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The Asset Allocation Puzzle Is Still A Puzzle.

By

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Abstract

Intertemporal hedging activity has been advocated by the Finance literature as the solution to the Asset Allocation Puzzle. A standing assumption made in the dynamic asset allocation literature is that one or several bonds could perfectly hedge the interest rate risk in the Economy. Moreover, when the market prices of risk are allowed to be time varying, it is always assumed that they are driven by the same state variables as the spot rate and that bonds are enough to hedge the risk stemming from these market prices of risk. In a multi asset world, ceteris paribus, such assumptions bias the portfolio allocation of a non-myopic investor towards bonds which are then the only vehicle that should be used for the intertemporal hedging of the interest rate risk and market prices of risk: this leads to a solution of the puzzle.

Such assumptions have no empirical support. On the contrary it has been shown that bonds alone can not span all the uncertainty in the bond market. In addition, it is likely that the market prices of risk are driven by some stock specific source of risk and therefore bonds cannot hedge this particular source of risk. We solve explicitly in this paper for the optimal portfolio choice of a rational investor when bond markets are incomplete. It is shown that investors with a high level of risk aversion could invest more in stocks than in bonds. A calibration of the results show that the stock/bond mix is in general a U shaped function of the parameter of risk aversion in absolute value.

1. Introduction

In an influential contribution, Canner et al. (1997) raised the Asset Allocation Puzzle. The empirical evidence they documented is that popular advice on strategic asset allocation recommends a high ratio of bonds to stocks for conservative investors and a low ratio for less risk averse (aggressive) investors. This finding is in contrast to the basic mutual fund separation theorem of Finance Theory that is related to the standard Capital Asset Pricing Model (CAPM). According to the separation theorem, all risk averse investors should hold the same ratio of bonds to stocks. Since then, many solutions to this puzzle have been suggested. The most popular one has tried to show that there are contexts in which such a behavior is compatible with rational expected-utility maximization. The path followed by the Finance literature, which was actually suggested by Canner et al. (1997) among other possibilities, lies in the intertemporal dimension nature of the asset allocation issue that the CAPM ignores since it is a static model.

While it was common place to assume in the literature on asset allocation that interest rates are constant or deterministic, the puzzle motivated several studies that deal explicitly with stochastic interest rates and their impact on the optimal asset allocation of a rational expected utility maximizer. Part of these studies to be discussed below aim at solving the raised puzzle. For example, Bajeux-Besnainou et al. (2001, 2003) and Brennan and Xia (2000, 2002) explicitly solved the intertemporal portfolio allocation for an expected utility maximizer under different settings. Moreover these studies show that the portfolio allocation not only contains a speculative component, but also intertemporal hedging components that hedge against changes in the investment opportunity set until the investor's horizon is reached. While such a feature was known at least since Merton (1973), these studies explicitly related some of the hedging components to stochastic interest rates. They were able to show that risk averse investors invest more in bonds than in stocks as their risk aversion grows. This offered an elegant solution to the puzzle. Such an explanation for the puzzle, grounded on rigorous economic theory, is widely accepted among academics and practitioners as well.

Our purpose in this paper is to show that this enthusiasm may be premature and that the suggested explanation is highly contingent on the set of assumptions that have been made so far. These assumptions lack empirical support and drive the results. When they are relaxed, the puzzle turns out to be even more accentuated: investors with a high risk aversion invest more in stocks. It turns out that the première intuition of Canner et al. (1997) that intertemporal hedging activity is unlikely to provide a satisfactory answer to the puzzle was right.

One standing characteristic of dynamic asset allocation models that explicitly deal with interest rate risk is their assumption about the dynamics of the short term interest rate that makes the bonds the only instrument for the purpose of intertemporal hedging of interest rate risk. For example, Detemple et al. (2003)[1] and Liu (2001)[2] assume that the dynamics of the bond which is traded are perfectly correlated with the spot rate. Lioui and Poncet (2001) and Bajeux-Besnainou et al. (2003) also consider a multi-asset world where the short term interest rate is the only state variable which is perfectly correlated with a discount bond available for trading. In Brennan and Xia (2000, 2002), the correlation between any discount bond and the short term interest rate is imperfect. They however assume that two bonds are available for trading so that the combination of these two bonds with the (locally) riskless asset yields a perfect hedge of the interest rate risk.

While most papers assumed constant market prices of risk, some papers solved for the optimal strategic asset allocation of a power utility function investor when market prices

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[1] In their multi-asset setting, equations (37) and (38).
[2] In his section 6, equations (76) and (77).

of risk are stochastic. However, when it is the case, these market prices of risk are assumed to be driven by the same state variables as the spot rate and bonds are enough to hedge the risk stemming from these market prices of risk. For example, Lioui and Poncet (2001) assume that the market price of risk is an affine function of the spot rate and thus it is perfectly correlated with the spot rate which is perfectly correlated with traded bonds. Sangvinatsos and Wachter (2005) extended the analysis to a multifactor setting. They assume that the market prices of risk are a linear function of m state variables, and these m state variables also drive the spot rate in the Economy. Since m non redundant bonds are traded, these assets are enough to hedge the risk stemming from these market prices of risk.

