Equilibrium Asset Pricing and Portfolio Choice with Heterogeneous Preferences *

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Abstract

We provide general representations for the rate of return and the volatility of a risky asset and for the optimal portfolios in equilibrium with heterogeneous agents. Our universal representations allow for arbitrary utility functions and an arbitrary diffusion process for the state variable. The key element is a new object that we call the "rate of macroeconomic fluctuations": In equilibrium, pricing of all macroeconomic risks requires discounting them at this rate. We use the obtained representations to establish quantitative and qualitative properties of equilibrium dynamics and to generate a number of empirical predictions.

Keywords: equilibrium, heterogeneous agents, volatility, optimal portfolios, stochastic opportunity set.

JEL Classification. D53, G11, G12

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1 Introduction

This paper provides universal representations for equilibrium non-myopic optimal portfolios, market price of risk (MPR) and stock price volatility in a general, continuous-time CAPM with agents consuming only at a terminal horizon. We relax the assumptions of the classical Merton (1973) CAPM by allowing for arbitrary, heterogeneous utility functions, stochastic interest rates, and the dividend following a general diffusion process. Our representations are universal in the sense that they do not explicitly involve the representative agent's utility.[1] We use the obtained representations to generate various empirical predictions that should hold in any continuous-time CAPM in which agents derive utility only from terminal consumption. We also apply these representations to establish various static and dynamic properties of equilibrium quantities.

The main difficulty in studying dynamic equilibrium comes from the non-myopic behavior of the optimal portfolios in equilibrium. Merton introduced the myopic portfolio and defined the remainder of the optimal portfolio as the non-myopic intertemporal hedging demand. The literature contains sophisticated results on representations of single agent optimal portfolios in partial equilibrium.[2] However, except for several very special cases, we still know little about the equilibrium behavior of the hedging demand. From an investor's point of view, the problem is to construct a long-run (or strategic) portfolio that reacts dynamically to future trends in return and volatility. Siegel (2008) suggests estimating the long-run risk/return tradeoff by considering the buy-and-hold strategies over long periods, but this is not necessarily appropriate, because long-horizon investors would want to dynamically rebalance their portfolios. We derive a simple and intuitive representation for the non-myopic optimal portfolio. A key element of this representation is a new object we call the "rate of macroeconomic fluctuations" (RMF).[3] We show that the intertemporal hedge against the market price of risk fluctuations has two components: a *wealth hedge* and a *risk tolerance hedge*.[4] The wealth (risk tolerance) hedge is given by the covariance of the future wealth (risk tolerance) of the agent with

[1] Just as in the classical CAPM, the representative agent's risk aversion does appear in the expressions for equilibrium quantities, but can be replaced by the return-to-risk ratio of the market portfolio.

[2] See Detemple, Garcia and Rindisbacher (2003) and Detemple and Rindisbacher (2009).

[3] The rate of macroeconomic fluctuations appears in all the representations derived in this paper – in equilibrium, pricing of all macroeconomic risks requires discounting them at this rate. RMF is an intrinsic characteristic of the state variable dynamics and can be calculated explicitly. Effectively, it is the rate at which the growth rate of the economy is changing.

[4] In addition to the market price of risk hedging demand, we also derive a representation for the interest rate hedge.

future market price of risk, discounted at the rate of macroeconomic fluctuations.

In equilibrium, the decomposition of optimal portfolios into myopic and non-myopic components translates into analogous decompositions of other quantities. We call "myopic" those quantities that are determined as the *present market value* of aggregate quantities, such as dividend volatility and market price of risk. By contrast, the "non-myopic" quantities are those determined by the *future fluctuations* of aggregate quantities, or more precisely, by covariances thereof.

The equilibrium market price of risk is myopic by its nature because its current level only impacts the myopic part of the agents' portfolios. In equilibrium, the size of the market price of risk should be determined by the size of the macroeconomic risk in the economy, that is, by the future dividend and interest rate volatilities. We show that the appropriate measure of the future risks is given by the present market value of future dividend volatility, discounted at the rate of macroeconomic fluctuations. We call the latter "myopic volatility" because it only accounts for the size of future dividend risk and not for its future fluctuations. If the aggregate risk aversion is constant, the equilibrium market price of risk is given by the product of that risk aversion and the myopic volatility, net of the present market value of future interest rate volatility, also discounted at the rate of macroeconomic fluctuations.

Unlike market price of risk, the equilibrium stock volatility is a non-myopic quantity. It depends not only on the present value of future (discounted) dividend volatility, but also on the cyclical properties of the market price of risk. Therefore, just like optimal portfolios, stock volatility can be decomposed into the myopic and the non-myopic component. We show that the non-myopic volatility is given by the negative covariance of the aggregate dividend with future market price of risk net of dividend volatility, discounted at the rate of macroeconomic fluctuations.

Several new testable implications follow from our results.

First, we uncover a new concept, the rate of macroeconomic fluctuations, and show that future macroeconomic shocks should be valued discounted at this rate.[5] The rate of macroeconomic fluctuations is equal to the difference between the drift of the market price of risk and the product of the market price of risk and its volatility[6] and can be

[5]There is a large literature devoted to pricing of macroeconomic risks. See, e.g., Borovička, Hansen, Hendricks and Scheinkman (2009). The rate of macroeconomic fluctuations provides a tool for studying this problem from an equilibrium point of view.

[6]That is, $d\lambda_t = \lambda_t\,(\mu_t^\lambda\,dt + \sigma_t^\lambda\,dB_t)$ with the drift $\mu_t^\lambda = \rho_t + \lambda_t\,\sigma_t^\lambda$ where ρ_t is the rate of

directly estimated from data.

Second, we show that, when the dividend risk is stochastic, the natural analog of fundamental (aggregate dividend) volatility is the myopic volatility that can be estimated from data given the dividend volatility and the rate of macroeconomic fluctuations. Long-run investors should use the myopic volatility to estimate the size of macroeconomic risks.

Third, we show that the variation in the risk-return tradeoff is due to the variation in the ratio between the myopic and the non-myopic volatilities. A fundamental result of Merton (1973) is that the equilibrium excess return is a linear combination of the stock volatility (the risk component) and its covariance with investment opportunities (the hedge component). Recent empirical findings (see Guo and Whitelaw (2006)) indicate that expected returns are driven primarily by the hedge component. Our results imply that the hedge component of the equilibrium market price of risk precisely cancels the non-myopic part of stock volatility. The latter, as mentioned above, is given by the covariance of the future discounted dividend with the future market price of risk net of future dividend volatility. This representation then generates an additional prediction that the variation in the non-myopic volatility is explained by the variation in the covariance. These predictions can be tested analogously to similar predictions for forward-futures spreads.[7] Furthermore, our myopic/non-myopic volatility decomposition can be used to construct better empirical variance decompositions.[8]

Fourth, if we adopt the common hypothesis that agents have HARA utilities, our results generate the prediction that the variation in the intertemporal market price of risk hedging demand is explained by the variation in the co-movement of agent's wealth and the discounted market price of risk. This prediction can be tested directly using the PSID data. It would allow us to gain a deeper understanding of the joint fluctuations of household wealth and optimal portfolios. For example, Brunnermeier and Nagel (2008) find that the fraction of wealth that households invest into risky assets does not change with the level of wealth. They use this finding to conclude that the households' relative risk aversion is constant. However, if this risk aversion is different from one, the optimal portfolio must be non-myopic and contain a hedging component. Given our representation for the hedging demand, the findings of Brunnermeier and Nagel (2008) imply, in our model, that the degree of co-movement of households' wealth

macroeconomic fluctuations.

[7]See Meulbroek (1992) and Gupta and Subrahmanyam (2000).

[8]See Campbell (1991).

with the market price of risk is stable and does not fluctuate though time. It would be of interest to test this prediction and determine how much of households' investments is driven by the non-myopic demand.

In addition to the above empirical predictions, our representations can be used to derive static and dynamic properties of equilibrium quantities. In particular, we obtain results on the cyclicality of the market price of risk, excess volatility and hedging portfolios.

Our most important qualitative prediction is a precise condition for the positivity of the market price of risk hedging demand. We show that, when the market price of risk is countercyclical, the hedge against market price of risk fluctuations is positive if and only if the *product of the agent's prudence and risk tolerance is below* 2. We provide an intuitive explanation of this surprising result and relate it to the precautionary savings motive.

We also provide tight conditions for the market price of risk countercyclicality and positivity of the non-myopic (excess) volatility. We show that, when interest rate volatility is small, the market price of risk is countercyclical if the product of the aggregate risk aversion and dividend volatility is countercyclical and the rate of macroeconomic fluctuations is procyclical.[9] This result extends the characterization of He and Leland (1993) to general dividend processes and stochastic interest rates. It particular, we show that, in contrast to the sufficient conditions in the framework of that paper, even countercyclicality of both dividend volatility and aggregate risk aversion may *not be sufficient* for market price of risk to be countercyclical, as its equilibrium behavior depends on the dynamics of the rate of macroeconomic fluctuations.

The non-myopic volatility is positive under the same conditions under which the market price of risk is countercyclical. As above, these conditions are tight.

In the special case when the log-dividend follows an autoregressive process and all agents have CRRA preferences, our results can be further strengthened. We find explicit bounds on the market price of risk, excess volatility, risk-return tradeoff and the size of equilibrium optimal portfolios in terms of the spread between the highest and the lowest risk aversion in the economy. This is a natural measure of heterogeneity and our bounds can be used to estimate this measure from data.

We conclude the introduction with a discussion of related literature. Most of the

[9] These conditions are tight in the sense that none of them can be dispensed with.

existing work devoted to standard CAPM with heterogeneous risk preferences is done in special models with two CRRA agents only. Dumas (1989) considers a production economy of that type. In this case, the stock returns coincide with the exogenously specified returns on the production technology, and only the risk-free rate is determined endogenously. Wang (1996) studies the term structure of interest rates in an economy populated by two CRRA agents, maximizing time-additive utility from intermediate consumption and the aggregate dividend following a geometric Brownian motion. Basak and Cuoco (1998) study equilibria with two agents and limited stock market participation. Bhamra and Uppal (2009a) also consider an economy with two agents, and derive conditions under which the market volatility is higher than the dividend volatility.[10] Bhamra and Uppal (2009b) derive closed form expressions in terms of convergent power series for an economy populated by two CRRA agents with arbitrary risk aversions, discount rates and heterogeneous beliefs. Cvitanić, Jouini, Malamud and Napp (2009) consider the same model as Bhamra and Uppal, but with an arbitrary number of agents. Cvitanić and Malamud (2009a, 2009b) study asymptotic equilibrum dynamics with an arbitrary number of CRRA agents maximizing utility from terminal consumption, as the horizon becomes large.

Only few papers study general properties of equilibria with non-CRRA preferences and/or a general dividend process.[11] Our paper is partially related to He and Leland (1993).[12] They also allow for an arbitrary dividend process and describe the set of viable price processes, that is, all price processes that can be attained by varying the utility of the representative agent. One of their main results is that, when the dividend is a geometric Brownian motion, the market price of risk is countercyclical if and only if the risk aversion is countercyclical. Our representation allows us to prove the most general result of this kind. More precisely, we show that countercyclical risk aversion is not enough in general, and countercyclicality of dividend volatility and procyclicality of the rate of macroeconomic fluctuations are needed.

In a number of models heterogeneity comes from beliefs and asymmetric information, as in Basak (2005), Jouini and Napp (2009), Biais, Bossaerts and Spatt (2009) and

[10] The main message of Bhamra and Uppal (2009a) is that allowing the agents to trade in an additional derivative security, making the market complete, may actually increase the market volatility. Because of completeness, their equilibrium coincides with the Arrow-Debreu equilibrium of Wang (1996).

