right]\nu(dz)\right), \tag{B.5}$$

and the optimal consumption choice is therefore

$$C_t^* = rX_t - \frac{1}{q}\log(rK). \tag{B.6}$$

Finally, we evaluate equation (B.4) at C^* and $\boldsymbol{\omega}^*$ to identify K,

$$K = \frac{1}{r}\exp\left(1 - \frac{\beta}{r} - q\boldsymbol{\omega}^{*\prime}\mathbf{R} + \frac{1}{2}rq^2\boldsymbol{\omega}^{*\prime}\Sigma\boldsymbol{\omega}^* + \frac{\lambda}{r}\int\left[e^{-q\boldsymbol{\omega}^{*\prime}\mathbf{J}z} - 1\right]\nu(dz)\right). \tag{B.7}$$

C. Log Utility

Finally, consider an investor with log utility, $U(x) = \log(x)$. We can look for a solution to (2.8) in the form

$$L(x) = K_1^{-1} \log(x) + K_2 \tag{C.1}$$

where K_1 and K_2 are constant, so that

$$\frac{\partial L(x)}{\partial x} = K_1 x^{-1}, \quad \frac{\partial^2 L(x)}{\partial x^2} = -K_1 x^{-2}. \tag{C.2}$$

Then (2.8) reduces to

$$\begin{aligned} 0 = \max_{\{C_t, \boldsymbol{\omega}_t\}} &\Big\{ \log(C_t) - \beta K_1^{-1} \log(X_t) - \beta K_2 + K_1^{-1} X_t^{-1} \left(r X_t + \boldsymbol{\omega}_t' \mathbf{R} X_t - C_t \right) \\ & - \frac{1}{2} K_1^{-1} \boldsymbol{\omega}_t' \Sigma \boldsymbol{\omega}_t + \lambda K_1^{-1} \int \log\left(1 + \boldsymbol{\omega}_t' \mathbf{J} z\right) \nu(dz) \Big\} \end{aligned} \tag{C.3}$$

that is

$$\begin{aligned} 0 = \min_{\{C_t, \boldsymbol{\omega}_t\}} &\Big\{ -\ln(C_t) + \beta K_1^{-1} \ln(X_t) + \beta K_2 - K_1^{-1} r - K_1^{-1} \boldsymbol{\omega}_t' \mathbf{R} + K_1^{-1} X_t^{-1} C_t \\ & + K_1^{-1} \frac{1}{2} \boldsymbol{\omega}_t' \Sigma \boldsymbol{\omega}_t - \lambda K_1^{-1} \int \ln\left(1 + \boldsymbol{\omega}_t' \mathbf{J} z\right) \nu(dz) \Big\} \end{aligned} \tag{C.4}$$

The optimal policy of $\boldsymbol{\omega}_t$ is given by the objective function,

$$\min_{\{\boldsymbol{\omega}_t\}} \left(-\boldsymbol{\omega}_t' \mathbf{R} + \frac{1}{2} \boldsymbol{\omega}_t' \Sigma \boldsymbol{\omega}_t - \lambda \int \log\left(1 + \boldsymbol{\omega}_t' \mathbf{J} z\right) \nu(dz) \right), \tag{C.5}$$

and the optimal consumption choice is therefore

$$C_t^* = K_1 X_t. \tag{C.6}$$

To identify K_1 and K_2, we evaluate equation (C.4) at C^* and $\boldsymbol{\omega}^*$,

$$K_2 = \frac{1}{\beta} \left\{ \log(\beta) + \frac{r}{\beta} + \frac{1}{\beta} \boldsymbol{\omega}^{*\prime} \mathbf{R} - 1 - \frac{1}{2\beta} \boldsymbol{\omega}^{*\prime} \Sigma \boldsymbol{\omega}^* + \frac{\lambda}{\beta} \int \log\left(1 + \boldsymbol{\omega}^{*\prime} \mathbf{J} z\right) \nu(dz) \right\} \tag{C.7}$$