[11] Some papers study static, one-period economies with heterogeneous preferences; see, e.g., Benninga and Mayshar (2000), Gollier (2001), Hara, Huang and Kuzmics (2007). There are also conditions for zero equilibrium trading volume; see Berrada, Hugonnier and Rindisbacher (2007).

[12] See also Bick (1990) and Wang (1993).

Dumas, Kurshev and Uppal (2009). In particular, latter paper studies excess volatility and non-myopic optimal portfolios in a model with two CRRA agents with identical preferences but heterogeneous beliefs. Methodologically, our paper is close to Detemple and Zapatero (1991), Detemple, Garcia and Rindisbacher (2003), Detemple and Rindisbacher (2009), Dumas, Kurshev and Uppal (2009) and Bharma and Uppal (2009a,b), as we use Malliavin calculus to obtain the main results. In particular, Detemple, Garcia and Rindisbacher (2003) and Detemple and Rindisbacher (2009) derive representations for intertemporal hedging demand in a single agent partial equilibrium setting and with a general, multi-dimensional diffusion state variable. Due to having a single state variable and due to a particular nature of equilibrium market price of risk dynamics, our representations for hedging demand have a simpler form.

Chan and Kogan (2002) and Xiouros and Zapatero (2009) perform a numerical analysis of equilibrium asset prices in economies populated by a large number (continuum) of agents with heterogeneous risk aversions and habit formation preferences.[13] However, these papers do not analyze equilibrium optimal portfolios.

Several papers investigate, both theoretically and empirically, the equilibrium relation between risk and return. For example, Abel (1988), Gennotte and Marsh (1993), Whitelaw (2000) and Guo and Whitelaw (2006) show that, with stochastic consumption opportunities and a representative CRRA agent, it is possible to generate a highly non-linear risk-return relationship.[14] Our model provides a general expression for the risk-return profile that holds in *any* continuous-time CAPM with only terminal consumption. As mentioned above, we show that the source of the non-linearity is the relation between myopic and non-myopic volatilities.

The paper is organized as follows. Section 2 describes the setup and notation. In Section 3 we introduce the rate of macroeconomic fluctuations and in Section 4 we derive representations for the market price of risk and stock volatility, and study their behavior. Section 5 is devoted to equilibrium optimal portfolios. Section 6 studies quantitative implications of the general results. Section 7 illustrates the results in the special case of autoregressive log-dividend process and CRRA agents. Section 8 concludes.

[13]Xiouros and Zapatero (2009) is the first paper to derive a closed form expression for the equilibrium state price density in a market with many agents with heterogeneous risk aversions.

[14]See also Glosten, Jagannathan and Runkle (1993) and Harvey (2001).

2 Setup and Notation

2.1 The Model

We consider a standard setting similar to that of Wang (1996). The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion $B_t,\ t \in [0\,,\,T]$ on a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\mathcal{F}$ is the augmented filtration generated by B_t. There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend D_T such that

$$D_t^{-1}\, dD_t \;=\; \mu^D(D_t)\, dt \;+\; \sigma^D(D_t)\, dB_t.$$

This diffusion process lives on $(0, +\infty)$ and $\sigma^D(D_t) > 0$.[15]

We also assume that a zero coupon bond with instantaneous risk-free rate $r_t = r(D_t)$ is available in zero net supply.[16] In particular, Ito's formula implies that

$$dr(D_t) \;=\; \mu^r(D_t)dt \;+\; \sigma^r(D_t)\, dB_t$$

with

$$\sigma^r(D_t) \;=\; r'(D_t)\, D_t\, \sigma^D(D_t)\,.$$

Note that $|\sigma^r|$ is the instantaneous volatility of the risk-free rate and the sign of σ^r coincides with the instantaneous correlation of the interest rate with the dividend D_t. We say that the interest rate is procyclical if σ^r is positive, i.e., $r'(D_t) > 0$, and countercyclical otherwise.

There are K competitive agents behaving rationally. Agent k is initially endowed with $\psi_k > 0$ shares of stock, and the total supply of the stock is normalized to one,

$$\sum_{k=1}^{K} \psi_k \;=\; 1\,.$$

Agent k chooses portfolio strategy $\pi_{k\,t}$, the portfolio weight at time t in the risky asset, as to maximize expected utility[17]

$$E\,[u_k(W_{k\,T})]$$

[15] We assume that μ^D and σ^D are such that a unique strong solution exists, and also that $\mu^D \in C^1(\mathbb{R}_+)$, $\sigma^D \in C^2(\mathbb{R}_+)$. In general, whenever we use a derivative of a function, we implicitly assume it exists.

[16] We assume that the function $r(x)$ is C^2 and bounded from below.

[17] We assume that u_k is $C^3(\mathbb{R}_+)$ and satisfies standard Inada conditions.

of the terminal wealth W_{kT}, where the wealth W_{kt} of agent k evolves as

$$dW_{kt} \;=\; W_{kt}\left[r_t\, dt \;+\; \pi_{k\,t}\,(S_t^{-1}\, dS_t \;-\; r_t\, dt)\right].$$

In this equation, S_t is the stock price at time t. The instantaneous drift and volatility of the stock price S_t are denoted by μ_t^S and σ_t^S respectively, so that

$$\frac{dS_t}{S_t} \;=\; \mu_t^S dt \;+\; \sigma_t^S dB_t\,.$$

The market price of risk (MPR) λ_t is given by

$$\lambda_t \;=\; \frac{\mu_t^S \;-\; r_t}{\sigma_t^S}\,.$$

2.2 Equilibrium

Definition 2.1 *We say that the market is in equilibrium if the agents behave optimally and both the risky asset market and the risk-free market clear.*

It is well known that the above financial market is complete, if the volatility process σ_t^S of the stock price is almost everywhere strictly positive.[18]

When the market is complete, there exists a unique state price density process $\xi \;=\; (\xi_t)$ such that the stock price is given by

$$S_t \;=\; \frac{E_t[\xi_T\, D_T]}{\xi_t}\,, \tag{1}$$

where

$$\xi_t \;=\; e^{-\int_0^t r_s\, ds}\, M_t$$

and M_t is the density process of the equivalent martingale measure Q,

$$\left(\frac{dQ}{dP}\right)_t \;=\; M_t \;=\; E_t[\,M_T\,]\,.$$

Thus, we can rewrite (1) in the form

$$S_t \;=\; E_t^Q[\,e^{-\int_t^T r_s\, ds}\, D_T\,]\,. \tag{2}$$

[18]This can be verified under some technical regularity conditions on the model primitives. See, Hugonnier, Malamud and Trubowitz (2009).

Because of the market completeness, any equilibrium allocation is Pareto-efficient and can be characterized as an Arrow-Debreu equilibrium. See, e.g. Duffie and Huang (1986), Wang (1996).[19]

Introduce the inverse of the marginal utility

$$I_k(x) \;:=\; (u_k')^{-1}(x)\,. \tag{3}$$

It is well known that in this complete market setting the optimal terminal wealth is of the form

$$W_{k\,T} \;=\; I_k(y_k\,\xi_T)$$

where y_k is determined via the *budget constraint*[20]

$$E[I_k(y_k\,\xi_T)\,\xi_T] \;=\; W_{k\,0} \;=\; \psi_k\,S_0 \;=\; \psi_k\,E[\xi_T\,D_T].$$

Since in equilibrium the final wealth amounts of all the agents have to sum up to the aggregate dividend, equilibrium state price density ξ_T is uniquely determined by the consumption market clearing equation

$$\sum_{k=1}^{K}\; I_k(y_k\,\xi_T) = \; D_T\,. \tag{4}$$

3 Rate of Macroeconomic Fluctuations

In this section we introduce a new object, the rate of macroeconomic fluctuations. This rate will play a crucial role in the sequel as it will appear in expressions for all equilibrium quantities (market price of risk, volatility and optimal portfolios) and impact all static and dynamic properties of the equilibrium.

There are two forces driving the stochastic dynamics of the dividend D_t : stochastic volatility $\sigma^D(D_t)$ and stochastic drift $\mu^D(D_t)$. The volatility σ^D determines the *size of macroeconomic fluctuations.* In order to understand the role of μ^D, we will “extract” all the stochastic volatility from D_t.

[19] Because the endowments are co-linear (all agents hold shares of the same single stock), it can be shown that, under some conditions on the agents' utility functions, the equilibrium is in fact unique up to a multiplicative factor, and unique if we fix the risk-free rate. See, e.g., Dana (1995), Dana (2001). If the dividend is neither bounded away from zero nor from infinity, some additional care is needed to verify the existence of equilibrium. See, e.g., Dana (2001) and Malamud (2008). We implicitly assume throughout the paper that an equilibrium exists.

[20] We assume a unique such y_k exists.

Fix an $x_0 \in \mathbb{R}_+$ and introduce the function

$$F(x) \stackrel{def}{=} \int_{x_0}^{x} \frac{1}{y\,\sigma^D(y)}\,dy\ ,\ x > 0\,.$$

Then, a direct calculation shows that the diffusion coefficient of the process

$$A_t \stackrel{def}{=} F(D_t)$$

is equal to one. Since A_t is also an Ito diffusion, its dynamics are given by

$$dA_t = C(A_t)\,dt + dB_t\,.$$

Definition 3.1 *The negative instantaneous expected change of the volatility-extracted process A_t will be referred to as the rate of macroeconomic fluctuations, or RMF, and denoted ρ_t. More precisely, we define*

$$\rho_t \stackrel{def}{=} -\,C'(A_t)\,.$$

Note that the rate of macroeconomic fluctuations is a function of the current dividend: if we define the function

$$c(x) = c^D(x) \stackrel{def}{=} -C'(F(x))$$

then

$$\rho_t = c^D(D_t)\,.$$

It follows directly from the definition that the function c is invariant under functional transformations. [21] That is, if $D_t = g(\tilde{D}_t)$ for some process $\tilde{D}_t$ and a smooth, increasing function g then

$$c^D(g(x)) = c^{\tilde{D}}(x)\,.$$

In particular, the following is true

[21] It is possible to verify that function $c(x)$ is given by

$$c(x) = -x\,(\mu^D)'(x) + x\,(\sigma^D)'(x)\,\sigma^D(x)^{-1}\,\mu^D(x) + (\sigma^D)'(x)\,\sigma^D(x)\,x + 0.5\,(\sigma^D)''(x)\,x^2\,\sigma^D(x). \quad (5)$$

See Appendix. We note that the right-hand side of (5) appears also in the paper Detemple, Garcia and Rindisbacher (2003) in a partial equilibrium setting as an auxiliary process, without having a direct economic interpretation.

Corollary 3.2 *The rate of macroeconomic fluctuations ρ_t is constant if and only if the dividend D_t is a smooth and strictly increasing function of an autoregressive process*

$$dA_t \;=\; (a \,-\, b\, A_t)\, dt \;+\; dB_t. \tag{6}$$

In this case, $\rho_t \;=\; b$.

We will also need the following

Definition 3.3 *We will refer to*

$$\sigma^D(t\,,\,\tau) \;\stackrel{def}{=}\; e^{-\int_t^\tau \rho_s\, ds}\, \sigma^D(D_\tau)$$

as the discounted dividend volatility, to

$$\sigma^r(t\,,\,\tau) \;\stackrel{def}{=}\; e^{-\int_t^\tau \rho_s\, ds}\, \sigma^r(D_\tau)$$

as the discounted interest rate volatility, to

$$\lambda(t\,,\,\tau) \;\stackrel{def}{=}\; e^{-\int_t^\tau \rho_s\, ds}\, \lambda_\tau$$

as the discounted market price of risk, and to

$$\sigma_t^{\text{myopic}} \;\stackrel{def}{=}\; E_t^Q\left[\,\sigma^D(t\,,\,T)\,\right] \tag{7}$$

as the (equilibrium) myopic volatility.

In general, we call *myopic* all the quantities that are determined by the present market value of their (possibly discounted) end-of-horizon value. In other words, their current value does not depend on their future co-movement with other market variables, but only on the "myopic estimate" of the future value.

As our benchmark example throughout the paper, we will consider a simple generalization of the Geometric Brownian Motion (GBM) as the dividend process, by having $A_t = \log D_t$ satisfy (6). Then, if $b > 0$, the growth rate of the economy (the log dividend) is mean reverting. On the other hand, if $b < 0$, the growth rate of the economy departs from the "mean" a at the rate equal to $-b$.

4 Non-Myopic Volatility and Risk-Return Tradeoff

In our economy the markets are complete. This implies that the prices coincide with those in an artificial economy populated by a single, representative agent with a utility function U (see Duffie (2001)). The equilibrium state price density equals the marginal utility of the representative agent, evaluated at the aggregate endowment,

$$\xi_T \;=\; U'(D_T). \tag{8}$$

That is, the function $U'(x)$ satisfies the equation

$$\sum_k \; I_k(y_k\, U'(x)) \;=\; x. \tag{9}$$

Let

$$\gamma^U(x) \;=\; -\frac{x\, U''(x)}{U'(x)}$$

be the relative risk aversion of the representative agent.

4.1 Equilibrium Market Price of Risk

The next result provides a representation of the market price of risk in terms of the quantities introduced in Definition 3.3 and highlights the role of myopic volatility.

Theorem 4.1 *The equilibrium market price of risk is given by the present market value of future discounted market price of risk net of the present cumulative value of future discounted interest rate volatility,*

$$\lambda_t \;=\; E_t^Q\,[\,\lambda(t\,,\,T)\,] \;-\; E_t^Q\left[\int_t^T \sigma^r(t,s)\,ds\right] \tag{10}$$

where

$$\lambda(t\,,\,T) \;= e^{-\int_t^T \rho_s\,ds}\,\lambda_T \;=\; e^{-\int_t^T \rho_s ds}\,\gamma^U(D_T)\,\sigma^D(D_T)\,. \tag{11}$$

In particular, if the representative agent's risk aversion $\gamma^U = \gamma$ is constant, then

$$\lambda_t \;=\; \gamma\,\sigma_t^{\text{myopic}} \;-\; E_t^Q\left[\int_t^T \sigma^r(t,s)\,ds\right]. \tag{12}$$

It is instructive to compare the result of Theorem 4.1 with the analogous result for Merton's CAPM. When the interest rate is constant, the dividend follows a GBM with volatility σ and $\gamma^U = \gamma$, we have

$$\lambda_t \;=\; \gamma\,\sigma.$$

Formula (12) shows that, when the dividend is not a GBM, the role of the fundamental volatility σ is played by the myopic volatility σ_t^{myopic}. The latter is given by the present market value of future dividend volatility, discounted at the rate of macroeconomic fluctuations ρ_t, and thus depends on the dynamic properties of D_t in a nontrivial way. In particular, dividend volatility by itself is not enough for determining the size of the equilibrium market price of risk. We need to know the rate ρ_t. For example, if ρ_t is always positive (negative), the level of fundamental volatility leads to overestimating (underestimating) the size of the market price of risk.

In order to understand the intuition behind this discounting, consider a simple modification of Merton's CAPM, for which the log dividend follows (6) with $b > 0$ and both $\gamma^U = \gamma$ and r are constant. Then, $\rho_t = b$ and

$$\lambda_t \;=\; \gamma\,\sigma\,e^{-b\,(T-t)}\,.$$

That is, the equilibrium riskiness of the stock is given by $\sigma\,e^{-b\,(T-t)}$. On the other hand, the stock riskiness is determined by its sensitivity to changes in the state variable D_t : If the stock price moves a lot in response to a change in the dividend, stock is risky, and if the sensitivity is small, stock price fluctuations are small and the stock is less risky. As recalled in (2), the stock price is the expectation (under the risk neutral measure) of future dividends. If $b > 0$, log dividends are mean reverting, and therefore, stock price sensitivity to dividend changes is smaller than in the case $b = 0$. This drives the stock riskiness and the equilibrium market price of risk down. However, as the time t approaches terminal horizon T, there is not enough time left for the dividends to mean-revert and the sensitivity of expected future dividends to changes in the current value of the dividend grows exponentially at the rate b, equal to the rate of mean-reversion. Thus, stock riskiness also grows exponentially, and converges to the riskiness of the dividend when $t \uparrow T$.

The situation is opposite when the rate $\rho_t = b$ is negative. Then, the log-dividend is non-stationary and every small fluctuation around the mean may force the dividend to deviate substantially. Hence, stock is riskier than in the case $b = 0$, and the market price of risk goes up. However, as time t approaches T, there is not enough time left for the dividend to deviate, riskiness decreases at the rate b and the equilibrium market price of risk goes down.

When the interest rate is stochastic, another term, $E_t^Q\left[\int_t^T \sigma^r(t,s)\,ds\right]$, appears in (12). To understand the intuition behind this term, suppose that the interest rate is procyclical. Then, the bond prices are countercyclical. Thus, bonds are expensive when stocks are cheap. This drives the demand for stocks up, pushing the equilibrium stock price up, and, consequently, the equilibrium stock returns and the market price of risk decrease.

There is some empirical evidence that interest rates are procyclical. However, there is also evidence that interest rates are negatively correlated with some market indices (e.g., S&P 500). See Campbell (1987). In our model, dividends and stock returns are perfectly correlated. Thus, if we interpret the market portfolio as the S&P 500, we may assume that the interest rates are countercyclical, which will lead to an increase in the equilibrium market price of risk. Note however that the empirical interest rate volatility is small and therefore the correction to the market price of risk, arising from stochastic interest rates, is also relatively small.

The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.2 *Assume that r is constant.*[22] *Under the equilibrium risk neutral measure, the drift of the market price of risk is independent of the representative agent's utility, and is always equal to ρ_t.*

Corollary 4.2 has direct empirical implications, because it is *model-independent.* More precisely, it implies that the market price of risk evolves as

$$d\lambda_t \;=\; \lambda_t\,(\mu_t^\lambda\,dt \;+\; \sigma_t^\lambda\,dB_t)$$

with the drift μ_t^λ given by

$$\mu_t^\lambda \;=\; \rho_t \,+\, \lambda_t\,\sigma_t^\lambda.$$

If we estimate the market price of risk λ_t, its volatility σ_t^λ and drift μ_t^λ from data, we immediately obtain an estimate for the rate ρ_t :

$$\rho_t \;=\; \lambda_t\,\sigma_t^\lambda \;-\; \mu_t^\lambda\,.$$

In this sense, the rate of macroeconomic fluctuations is directly observable.[23]

[22] An analogous result also holds when r is stochastic, but it involves an additional term.

[23] Note however that, generally, estimating the drift is not easy and requires sophisticated econometric procedures.

4.2 Equilibrium Volatility

Unlike the market price of risk, the stock price volatility is a non-myopic object. It depends not only on the size of the aggregate risk, but also on its future fluctuations relative to other variables. More precisely, the following is true:

Theorem 4.3 *The equilibrium stock price volatility is given by*

$$\sigma_t^S \;=\; \sigma_t^{\text{myopic}} \;+\; \sigma_t^{\text{non-myopic}} \;-\; E_t^Q\left[\int_t^T \sigma^r(t\,,\,s)\,ds\right]\,,$$

where[24]

$$\sigma_t^{\text{non-myopic}} \;=\; -\,\frac{1}{S_t}\,\mathrm{Cov}_t^Q\Big(\big(\lambda(t\,,\,T)\;-\;\sigma^D(t\,,\,T)\big)\,,\,e^{-\int_t^T r(D_s)ds}\,D_T\Big)\,. \tag{13}$$

Furthermore, σ_t^S is positive if r' is sufficiently small (the interest rate is not highly procyclical).

To understand the intuition behind Theorem 4.3, we first discuss our benchmark extension of Merton's CAPM when r, γ^U are constant, and the log dividend follows the autoregressive process (6). Then, σ^D and the rate $\rho_t = b$ are constant and the non-myopic volatility $\sigma_t^{\text{non-myopic}}$ vanishes. Consequently, by Theorems 4.1 and 4.3, the risk-return tradeoff coincides with that of Merton's CAPM:

$$\lambda_t \;=\; \gamma\,\sigma_t^S.$$

However, the stock volatility does not coincide with the fundamental dividend volatility. Rather, we have

$$\sigma^S \;=\; \sigma_t^{\text{myopic}} \;=\; e^{-b(T-t)}\sigma^D\,.$$

Stock volatility is higher than the dividend volatility if the log dividend is non-stationary (i.e., $b < 0$) and lower if the log dividend is stationary and mean-reverting (i.e., $b > 0$), because stock price volatility is the sensitivity of expected future dividends to changes in D_t, as seen from (15) below. This sensitivity is large (small) if the log dividend is non-stationary (stationary). See also the discussion after Theorem 4.1.

Expression (13) implies that, if the interest rate is constant, the spread between the stock price volatility and the myopic volatility is given by the covariance of the

[24] We use $\mathrm{Cov}_t^Q(X, Y)$ to denote conditional covariance of random variables X and Y under the equilibrium risk-neutral measure Q.

discounted future aggregate dividend with the future discounted market price of risk net of the discounted future dividend volatility. The term "non-myopic" is most easily justified if we recall that

$$\lambda(t\,,\,T)\;-\;\sigma^D(t\,,\,T)\;\;=\;\;\big(\gamma^U(D_T)\;-\;1\big)\,\sigma^D(t\,,\,T)\,. \tag{14}$$

Formulae (14) and (13) imply that, generically, the non-myopic volatility vanishes if and only if $\gamma^U \equiv 1$. That is, precisely when the representative agent is completely myopic. Hence, the nature of the non-myopic volatility is necessarily driven by the agents' non-myopic equilibrium behavior. When the market price of risk is stochastic, non-myopic agents increase or decrease stock investment depending on the expected future fluctuations of the market price of risk. The equilibrium hedging demand raises or decreases the total equilibrium demand for stocks and therefore drives the equilibrium stock price up or down. Since D_t is the single state variable in our model, standard results imply that $S_t = S(t, D_t)$ is a smooth function[25] of D_t and therefore, by Ito's formula, we get

$$\sigma_t^S\;\;=\;\;\sigma^D(D_t)\,D_t\,\frac{\partial}{\partial D_t}\,\log S(t, D_t)\,. \tag{15}$$

Thus, stock price volatility is nothing but the sensitivity of the stock price to the changes in the dividend. Since equilibrium optimal portfolios respond to changes in D_t in a non-myopic way, so does the equilibrium stock price, giving rise to non-myopic volatility.

Formula (13) for the non-myopic volatility is similar to the formula for the forward-futures spread, which says that the forward-futures spread is given by the covariance under the risk-neutral measure of the future payoff with the cumulative return on the short-term bonds (the bank account). As has been recognized in the literature, such a representation immediately generates two empirical predictions for the spread: (1) a weak prediction that the sign of the spread should coincide with the sign of the covariance, and (2) a strong prediction that the variation in the spread is explained by the covariance (See Meulbroek (1992)). Based on formula (13), we can make analogous predictions for the dynamics of the non-myopic volatility and the co-movement of the dividends with the discounted market price of risk $\lambda(t,T)$ net of the discounted dividend volatility $\sigma^D(t,T)$.