$$K_1 = \beta. \tag{C.8}$$

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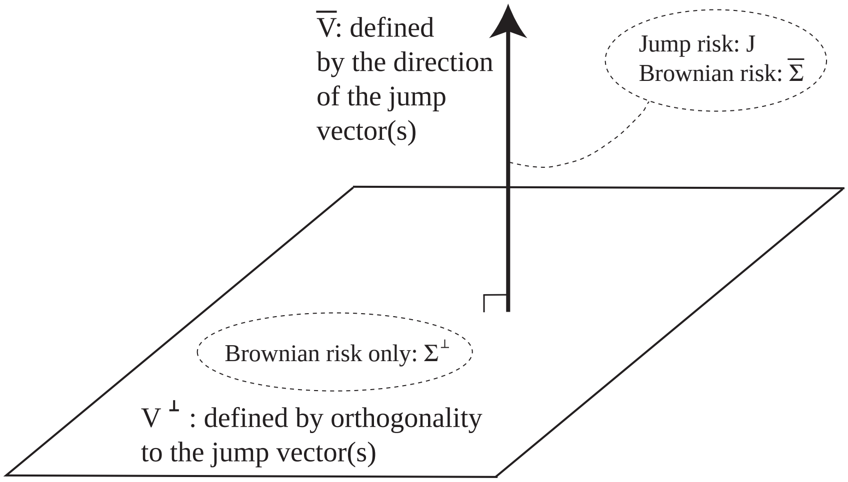

Figure 1: Orthogonal decomposition of the returns space $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ where $\bar{V}$ is the span of the jump vector(s) and $V^{\perp}$ is the orthogonal hyperplane. By construction, only $\bar{V}$ contains jump risk. However, Brownian risk is contained throughout $\mathbb{R}^n$: $\bar{V}$ contains the part $\bar{\Sigma}$, and $V^{\perp}$ the part $\Sigma^{\perp}$.

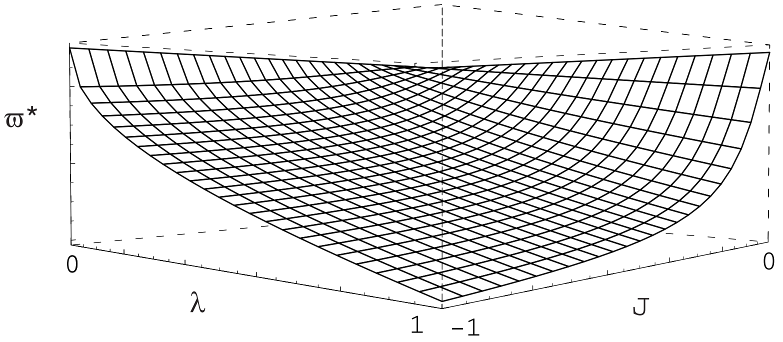

Figure 2: Optimal portfolio weight ϖ^*_∞ as a function of $\bar{J}$ and λ.

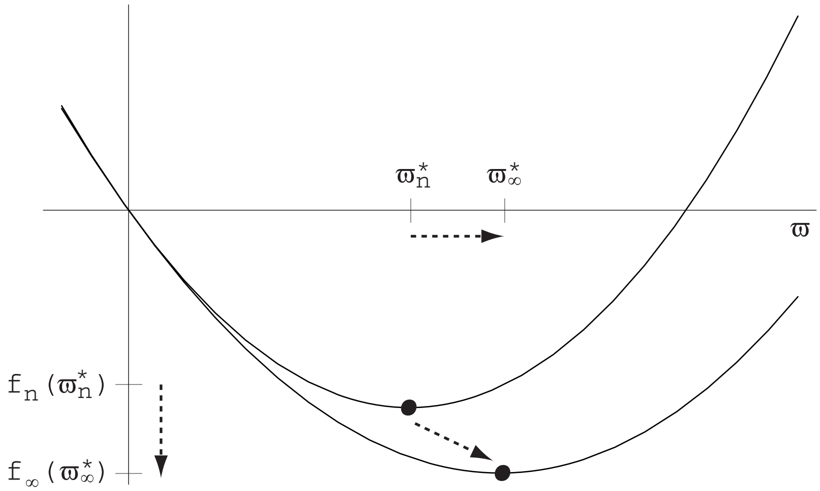

Figure 3: Scalar objective function used to determine the optimal portfolio weight ϖ_n^* and its large asset asymptotic limit, ϖ_∞^*.

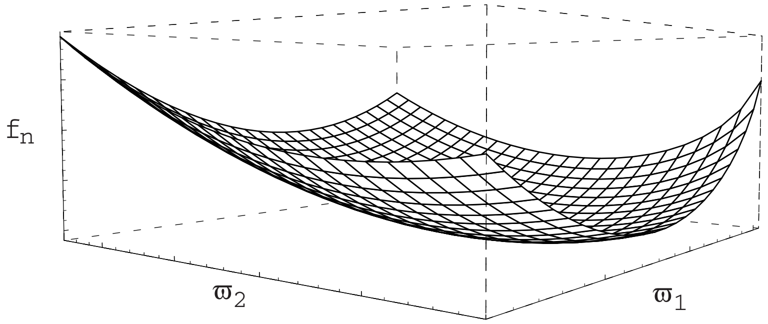

Figure 4: Bivariate objective function in a two-sector economy.