Theorems 4.1 and 4.3 also have direct implications for the dynamics of risk-return tradeoff. Empirical evidence suggests that expected stock returns are weakly related to volatility. The results of Gennotte and Marsh (1993), Glosten, Jagannathan and Runkle (1993), Whitelaw (2000) and Harvey (2001) suggest that the relation between expected

[25] Under some technical conditions. See Hugonnier, Malamud and Trubowitz (2009).

returns and volatility is highly complex and non-linear.[26] As Guo and Whitelaw (2006) show, one possible explanation of this phenomenon is equilibrium hedging behavior. Suppose for simplicity that the risk aversion $\gamma^U = \gamma$ of the representative agent and the interest rate r are constant. Then, by a fundamental result of Merton (1973), we have

$$\lambda_t \;=\; \gamma\,(\sigma_t^S \,+\, \lambda_t^{\text{hedge}})$$

where λ_t^{hedge} is the so-called hedge component appearing due to the equilibrium hedging demand. Guo and Whitelaw (2006) show that the market price of risk is driven primarily by λ_t^{hedge}. Theorems 4.1 and 4.3 imply that $\lambda_t^{\text{hedge}} \;=\; -\,\sigma_t^{\text{non-myopic}}$, providing a direct economic interpretation for Merton's hedge component. Furthermore, when interest rate r is constant, the risk-return relation is driven by the nonlinear relation between the myopic and the non-myopic volatility,

$$\frac{\lambda_t}{\sigma_t^S} \;=\; \gamma\,\frac{\sigma_t^{\text{myopic}}}{\sigma_t^{\text{myopic}} \,+\, \sigma_t^{\text{non-myopic}}}\,. \tag{16}$$

Thus, calculating the risk-return tradeoff amounts to calculating the ratio of the non-myopic and the myopic volatility. One can explore empirically this ratio using representations (7) and (13). It is particularly convenient that the latter does not explicitly depend on the utility of the representative agent and can be directly estimated from data using the empirical state price density (see Chernov and Ghysels (2000) and Rosenberg and Engle (2002)).

5 Equilibrium Optimal Portfolios

In this section we derive various properties of equilibrium optimal portfolios. Let

$$U_{k\,t}(x) \;\stackrel{def}{=}\; \sup_{\pi}\; E_t\,[\,u_k(W_{kT})\,|\,W_{kt} \;=\; x\,]$$

be the value function of agent k.[27] Following Merton (1971,1973), we define the *effective relative risk aversion* of agent k at time t via

$$\gamma_{k\,t} = \gamma_{k\,t}(W_{kt}) \;\stackrel{def}{=}\; -\,\frac{W_{kt}\,U''_{k\,t}(W_{kt})}{U'_{k\,t}(W_{kt})}.$$

[26] Bali and Peng (2006) and Ghysels, Santa-Clara and Valkanov (2005) find a positive relation between returns and volatility, but this relation is also non-linear.

[27] Note that $U_{k\,t}$ depends on D_t but we suppress this dependence.

It is known (see Merton (1971)), that, when the investment opportunity set is non-stochastic, the optimal portfolio is myopic, instantaneously mean-variance efficient and is given by

$$\pi_{k\,t}^{\text{myopic}} \stackrel{def}{=} \frac{\lambda_t}{\gamma_{k\,t}\,\sigma_t^S}\,.$$

A direct calculation implies the following

Proposition 5.1 *We have*

$$\gamma_{k\,t} \;=\; \frac{W_{k\,t}}{E_t^Q[\gamma_{kT}^{-1}\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}]}. \tag{17}$$

Representation (17) implies that the effective absolute risk tolerance

$$R_{kt} \;=\; \frac{W_{k\,t}}{\gamma_{k\,t}}$$

(i.e., the absolute risk tolerance of the value function) is a discounted martingale under the risk-neutral measure. Consequently,

$$W_{k\,t}\pi_{k\,t}^{\text{myopic}} \;=\; \frac{\lambda_t}{\sigma_t^S}R_{kt} = \frac{\lambda_t}{\sigma_t^S}E_t^Q[\,e^{-\int_t^T r_s\,ds}\,R_{kT}]\,.$$

The last expression shows that the absolute amount that the myopic portfolio component holds in the risky asset in equilibrium is equal to the current return-risk ratio times the market value of the future absolute risk tolerance, in agreement with our definition of "myopic".

We denote

$$\pi_{k\,t}^{\text{hedging}} \stackrel{def}{=} \pi_{k\,t} \;-\; \pi_{k\,t}^{\text{myopic}}$$

and refer to it as the hedging portfolio. This is the non-myopic component of the optimal portfolio that the agent uses to hedge against (or, take advantage of) future fluctuations in the stochastic opportunity set. The main result of this section is the following

Theorem 5.2 *We have*

$$\pi_{k\,t}^{\text{hedging}} \;=\; \pi_{k\,t}^{\text{MPR hedge}} \;+\; \pi_{k\,t}^{\text{IR hedge}}$$

with the market price of risk (MPR) hedging component

$$\begin{aligned}\pi_{k\,t}^{\text{MPR hedge}} \;&=\; -\;\frac{1}{\sigma_t^S\,W_{k\,t}}\,\times\\ &\text{Cov}_t^Q\Big(\lambda(t\,,\,T)\,,\,e^{-\int_t^T r(D_s)ds}\,(W_{k\,T}\;-\;R_{k\,T})\Big)\,.\end{aligned} \tag{18}$$

and the interest rate (IR) hedging component

$$\pi_{k\,t}^{\text{IR hedge}} \;=\; \frac{1}{\sigma_t^S}\left(\frac{1}{\gamma_{k\,t}}\,-\,1\right)\,E_t^Q\left[\int_t^T \,\sigma^r(t,\tau)\,d\tau\right]. \tag{19}$$

Formulae (18) and (19) provide closed-form expressions for equilibrium hedging portfolios and show that the market price of risk hedge and the IR hedge have a very different nature. The IR hedge is a product of two simple terms: one that depends on the current level of the agent's effective risk aversion, another given by the market value of the (discounted) cumulative interest rate volatility, normalized by the stock volatility. In particular, if the interest rate is procyclical,[28] IR hedge is positive (negative) when effective risk aversion is below (above) one.[29] Furthermore, IR hedges of two CRRA agents differ only by a constant multiple.[30]

The market price of risk hedge is more complex. It is determined by joint fluctuations of the discounted future market price of risk with agent-specific characteristics: his future wealth net of his absolute risk tolerance. Suppose for simplicity that the interest rate is constant. We can then write (18) as

$$e^{r(T-t)}\,\sigma_t^S\,W_{k\,t}\,\pi_{k\,t}^{\text{MPR hedge}} \;=\; \underbrace{-\text{Cov}_t^Q(\lambda(t\,,\,T)\,,\,W_{k\,T})}_{\textit{wealth hedge}} \;+\; \underbrace{\text{Cov}_t^Q(\lambda(t\,,\,T)\,,\,R_{k\,T})}_{\textit{risk tolerance hedge}}. \tag{20}$$

Both terms on the right-hand side have a natural interpretation. The first one is the covariance of agent's wealth with the (discounted) market price of risk. We will refer to it as the wealth hedge. Similarly, the second one will be referred to as the risk tolerance hedge.

To understand the nature of the wealth hedge, suppose that the future market price of risk and the agent's wealth are positively correlated. When that wealth is low, any additional unit of wealth is valuable to the agent. Since the market price of risk is also low in those states, the agent prefers to invest in the risk-free asset, because it offers a better hedge against the low-wealth states. This generates a negative hedging demand. When the agent's wealth is high, the utility of getting an additional unit of wealth is low and therefore, even though the market price of risk is high in those states, it does not make stocks that much more attractive. [31]

[28]However, not too procyclical so that σ^S is positive.

[29]Detemple, Garcia and Rindisbacher (2003) derive an analogous result in a partial equilibrium setting.

[30]By (17), $\gamma_{kt} = \gamma_k$ if agent k has a CRRA utility. This also follows from a simple homogeneity argument.

[31]The argument for the case of negative correlation between wealth and the market price of risk is analogous.

To illustrate the intuition behind the risk tolerance hedge, suppose that the market price of risk and risk tolerance are positively correlated. Then, when risk tolerance is high, the agent is willing to take more risk and this effect is magnified since the market price of risk is high in those states, driving the demand for stock up. When risk tolerance is low, the market price of risk is also low. This generates an incentive to hedge against those states and reduce the long position in the stock, but not go short too much, because of low risk tolerance. As a consequence, stock investment is altogether more attractive (the impact of the states with high market price of risk and high risk tolerance prevails) and this generates a positive hedging demand.[32]

If we adopt Arrow's hypothesis that absolute risk aversion is monotone decreasing in wealth, risk tolerance $R_{k\,T} = R_k(W_{k\,T})$ is increasing in wealth and therefore, generally speaking, the wealth hedge and the risk tolerance hedge will have opposite signs and represent competing effects for determining the size of the hedging demand. Note also that, if the agent's utility is of CARA (exponential) class, $R_{k\,T}$ is constant and, consequently, the risk tolerance hedge vanishes.

There is a clear similarity between expressions (18) and (13) for the market price of risk hedge and the non-myopic volatility respectively. In fact, the hedging demand (18) is the source of non-myopic volatility (13). Both of them arise as a non-myopic response to the future fluctuations of the market price of risk. Similarly to (13), expression (18) does not directly depend on the utility function of the representative agent. In particular, if we adopt the common hypothesis that the agents have hyperbolic absolute risk aversion (HARA) preferences,

$$R_{k\,T} \;=\; a\,W_{k\,T} \,+\, b$$

for some constants a, b and the market price of risk hedge takes the form

$$e^{r(T-t)}\,\sigma_t^S\,W_{k\,t}\,\pi_{k\,t}^{\text{MPR hedge}} \;=\; (a\,-\,1)\,\mathrm{Cov}_t^Q(\lambda(t\,,\,T)\,,\,W_{k\,T})\,.$$

This formula generates the empirical prediction that the market price of risk hedge is driven by the co-movement of agent's wealth with the discounted market price of risk. This prediction can be tested using PSID data; see Brunnermeier and Nagel (2008).

One could also use this formula together with the empirical state price density method of Rosenberg and Engle (2002) and Chernov and Ghysels (2000) to estimate the empirical optimal hedging portfolio.

[32]The case when the market price of risk and risk tolerance are negatively correlated is analogous.

6 Quantitative Implications

In this section we derive several specific quantitative implications of our general results. We start with a simple bound on the equilibrium market price of risk. Let $\gamma_k^{\inf}$, $\gamma_k^{\sup}$ denote the infinum and supremum of the relative risk aversion of agent k. Then,[33]

$$\min_k \gamma_k^{\inf} \;\le\; \gamma^U(x) \;\le\; \max_k \gamma_k^{\sup} \qquad \text{for all } x\,.$$

Therefore, Theorem 4.1 immediately yields

Proposition 6.1 *Suppose that the risk free rate r is constant. Then, the equilibrium market price of risk λ_t satisfies*

$$\min_k \gamma_k^{\inf} \;\le\; \frac{\lambda_t}{\sigma_t^{\text{myopic}}} \;\le\; \max_k \gamma_k^{\sup}.$$

When the relation between risk and return is linear, the line $\lambda = \gamma\,\sigma$ in the (λ,σ)-plane is often referred to as the security market line. Proposition 6.1 shows that, when the risk aversion of the representative agent is stochastic, there are two security market lines in the $(\lambda,\sigma^{\text{myopic}})$ plane and the equilibrium risk-return profile stays inside the stripe between them. This phenomenon is similar to that discovered by Stulz (1981) in the context of an equilibrium model of segmented markets and can be tested empirically, providing empirical lower and upper bounds for risk aversion in the economy.

To address further dynamic properties of equilibrium, we will need to make additional assumptions about the evolution of D_t. It has become common in the literature to use consumption volatility as a measure of macroeconomic uncertainty. There is empirical evidence that macroeconomic uncertainty is countercyclical. See, e.g., French and Sichel (1993), Kim et al. (2009).

Definition 6.2 *We say that our model exhibits countercyclical macroeconomic uncertainty if the dividend volatility σ_t^D is countercyclical and the rate of macroeconomic fluctuations ρ_t is procyclical.*[34]

This definition is natural because volatility σ^D and $-\rho_t$ determine respectively the size and the speed of macroeconomic fluctuations. From now on we make the following

[33] It is known (see, Wilson (1968) and Hara, Huang and Kuzmics (2007)) that the representative agent's risk aversion is a weighted average of individual risk aversions.

[34] We call an economic variable countercyclical if it is monotone decreasing in D_t and procyclical otherwise.

Assumption 6.3 *The economy exhibits countercyclical macroeconomic uncertainty and the aggregate risk aversion γ^U is countercyclical.*

By Definition 3.1, the economy exhibits countercyclical macroeconomic uncertainty if and only if $\log D_t = f(A_t)$ where f is increasing and concave, and the process A_t has constant volatility and concave drift. For example, a simple family of such processes arises when

$$dA_t = (a - b\,A_t)\,dt + \sigma\,dB_t.$$

and the dividend process is given by $D_t = e^{f(A_t)}$ for some increasing and concave function f.

The assumption of countercyclical risk aversion is natural and there is strong empirical evidence supporting it; see, e.g., Campbell and Cochrane (1999), Smith and Whitelaw (2009). Furthermore, countercyclical aggregate risk aversion can be generated from simple microeconomic assumptions about individual agents' preferences. In fact it is known that if all agents in the economy have (heterogeneous) CRRA preferences the representative agent's utility will have a decreasing (i.e., countercyclical) relative risk aversion (DRRA). See Benninga and Mayshar (2000), Cvitanić and Malamud (2009b). A modification of the arguments from the latter paper also implies that the following slight generalization is true:

Proposition 6.4 *If all agents have DRRA preferences then the aggregate risk aversion is countercyclical.*

In fact, it is possible to show that Proposition 6.4 still holds if individual risk aversions are sufficiently heterogeneous and "not too increasing".

It is a conventional wisdom that countercyclical risk aversion generates countercyclical market price of risk.[35] The following result confirms this intuition.

Proposition 6.5 *Under Assumption 6.3, the market price of risk is countercyclical if both r' and r'' are sufficiently small.*[36]

[35] See Campbell and Cochrane (1999). There is also empirical evidence supporting countercyclicality of both market price of risk and aggregate risk averion. See Fama and French (1989), Ferson and Harvey (1991).

[36] In fact, we show that the following is true: for any interval $[d_1, d_2] \subset \mathbb{R}_+$ there exists an $\varepsilon > 0$ such that λ_t is monotone decreasing in D_t in the interval, as long as $r' < \varepsilon$ and $|r''| < \varepsilon$.

He and Leland (1993) show that, when r is constant and the dividend follows a geometric Brownian motion, the market price of risk is countercyclical if the aggregate risk aversion is countercyclical. Proposition 6.5 provides general sufficient conditions for the market price of risk countercyclicality. These conditions are tight, in the sense that even if dividend volatility is constant and risk aversion is countercyclical, a sufficiently countercyclical rate of macroeconomic fluctuations can generate procyclical behavior of the market price of risk.

The requirement that r' and r'' be small is not too restrictive, given that, empirically, interest rate volatility is very small, especially relative to stock volatility. However, from a theoretical perspective, it is interesting to note that the effect of stochastic interest rates on the market price of risk might lead to unexpected equilibrium dynamics if countercyclicality of the aggregate risk aversion is not strong enough. Consider for example the case when $\gamma^U = \gamma$ is constant and the dividend is GBM. Then, the market price of risk is given by

$$\lambda_t \;=\; \sigma\left(\gamma \;-\; E_t^Q\left[\int_t^T r'(D_\tau)\, D_\tau\, d\tau\right]\right) .$$

Consequently, the dynamics of the market price of risk are determined exclusively by the interest rate process. If $r'(D_t)\, D_t$ is larger than γ with high probability, the market price of risk may become negative, because the bond prices become so countercyclical that the investors are eager to buy stock for hedging purposes, even if the stock offers a negative excess return. Furthermore, if $r'(D_t)\, D_t$ is decreasing in D_t, the market price of risk becomes procyclical.

By Proposition 6.5, countercyclical macroeconomic uncertainty together with countercycical risk aversion generate countercyclical dynamics of the market price of risk. The latter dynamics, in turn, determine the sign of non-myopic volatility, as can be seen from expression (13) and the definition of $\lambda(t,T)$. More precisely, we arrive at the following result:

Proposition 6.6 *Under Assumption 6.3, the non-myopic volatility is positive if* $\gamma^U \geq 1$ *and* r' *is sufficiently small.*

The reasoning behind this result is as follows. Under Assumption 6.3, the market price of risk is low in good states, but the future aggregate risk aversion is also (relatively) low and the expected future dividends are high. Therefore, the agent is willing to hold the stock despite low market price of risk. This makes the price go up in good states and, by similar arguments, go down in bad states, driving the non-myopic part of the price volatility up. Put differently, the tension between the movements in future dividend and the market price of risk creates non-myopic (excess) volatility, because the larger the dividend, the larger the demand for the stock, while at the same time the lower the market price of risk , the lower the demand. The requirement that r' be not too large is important: if the interest rate is highly procyclical, discounted dividends $e^{-\int_t^T r_s\,ds}\,D_T$ may exhibit countercyclical behavior, which, by (13), gives rise to a negative non-myopic volatility.

The non-myopic volatility is the natural analog of excess volatility for the case of stochastic dividend risk.[37] Our representation illustrates that dynamical properties of aggregate risk aversion and dividend volatility by themselves are *not sufficient* to generate excess volatility and the outcome is strongly influenced by the dynamics of the rate of macroeconomic fluctuations. This result should be contrasted with a related result of Bhamra and Uppal (2009a). They analyze equilibrium with two CRRA agents and a GBM dividend and show that the stock price volatility is higher than the dividend volatility if agent's elasticity of intertemporal substitution (EIS) is not too large. Their model is different from ours because they allow for intermediate consumption and, therefore, the interest rate is determined endogenously by the agents' EIS.[38] Our volatility representation for the non-myopic (excess) volatility (see Theorem 4.3) allows us to look at the result of Bhamra and Uppal (2009a) from a different perspective. There are two sources generating deviations from fundamental (myopic) volatility in their model. Countercyclical market price of risk drives the non-myopic volatility up, whereas the procyclical risk-free rate drives the volatility down, and the constraint of EIS being not

[37]Stock prices are said to exhibit excess volatility if they are substantially more volatile than the underlying dividends. Excess volatility is a well known stylized fact. See, e.g., Shiller (1981), LeRoy and Porter (1981), Mankiw, Romer, and Shapiro (1985, 1991) and West (1988).

[38]The simplifying assumption of no intermediate consumption is frequently used in equilibrium asset pricing literature, including the Sharpe and Lintner's CAPM. See, e.g., Bick (1990), He and Leland (1993), Grossman and Zhou (1996), Kogan et al. (2006). Since agents do not have to substitute consumption for investment, the interest rate process is not determined in equilibrium and can be specified exogenously. This allows to isolate interest rate effects on the equilibrium dynamics. One can contrast this to production economies, in which the market price of risk is specified exogenously and only the interest rate is determined in equilibrium. See, e.g., Dumas (1989). Since dividend payments are relatively infrequent, assuming a continuous flow of dividends is often counterfactual, especially when the horizon T is small. In that case, a model with only terminal dividends is more appropriate.

too large diminishes the second effect.[39]

Another important consequence of Proposition 6.6 and Theorem 4.1 is that, under their assumptions, the following *risk-return inequality*[40] holds:

$$\lambda_t \;<\; \sigma_t^S \max_x \gamma^U(x)\,. \tag{21}$$

The reason is that the market price of risk is driven by the myopic volatility, which is smaller than the stock volatility by Theorem 4.3 and Proposition 6.6. Due to its myopic nature, the equilibrium market price of risk "underestimates" the true stock price volatility and is unable to account for future stock price and return fluctuations. Since the market price of risk is countercyclical, the stock is cheap in bad states (those with low D_t) and offers a high instantaneous return. This motivates the agents to buy more shares in those states, and the stock becomes more volatile, which, in turn, stabilizes the demand. A similar argument applies in good states. Risk-return inequality is a much weaker requirement than the risk-return equality of Merton (1973) and we can use it together with empirical data to estimate the maximal risk aversion in the economy.

We now study the implications of the market price of risk countercyclicality for non-myopic equilibrium optimal portfolios. Recall that the absolute prudence and absolute risk tolerance of agent k are defined by

$$P_k(x) \;=\; -\frac{u_k'''(x)}{u_k''(x)} \qquad \text{and} \qquad R_k(x) \;=\; -\frac{u_k'(x)}{u_k''(x)}$$

respectively. The following is true:

Theorem 6.7 *Under Assumption 6.3, and supposing that r' is not too large, the market price of risk hedging component $\pi_t^{\text{MPR hedge}}$ is positive if and only if the product of prudence and risk tolerance is below two*[41]*, that is*

$$P_k(x)\,R_k(x) \;\le\; 2 \qquad \text{for all } x.$$

[39] It would be of significant interest to study the validity of our general results in economies with intermediate consumption and endogenously determined interest rate. However, the equilibrium structure in such economies is much more complex and we leave it for future research. The advantage of an exogenous risk-free rate process is that we can isolate and study the effect of the stochastic interest rate on the dynamics of asset prices and optimal portfolios and derive general necessary and sufficient conditions for excess volatility.

[40] This is indirectly related to the state price density inequality of Constantinides and Duffie (1996).

[41] Note that the product of prudence and risk tolerance is exactly two for the log investor, for whom the hedging portfolio is zero.

The result of Theorem 6.7 is somewhat unexpected at first glance. Since the optimal portfolio of a log investor is always myopic, one would expect that the sign of the hedging component only depends on whether risk aversion is above or below one (as it does for the IR hedging component). However, Theorem 6.7 shows that the hedging motive depends on properties of *three* derivatives of the utility function.

For simplicity, let r be constant and recall expression (20):

$$e^{r(T-t)}\,\sigma_t^S\,W_{k\,t}\,\pi_{k\,t}^{\text{MPR hedge}} \;=\; -\text{Cov}_t^Q\big(\lambda(t\,,\,T)\,,\,(W_{k\,T}\;-\;R_{k\,T})\big)\,.$$

By Assumption 6.3, $\lambda(t\,,\,T)$ is countercyclical. Since markets are complete, all terminal wealth amounts $W_{k\,T}$ are co-monotone, increasing in D_T and, consequently, the wealth hedge $-\text{Cov}_t^Q(\lambda(t\,,\,T)\,,\,W_{k\,T})$ is positive. The size of the risk tolerance hedge depends on the cyclical properties of agent's risk tolerance $R_{k\,T}$. A direct calculation shows that

$$\frac{d}{dx}\,R_k(x) \;=\; -1\;+\;P_k(x)\,R_k(x).$$

If the product $P_k\,R_k$ is large, the risk tolerance $R_{k\,T}$ is highly sensitive to wealth fluctuations and, consequently, highly procyclical. Then, the risk tolerance hedge

$$\text{Cov}_t^Q\big(\lambda(t\,,\,T)\,,\,R_{k\,T}\big)$$

is very negative, dominates over the wealth hedge

$$-\,\text{Cov}_t^Q\big(\lambda(t\,,\,T)\,,\,W_{k\,T}\big)$$

and the market price of risk hedge is negative. By contrast, if $P_k\,R_k$ is small, the risk tolerance hedge is not very negative. Therefore, the wealth hedge dominates and the market price of risk hedge is positive. The threshold of two appears because $W_{k\,T}\;-\;R_k(W_{k\,T})$ is monotone increasing in $W_{k\,T}$ if and only if $P_k\,R_k\;\le 2$.

The appearance of prudence is also related to precautionary savings. As Kimball (1990) showed in a static, one period model, the strength of the precautionary savings motive for an agent anticipating stochastic fluctuations in his future income is determined by his prudence P_k. Here, P_k plays a similar role, determining the strength of savings/investment motive for an agent, anticipating future changes in the stochastic investment opportunity set.

If we consider the Arrow (1965) hypothesis that $\gamma_k(x)$ is increasing, a direct calculation shows that it holds if and only if $x\,P_k(x)\;\le\;\gamma_k(x)\;+\;1$. If $\gamma_k(x)\ge 1$, we get $P_k(x)\,R_k(x)\;=\;x\,P_k(x)/\gamma_k(x)\;\le\;2$. Theorem 6.7 then implies the following

Corollary 6.8 *Suppose that $\gamma_k(x) \geq 1$ and is increasing and r' is not too large. Then, under Assumption 6.3, the market price of risk hedge is positive.*

For the benchmark CRRA utility, $P_k\, R_k \;=\; 1+\gamma_k^{-1}$ and the results take a simpler form

Corollary 6.9 *Suppose that $\gamma_k = \text{const}$, r' is not too large, and Assumption 6.3 holds. Then, the MPR hedge is positive if and only if $\gamma_k \geq 1$.*

The intuition behind Corollary 6.9 is as follows. The marginal utility $u_k'(x) \;=\; x^{-\gamma_k}$ of an agent with high risk aversion $\gamma_k > 1$ is high in bad states (low D_t). Therefore, any additional unit of consumption in those states is highly valuable for him. Since the market price of risk is countercyclical, it is high in bad states. This makes the stock a highly attractive instrument for agent k to hedge against those states and induces him to buy additional shares. On the other hand, an agent with low risk aversion $\gamma_k < 1$ is not "afraid" of the bad states and bets on the realization of good states with high D_t. Since the market price of risk is low in those states, it is optimal for him to sell some of his stock holdings, creating a negative hedging demand.

We complete our discussion of optimal portfolios with two monotonicity results. We will need the following

Definition 6.10 (Ross (1981)) *Agent i is more risk averse than agent j in the sense of Ross, if*

$$\inf_x \gamma_i(x) \;\geq\; \sup_x \gamma_j(x).$$

In this case we write $\gamma_i \geq_R \gamma_j$.

This definition was introduced by Ross (1981) in the context of a static, one period problem with two risky assets. Ross showed that the optimal investment in the riskier asset is generally not monotone in risk aversion, if we only require a weak, pointwise inequality in risk aversion. The reason is that the optimal portfolio choice with wealth dependent risk aversion and two risky assets becomes a non-local problem and the local properties of risk aversion are not sufficient for the analysis. A similar phenomenon arises in our dynamic, multi-period optimization: even though one of the assets is locally riskless, the amount of money invested into it changes over time and thus, effectively, we get a problem with two risky assets, and Definition 6.10 becomes the right concept to consider.

First, we note that the following is true.

Proposition 6.11 *If an agent is less (more) risk averse than the log agent then he invests more (less) into the risky asset then the log agent.*[42]

Proposition 6.11 is very general and holds for any economy. To compare portfolios of two non-logarithmic agents, we will need to make additional assumptions. We have

Proposition 6.12 *Suppose that risk aversions of agents i and j are above one and $\gamma_i \geq_R \gamma_j$. Then, under Assumption 6.3, agent i invests less than agent j into the stock.*

Monotonicity of optimal portfolios is important – almost all papers on heterogeneous equilibria use this monotonicity property as the basis for economic intuition. See, e.g., Dumas (1989), Wang (1996), Basak and Cuoco (1998), Basak (2005), Bhamra and Uppal (2009a,b). However, to the best of our knowledge, no proof of this property has ever been given even in an economy with only two CRRA agents. The result of Proposition 6.12 holds under very general conditions and is independent of the number of agents in the economy.

7 Example: Mean-Reverting Dividends and CRRA agents

We now illustrate the results from the previous sections in our benchmark example. That is, we assume

Assumption 7.1 *The log dividend $A_t = \log D_t$ follows an autoregressive process*

$$dA_t \;=\; (a - bA_t)dt \,+\, \sigma\, dB_t\,,$$

interest rate r is zero, and all agents in the economy have CRRA preferences, $\gamma_k(x) = \gamma_k$.

Under Assumption 7.1, $\sigma^D = \sigma,\ c = b$. Consequently, all the formulae substantially simplify and all the dynamic properties of equilibrium are determined solely by the properties of the aggregate risk aversion γ^U. The following proposition summarizes our main results.

Proposition 7.2 *Under Assumption 7.1,*

[42] A proof follows directly from Proposition B.1 in the appendix.

(1) the market price of risk

$$\lambda_t \;=\; E_t^Q[\,\gamma^U(D_T)\,]\,e^{-b\,(T-t)}\,\sigma$$

is counter-cyclical;

(2) the stock volatility is given by

$$\sigma_t^S \;=\; \sigma\,e^{-b\,(T-t)} \;+\; \sigma_t^{\text{non-myopic}}\,,$$

where the non-myopic (excess) volatility

$$\sigma_t^{\text{non-myopic}} \;=\; -\frac{e^{-b\,(T-t)}\,\sigma}{S_t}\,\text{Cov}_t^Q(\lambda_T\,,\,D_T)$$

is always positive;

(3) optimal portfolios are monotone decreasing in risk aversion for risk aversions above one. They are given by

$$\pi_{k\,t} \;=\; \frac{\lambda_t}{\gamma_k\,\sigma_t^S} \;+\; \pi_{k\,t}^{\text{hedging}}\,,$$

where the hedging component

$$\pi_{k\,t}^{\text{hedging}} \;=\; \frac{1}{W_{k\,t}}\,(\gamma_k^{-1}\;-\;1)\,\frac{e^{-b\,(T-t)}\,\sigma}{\sigma_t^S}\,\text{Cov}_t^Q(\lambda_T\,,\,W_{k\,T})$$

is positive if and only if $\gamma_k \;>\; 1$.

It is interesting to note that the hedging component depends on risk aversion in an asymmetric way. We have

$$W_{k\,T} \;=\; (\xi_T\,y_k)^{-\gamma_k^{-1}}\,.$$

Therefore, the sensitivity of wealth to changes in the state variable D_T is larger for the agents with small risk aversion. For example, for the agents with very small risk aversion, the coefficient $\gamma_k^{-1}-1$ controlling the size of the hedging demand is very large, and their hedging demand is very negative. By contrast, for large risk aversion, both effects are small and therefore their hedging demand is positive but relatively small.

We conclude this section with another interesting result related to heterogeneity in risk aversions, showing how the size of the risk-return tradeoff, the size of the ratio $\sigma_t^S/\sigma_t^{\text{myopic}}$ and the size of the equilibrium optimal portfolios depend on the magnitude of the heterogeneity. Let

$$\gamma \;\stackrel{def}{=}\; \min_k\,\gamma_k\;,\;\Gamma\;\stackrel{def}{=}\;\max_k\,\gamma_k.$$

Proposition 7.3 *The following is true:*

(1) the market price of risk satisfies

$$\gamma\,\sigma\,e^{-b(T-t)} \;\le\; \lambda_t \;\le\; \Gamma\,\sigma\,e^{-b\,(T-t)}\,;$$

(2) the stock volatility satisfies

$$\sigma\,e^{-b(T-t)} \;\le\; \sigma_t^S \;\le\; \sigma\,e^{-b(T-t)}\,\big(1\;+\;\Gamma\;-\;\gamma\big)\,;$$

(3) the risk-return tradeoff satisfies

$$\frac{\gamma}{1\,+\,\Gamma\,-\,\gamma} \;\le\; \frac{\lambda_t}{\sigma_t^S} \;\le\; \Gamma\,; \tag{22}$$

(4) the optimal portfolios satisfy

$$\begin{aligned}
\frac{1}{\gamma_k}\,\frac{\gamma}{1\,+\,\Gamma\,-\,\gamma} \;&\le\; \pi_{k\,t} \;\le\; \frac{1}{\gamma_k}\big(\Gamma\;+\;(\gamma_k-1)\,(\Gamma\,-\,\gamma)\big) && \text{if } \gamma_k > 1\,,\\
\frac{1}{\gamma_k}\,\frac{\gamma}{1\,+\,\Gamma\,-\,\gamma} \;&\le\; \pi_{k\,t} \;\le\; \frac{1}{\gamma_k}\,\Gamma && \text{if } \gamma_k < 1\,.
\end{aligned}$$

These results imply that the size $\Gamma - \gamma$ of heterogeneity plays a crucial role for determining the size of excess volatility, risk-return tradeoff and the size of equilibrium optimal portfolios. In the terminology of Dumas, Kurshev and Uppal (2009), investors are taking advantage of heterogeneous risk attitudes in the economy, which generates excess volatility and stochastic fluctuations of the market price of risk.

Bounds (22) and (23) are especially useful for empirical analysis because they do not depend on the parameters of the dividend process. For example, finding the empirical range

$$\left[\min\,\frac{\lambda_t}{\sigma_t^S}\;,\;\max\,\frac{\lambda_t}{\sigma_t^S}\right]$$

of the risk-return tradeoff immediately allows us to determine the *size* $\Gamma - \gamma$ *of heterogeneity* in the economy. Moreover, given the estimates for γ and Γ from risk-return tradeoffs, data on empirical portfolio holdings together with bounds (23) could be used to study the cross-sectional distribution of risk aversions in the economy.

8 Conclusions

We obtain general representations of the equilibrium optimal portfolios, market price of risk and stock volatility in terms of expected values and covariances of directly observable quantities, under the risk-neutral measure. In these representations, macroeconomic risks are priced discounted at a specific rate, that we call the rate of macroeconomic fluctuations. Our representations are universal because they do not explicitly involve the (unobservable) representative agent's utility and hold in any exchange economy with an arbitrary Markov dividend, arbitrary stochastic interest rates and arbitrary rational agents, maximizing utility from terminal consumption.

For the first time in the literature, we uncover the general structure of equilibrium non-myopic optimal portfolios of long-run investors. We show that the market price of risk hedge consists of two components, the wealth hedge and the risk tolerance hedge. Those are determined respectively by the covariances of agent's future wealth and future risk tolerance with future market price of risk, discounted at the rate of macroeconomic fluctuations.

We provide several theoretical predictions that can be tested empirically: (1) the drift of the market price of risk is equal the product of the market price of risk and its volatility plus the rate of macroeconomic fluctuations; (2) market price of risk is driven by the myopic volatility; (3) stock volatility consists of myopic and non-myopic volatility components, and both volatilities can be calculated empirically given the empirical dividend volatility, the rate of macroeconomic fluctuations and the empirical state price density; (4) both the optimal myopic and the optimal non-myopic portfolios can be calculated given the agent's desired wealth profile and the empirical quantities from item (3).

Furthermore, our results highlight general mechanisms behind the phenomena of market price of risk countercyclicality, non-myopic (excess) volatility and non-myopic optimal portfolios. In particular, we show that the non-myopic volatility is determined by the interplay between the market price of risk and the interest rate cyclicality, that the size and the sign of the interest rate hedge is determined by interest rate volatility and the size of agent's risk aversion, whereas the size and the sign of MPR hedge is determined by the interplay between prudence and risk tolerance.

We believe that our results are crucial for understanding continuous-time equilibrium CAPM and can be used both in theory for testing various versions of CAPM, and in practice for calculating optimal portfolios for long-run investors.

Appendix

A Proofs: Equilibrium Price Dynamics

Denote by $\mathcal{D}_t$ the Malliavin derivative operator. [43] The following is the main technical result of the paper.

Proposition A.1 *The drift and volatility of the stock price are given by*

$$\begin{aligned}
\mu_t^S &= r(D_t) + \sigma_t^S \Bigg(E_t^Q \left[\gamma^U(D_T)\,(\mathcal{D}_t D_T)/D_T\right] \\
&\quad - \int_t^T E_t^Q\left[r'(D_s)\,(\mathcal{D}_t D_s)\right]\,ds \Bigg), \\
\sigma_t^S &= \frac{1}{E_t[\xi_T\,D_T]} E_t[(1-\gamma_U(D_T)\xi_T \mathcal{D}_t D_T)] \\
&\quad - \frac{E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right] + E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t\xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]},
\end{aligned} \tag{23}$$

and the optimal portfolio of agent k is given by

$$\begin{aligned}
\pi_{k\,t} &= \frac{1}{\sigma_t^S}\frac{E_t\left[\mathcal{D}_t\,\xi_T\,(y_k\,\xi_T\,I_k'(y_k\,\xi_T) + I_k(y_k\,\xi_T))\right]}{E_t[\xi_T\,I_k(y_k\,\xi_T)]} \\
&\quad - \frac{E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right] + E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t\xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]},
\end{aligned} \tag{24}$$

where

$$\mathcal{D}_t D_T = D_t \sigma^D(D_t) e^{\delta_T - \delta_t} \tag{25}$$

with

$$\begin{aligned}
\delta_t &= \int_0^t [D_s(\mu^D)'(D_s) - 0.5(D_s(\sigma^D)'(D_s))^2 - D_s(\sigma^D)'(D_s)\sigma^D(D_s)]ds \\
&\quad + \int_0^t D_s(\sigma^D)'(D_s)dB_s
\end{aligned} \tag{26}$$

and

$$\mathcal{D}_t\xi_T = -\frac{1}{D_T}\gamma^U(D_T)\xi_T\mathcal{D}_t D_T\,.$$

[43] For an expedient introduction to Malliavin derivatives see Detemple, Garcia and Rindisbacher (2003).

Proof of Proposition A.1. By definition,

$$\xi_t \;=\; e^{-\int_0^t r(D_s)ds}\, M_t \;=\; e^{-\int_0^t r(D_s)ds}\, E_t[M_T] \;=\; E_t[e^{\int_t^T r(D_s)ds}\,\xi_T]\,.$$

The price S_t and the wealth of agent k satisfy

$$\log S_t \;=\; \int_0^t r(D_s)\,ds \,+\, \log E_t[\xi_T\, D_T] - \log E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]$$

and

$$\log W_{k\,t} \;=\; \int_0^t r(D_s)\,ds \;+\; \log E_t\left[\xi_T\, I_k(y_k\,\xi_T)\right] \;-\; \log E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]\,.$$

We get the volatility σ_t^S as the Malliavin derivative $\mathcal{D}_t \log S_t$ and we get $\sigma_t^S \pi_{k\,t}$ as the Malliavin derivative $\mathcal{D}_t \log W_{k\,t}$. Thus, we have

$$\pi_{k\,t} = \frac{\mathcal{D}_t \log W_{k\,t}}{\mathcal{D}_t \log S_t}\,. \tag{27}$$

We will now calculate the Malliavin derivatives. For process D, it is well known that the Malliavin derivative

$$Y_t := \mathcal{D}_t D_u\;, \quad u \ge t$$

satisfies the linear SDE

$$dY_u = (D_u(\mu^D)'(D_u) + \mu^D(D_u))\,Y_u\,du \;+\; (D_u(\sigma^D)'(D_u) + \sigma^D(D_u))\,Y_u\,dB_u\;,\; u \ge t$$

with

$$Y_t = D_t \sigma^D(D_t)$$

and (25) follows. Using this and (8), we can compute

$$\mathcal{D}_t \xi_T \;=\; U''(D_T)\mathcal{D}_t D_T\;. \tag{28}$$

Using the identity

$$\mathcal{D}_t E_t[X] = E_t[\mathcal{D}_t X]$$

we arrive at

$$\begin{aligned}\mathcal{D}_t \log W_{k\,t} \;=\;& \frac{1}{E_t[\xi_T I_k(y_k\,\xi_T)]} E_t\left[y_k \xi_T I_k'(y_k\,\xi_T)\mathcal{D}_t\xi_T + I_k(y_k\,\xi_T)\mathcal{D}_t\xi_T\right] \\ &- \frac{E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\,\xi_T\right] \;+\; E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t\xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]}\end{aligned} \tag{29}$$

and

$$\mathcal{D}_t \log S_t = \frac{1}{E_t[\xi_T D_T]} E_t[D_T \mathcal{D}_t \xi_T + \xi_T \mathcal{D}_t D_T]$$
$$- \frac{E_t\left[e^{\int_0^T r(D_s)ds} \left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right] + E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t \xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]}. \quad (30)$$

It remains to show the expression for the drift. By the martingale property, we have

$$\frac{dE_t[\xi_T D_T]}{E_t[\xi_T D_T]} = U_t\,dW_t\,, \quad \frac{dE_t[e^{\int_0^T r(D_s)ds}\,\xi_T]}{E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]} = V_t\,dW_t\,,$$

where, by the Clarke-Ocone formula and (28),

$$U_t = \frac{\mathcal{D}_t E_t[\xi_T D_T]}{E_t[\xi_T D_T]} = \frac{1}{E_t[\xi_T D_T]} E[\xi_T(1-\gamma_U(D_T))\mathcal{D}_t D_T]$$

and

$$V_t = \frac{\mathcal{D}_t E_t[e^{\int_0^T r(D_s)ds}\xi_T]}{E_t[e^{\int_0^T r(D_s)ds}\xi_T]}$$
$$= -\frac{E_t[e^{\int_0^T r(D_s)ds}\,(\gamma^U(D_T)\xi_T \mathcal{D}_t D_T)/D_T] - E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right]}{E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]}. \quad (31)$$

Applying Ito's formula, we get

$$d\log S_t = r(D_t)\,dt + d\log\frac{E_t[\xi_T D_T]}{E_t[\xi_T]} = \frac{1}{2}(2\,r(D_t) + V_t^2 - U_t^2)\,dt + (U_t - V_t)\,dW_t.$$

Therefore,

$$\mu_t^S = r(D_t) + \frac{1}{2}(V_t^2 - U_t^2 + (U_t - V_t)^2) = r(D_t) + V_t\,(V_t - U_t)$$

and thus, by (30),

$$\mu_t^S = r(D_t)$$
$$+ \frac{E_t[e^{\int_0^T r(D_s)ds}(\gamma^U(D_T)\xi_T \mathcal{D}_t D_T)/D_T] - E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right]}{E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]}$$
$$\times\ \mathcal{D}_t \log S_t$$
$$= r(D_t) + \sigma_t^S\left(E_t^Q\left[\gamma^U(D_T)\,(\mathcal{D}_t D_T)/D_T\right] - \int_t^T E_t^Q\left[r'(D_s)\,(\mathcal{D}_t D_s)\right]ds\right). \quad (32)$$

■

The following result allows us to rewrite the Malliavin derivative $\mathcal{D}_t D$ without involving stochastic integrals. It has also been proved by Detemple, Garcia and Rindisbacher (2003) in a slightly different form, but we present a derivation here for the reader's convenience.

Lemma A.2 *We have*

$$\mathcal{D}_t D_T \;=\; D_T\, \sigma^D(D_T)\, e^{-\int_t^T \rho_s\, ds}\,. \tag{33}$$

Proof. By Ito's formula,

$$\begin{aligned} \log(D_T\, \sigma^D(D_T)) \;-\; \log(D_t\, \sigma^D(D_t)) \;=\; & \int_t^T (\sigma^D(D_s) + D_s\,(\sigma^D)'(D_s))\, dB_s \\ & + \int_t^T \big((\sigma^D(D_s) \,+ D_s\,(\sigma^D)'(D_s))\,\sigma^D(D_s)^{-1}\,\mu^D(D_s)\big)\, ds \\ & + \frac{1}{2}\int_t^T \Big((2(\sigma^D)'(D_s) + D_s(\sigma^D)''(D_s))\,\sigma^D(D_s) \\ & - (\sigma^D(D_s) \,+ (D_s\,(\sigma^D)'(D_s))^2)\Big)\, ds. \end{aligned} \tag{34}$$

It remains to show that

$$\rho_s \;=\; c(D_s)$$

where

$$\begin{aligned} c(x) \;=\; & -x\,(\mu^D)'(x) \;+\; x\,(\sigma^D)'(x)\,\sigma^D(x)^{-1}\,\mu^D(x) \\ & +\; (\sigma^D)'(x)\,\sigma^D(x)\,x \;+\; 0.5\,(\sigma^D)''(x)\,x^2\,\sigma^D(x)\,. \end{aligned} \tag{35}$$

This claim can be verified by direct calculation. ■

Proof of Theorem 4.1. The proof follows directly by substituting (33) into (23). ■

We will need the following known

Lemma A.3 *For any one-dimensonal diffusion, the function*

$$G(t\,,\,x) \;=\; E[g(D_T)\,|\,D_t \;=\; x]$$

is monotone increasing (decreasing) in x for all $t \in [0,T]$ if and only if so does $g(x)$. Furthermore, if both $g(x)$ and $h(x)$ are increasing (or both decreasing), then

$$E_t[g(D_T)]\; E_t[h(D_T)] \;\le\; E_t[g(D_T)\, h(D_T)].$$

If both g,h are strictly increasing (or both strictly decreasing), then the inequality is also strict unless D_T is constant almost surely. If one function is increasing and the other is decreasing, then the inequality reverses.

Proof. See Herbst and Pitt (1991). ■

Lemma A.4 *Suppose that F and $G_1, \cdots, G_N$ are monotone increasing functions. Then, for any $N \in \mathbb{N}$ and any $\{t_1 \le \cdots \le t_N\} \subset [t\,,T]$,*

$$E[F(X_T)\,G_1(X_{t_1})\,\cdots\,G_N(X_{t_N})\,|\,X_t = x]$$

is monotone increasing in x and

$$E_t[F(X_T)\,G_1(X_{t_1})\,\cdots\,G_N(X_{t_N})] \;\ge\; E_t\,[F(X_T)]\;E_t\,[\,G_1(X_{t_1})\,\cdots\,G_N(X_{t_N})\,]\;.$$

Proof. The proof is by induction. For $N = 1$, we have

$$E_t[F(X_T)\,G_{t_1}(X_{t_1})] \;=\; E_t[\,E_{t_1}[F(X_T)]\,G_{t_1}(X_{t_1})].$$

By Lemma A.3, the function inside the expectation is increasing in X_{t_1} and another application of Lemma A.3 provides monotonicity of $E_t[F(X_T)\,G_{t_1}(X_{t_1})]$. Now, by Lemma A.3,

$$\begin{aligned} E_t[\,E_{t_1}[F(X_T)]\,G_1(X_{t_1})] \;&\ge\; E_t[\,E_{t_1}[F(X_T)]\,]\,E_t[G_{t_1}(X_{t_1})] \\ &=\; E_t[F(X_T)]\,E_t[G_{t_1}(X_{t_1})]\,, \end{aligned} \tag{36}$$

and we are done. Suppose now that the claim has been proved for N. Then,

$$\begin{aligned} &E_t[F(X_T)\,G_1(X_{t_1})\,\cdots\,G_N(X_{t_N})] \\ &\quad=\; E_t[\,G_1(X_{t_1})\,E_{t_1}[\,F(X_T)\,G_2(X_{t_2})\,\cdots\,G_N(X_{t_N})]\,] \\ &\quad\ge\; E_t[E_{t_1}[\,F(X_T)\,]\,E_{t_1}[G_1(X_{t_1})\,G_2(X_{t_2})\,\cdots\,G_N(X_{t_N})]\,] \end{aligned} \tag{37}$$

and the claim follows from Lemma A.3 and the induction hypothesis. ■

Lemma A.5 *Suppose that f and g are both increasing (decreasing). Then, the following is true*

(1) the function[44]

$$E\left[f(D_T)\,e^{\int_t^T g(D_s)\,ds}\,|\,D_t \;=\; x\right]$$

is also monotone increasing (decreasing);

(2) we have

$$\mathrm{Cov}_t\left(h(D_T)\,,\;f(D_T)\,e^{\int_t^T g(D_s)\,ds}\right) \;\ge\; 0$$

if h has the same direction of monotonicity as f and g, and the inequality reverses if h has the opposite direction of monotonicity.

[44] Item (1) of Lemma A.5 is contained in Mele (2007).

Proof of Lemma A.5. The claim follows from Lemma A.4, approximating the integral $\int_t^T g(D_s)\,ds$ by discrete integral sums. ■

Proof of Theorem 4.3. The proof follows directly from Proposition A.1 and (33). The fact that $\sigma_t^S > 0$ when r' is sufficiently small follows from (2), Lemma A.5 and

$$\sigma_t^S \;=\; \frac{\partial}{\partial D_t} S(t\,,\,D_t)\,\sigma^D(D_t)\,.$$

■

B Proofs: Optimal Portfolios

We start with the following auxiliary

Proposition B.1 *The optimal portfolio $\pi_{k\,t}$ is given by*

$$\sigma_t^S\,\pi_{k\,t} \;=\; \frac{E_t^Q\left[\lambda(t,T)\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}\,\left(\gamma_k^{-1}(W_{k\,T})\;-\;1\right)\right]}{W_{kt}} \;+\;\lambda_t\,. \tag{38}$$

Proof of Proposition B.1. The claim follows directly from Proposition A.1 and (33). ■

Proof of Proposition 6.12. By (38), we need to show that

$$\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T(\rho_s+r_s)ds}\,W_{kT}\,\left(1-\gamma_k^{-1}(W_{k\,T})\,\right)\right]}{E_t^Q[e^{-\int_t^T\,r_s\,ds}\,W_{kT}]}$$
$$\geq\;\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T\,(\rho_s+r_s)\,ds}\,W_{jT}\,\left(1-\gamma_j^{-1}(W_{j\,T})\right)\right]}{E_t^Q[e^{-\int_t^T\,r_s\,ds}\,W_{j\,T}]}\,. \tag{39}$$

We only prove case (1). Case (2) is analogous. Since, by assumption,

$$\inf\,(1\;-\;\gamma_{k\,T}^{-1})\;\geq\;\sup\,(1\;-\;\gamma_{j\,T}^{-1}),$$

it suffices to show that

$$\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T(\rho_s+r_s)ds}\,W_{kT}\right]}{E_t^Q[e^{-\int_t^T\,r_s\,ds}\,W_{kT}]}\;\geq\;\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T(\rho_s+r_s)\,ds}\,W_{j\,T}\right]}{E_t^Q[e^{-\int_t^T\,r_s\,ds}\,W_{jT}]}\,.$$

Introduce a new probability measure

$$dQ^k \;=\; \frac{e^{-\int_t^T\,r_s\,ds}\,W_{k\,T}}{E^Q[e^{-\int_t^T\,r_s\,ds}\,W_{k\,T}]}\,dQ$$

and let

$$f(x) \;=\; \frac{I_j(y_j x)}{I_k(y_k\, x)}.$$

and $z_i \;=\; I_i(\lambda_i\, x)\,,\; i\,\in\{j\,,\,k\}$. Then,

$$\begin{aligned} f'(x) \;&=\; \frac{I_j(y_j\,x)}{x\,I_k(y_k\,x)}\left(y_j\,x\,\frac{I_j'(y_j\,x)}{I_j(y_j\,x)} \;-\; y_k\,x\,\frac{I_k'(y_k\,x)}{I_k(y_k\,x)}\right) \\ &=\; \frac{I_j(y_j\,x)}{x\,I_k(y_k\,x)}\left(\frac{u_j'(z_1)}{z_1\,u_j''(z_1)} \;-\; \frac{u_k'(z_2)}{z_2\,u_k''(z_2)}\right) \\ &=\; \frac{z_1}{x\,z_2}\,(\gamma_k^{-1}(z_2) \;-\; \gamma_j^{-1}(z_1)) \;\le\; 0\,, \end{aligned} \tag{40}$$

that is, f is decreasing. Therefore, $f(U'(D_T))$ is increasing and, by Lemma A.5,

$$\begin{aligned} &\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T(\rho_s+r_s)\,ds}\,W_{j\,T}\right]}{E_t^Q[e^{-\int_t^T\,r_s\,ds}\,W_{jT}]} \\ &=\; \frac{E_t^{Q_k}\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T\,\rho_s\,ds}\,f(U'(D_T))\right]}{E_t^{Q_k}[f(U'(D_T))]} \\ &\le\; E_t^{Q_k}\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T\,\rho_s ds}\right] \;=\; \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T(\rho_s+r_s)\,ds}\,W_{kT}\right]}{E_t^Q[e^{-\int_t^T\,r_s\,ds}\,W_{kT}]}\,. \end{aligned} \tag{41}$$

■

Proof of Proposition 5.1. Let for simplicity $t\,=\,0$. By definition, the value function is

$$U_k(x) \;=\; E[\,u_k(I_k(\xi_T\,y_k))\,]$$

where $y_k = y_k(x)$ solves

$$x \;=\; E[\xi_T\,I_k(\xi_T\,y_k)].$$

Differentiating this identity, we get

$$y_k'(x) \;=\; E[\xi_T^2\,I_k'(\xi_T\,y_k)]^{-1}$$

and therefore

$$U_k'(x) \;=\; E\left[u_k'(I_k(\xi_T\,y_k))\,I_k'(\xi_T\,y_k)\,\xi_T\right]y_k'(x) \;=\; E\left[\,\xi_T\,y_k\,I_k'(\xi_T\,y_k)\,\xi_T\,\right]y_k'(x) \;=\; y_k(x)\,.$$

Consequently,

$$U_k''(x) \;=\; y_k'(x) \;=\; \frac{1}{E[\xi_T^2\,I_k'(\xi_T y_k)]}$$

and

$$\gamma_{k0}(x) \;=\; -\frac{x}{y_k\, E[\xi_T^2\, I_k'(\xi_T y_k)]} \;=\; -\frac{E[\xi_T\, I_k(\xi_T y_k)]}{E[\xi_T^2\, y_k\, I_k'(\xi_T y_k)]}\,.$$

Differentiating the identity

$$u_k'(I_k(x)) \;=\; x$$

we get

$$I_k'(x) \;=\; (u_k''(x))^{-1}$$

and therefore

$$y_k \xi_T\, I_k'(y_k \xi_T) \;=\; -\,\gamma_{kT}^{-1} W_{k\,T}.$$

■

Proof of Theorem 5.2. By Propositions B.1 and 5.1,

$$\begin{aligned}
\sigma_t^S\, \pi_{k\,t}^{\text{hedging}} \;&=\; \sigma_t^S\, \left(\pi_{k\,t} \;-\; \pi_{k\,t}^{\text{myopic}}\right)\\
&=\; \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\, e^{-\int_t^T r_s\,ds}\, W_{k\,T}\,\left(\gamma_{kT}^{-1} \;-\; 1\right)\right]}{E_t^Q[e^{-\int_t^T r_s\,ds}\, W_{kT}]} \;+\; \lambda_t\\
&-\; \lambda_t\, \frac{E_t^Q[e^{-\int_t^T r_s\,ds}\, \gamma_{kT}^{-1}\, W_{k\,T}]}{E_t^Q[e^{-\int_t^T r_s\,ds}\, W_{k\,T}]}\\
&=\; \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\, e^{-\int_t^T r_s\,ds}\, W_{k\,T}\,\left(\gamma_{kT}^{-1} \;-\; 1\right)\right]}{E_t^Q[e^{-\int_t^T r_s\,ds}\, W_{kT}]}\\
&-\; \lambda_t\, \frac{E_t^Q[e^{-\int_t^T r_s\,ds}\, (\gamma_{kT}^{-1} - 1)\, W_{kT}]}{E_t^Q[e^{-\int_t^T r_s\,ds}\, W_{kT}]}\\
&=\; \frac{1}{E_t^Q[W_{k\,T}]}\Big(E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\, e^{-\int_t^T r_s\,ds}\, W_{k\,T}\,\left(\gamma_{kT}^{-1} \;-\; 1\right)\right]\\
&-\; E_t^Q[\gamma^U(D_T)\,\sigma^D(t\,,\,T)]\, E_t^Q[(\gamma_{kT}^{-1} \;-\; 1)\, e^{-\int_t^T r_s\,ds}\, W_{k\,T}]\Big)\\
&+\; \frac{E_t^Q[e^{-\int_t^T r_s\,ds}\, (\gamma_{kT}^{-1} - 1)\, W_{kT}]}{E_t^Q[e^{-\int_t^T r_s\,ds}\, W_{kT}]}\, E_t^Q\left[\int_t^T \;\sigma^r(t,\tau)\, d\tau\right]\,,
\end{aligned} \tag{42}$$

which is what had to be proved. ■

Proofs of Proposition 6.5, Proposition 6.6 and Theorem 6.7. Propositions 6.5, 6.6 follows directly from Lemma A.5. Theorem 6.7 follows from Theorem 5.2 and Lemma A.5 since

$$f(x) \;=\; x \;-\; R_k(x)$$

is increasing if and only if

$$f'(x) \;=\; 1 \;+\; \frac{(u_k'')^2 \;-\; u_k'(x)\, u_k'''(x)}{(u_k''(x))^2} \;=\; \frac{1}{(u_k''(x))^2}\, (2 \;-\; P_k(x)\, R_k(x)) \;\geq\; 0\,,$$

and is decreasing otherwise. ■

Proof of Proposition 7.3. Item (1) follows from $\gamma^U \in [\gamma, \Gamma]$. Items (2) and (3) follows from

$$0 \;\le\; -(E_t^Q[D_T])^{-1}\,\mathrm{Cov}_t^Q(\gamma^U(D_T)\,,\,D_T) \\ = \;-\,\frac{E_t^Q[\gamma^U(D_T)D_T]}{E_t^Q[D_T]} \;+\; E_t^Q[\gamma^U(D_T)] \;\le\; -\gamma \,+\, \Gamma\,. \tag{43}$$

Similarly,

$$0 \;\le\; -(E_t^Q[W_{kT}])^{-1}\,\mathrm{Cov}_t^Q(\gamma^U(D_T)\,,\,W_{k\,T}) \\ = \;-\,\frac{E_t^Q[\gamma^U(D_T)W_{k\,T}]}{E_t^Q[W_{k\,T}]} \;+\; E_t^Q[\gamma^U(D_T)] \;\le\; -\gamma \,+\, \Gamma \tag{44}$$

and items (2)-(3) immediately yield the required inequality for $\gamma_k > 1$. The case $\gamma_k < 1$ follows from the directly verifiable identity

$$\pi_{k\,t} \;=\; \frac{\sigma\, e^{-b\,(T-t)}}{\sigma_t^S}\left(\gamma_k^{-1}\,\frac{E_t^Q[\gamma^U(D_T)\,W_{k\,T}]}{E_t^Q[W_{k\,T}]} \;-\; \mathrm{Cov}_t^Q(\gamma^U(D_T)\,,\,W_{k\,T})\right).$$

■

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