The implication of these assumptions for strategic asset allocation are best understood using the new portfolio decomposition suggested recently by Lioui and Poncet (2001) and Detemple et al. (2003). Theses authors have shown that the optimal portfolio strategy could be decomposed into three components: i) a standard myopic component, ii) a pure interest rate risk related component which intertemporaly hedges the risk stemming from a bond maturing at the investor's horizon and iii) a third intertemporal hedging component related to the risk stemming from the joint distribution of the spot rate and market prices of risk[3]. Assuming that the spot rate and the market prices of risk could be hedged with bonds puts actually most if not the whole burden of intertemporal hedging on the bonds.

At the empirical level, it has been widely documented that bonds alone cannot span the uncertainty in the Bond Market (see Collin-Dufresne and Goldstein (2002) and the

[3] More precisely, this component is related to the martingale that allows a change of numeraire from the historical probability to a martingale mesure under a numeraire which is the discount bond maturing at the investor's horizon. For details, see Lioui and Poncet (2001).

references therein). Actually, even interest rate derivatives could not help complete the Bond Market as recently shown by Fan et al. (2003). These Authors showed that swaptions and their underlying market are well integrated and therefore cannot help spanning the uncertainty in the Bond Market. It turns out that assets from other segments of the financial market may be necessary to complete the bond market. In addition, it is likely that the market prices of risk are driven by some stock specific source of risk and therefore bonds cannot hedge this particular source of risk. Since factors affecting bond returns also often affect stock returns[4], stocks are good candidate for interest rate risk hedging.

The objective of this paper is to build a framework where bond markets alone are incomplete and therefore some stocks are useful for intertemporal hedging of the interest rate risk and/or the market prices of risk. We look for an explicit solution so as to investigate the behavior of the hedging components and the stock/bond mix of a non-myopic investor. The investor is allowed to trade cash, a discount bond and a stock index.

Our setting has two key features. The correlation between a discount bond and the short term interest rate is imperfect and endogenous rather than determined exogenously. This is achieved by allowing the market price of risk of the interest rate risk to be stochastic[5]. A major consequence of our setting is that, even in a one factor term structure of interest rates, any discount bond will be perfectly correlated with the short term interest rate *but* the correlation can be positive or negative and it is time dependent. This is because the

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[4] See Li (2002) and d'Addona and Kind (2005).

[5] This contrasts with the studies of Bajeux-Besnainou et al. (2001, 2003), Brennan and Xia (2002), Munk and Sorensen (2004), and Mougeot (2003) where the market price of risk related to the interest rate risk and is assumed to be constant. In a recent paper, Sangvinatsos and Wachter (2005) solved for the optimal portfolio choice of an expected utility maximizer when the market prices of risk are stochastic in addition to the spot rate. However, like in Brennan and Xia (2002), there are m state variables and m non-redundant discount bonds are traded that allow perfect intertemporal hedging related to the presence of these state variables.

discount bond is not only affected by the short term interest rate risk but also by stochastic market price of risk. The second feature of our setting is that the stock market and the bond market are allowed to be correlated. Therefore, the stock index demand not only contains a hedging component related to the market prices of risk, but also an interest rate related hedging component.

The implications for dynamic asset allocation of this more realistic setting are as follows:

i) Even in the particular case where the bond alone serves to hedge the interest rate risk, shorting or buying the bond depends on the investor's horizon and the time until the horizon is reached. In addition, the correlation between the bond and the short term interest rate changes over time, and therefore may require a dramatic rebalancing of the investor's portfolio. For example, suppose that the correlation between the discount bond and the spot rate is perfect and negative for a horizon (maturity) of five years. Then the investor buys bonds to hedge the interest rate risk steming from her hedging component. However, if the sign of the correlation changes at any point in time between today and the investor's horizon[6], she will have to change dramatically her portfolio allocation by selling bonds held in the portfolio and short selling some more bonds for hedging purposes. This will yield changes in the stock/bond mix.

ii) Another consequence of our setting is that the demand for bonds contains two hedging components, one for the interest rate risk and one for the market prices of risk. Since they may be of opposite sign depending on the dynamics of these variables, we

[6] Although bonds are in general negatively correlated with the spota rate, Mele (2003) recently showed that in some cases and for certain maturities we may have a positive relation between the spot rate and the bond prices. In our setting this may happens due to stochastic market prices of risk.

may end up with a bond demand that is only constituted of its speculative component since the hedging components cancel out!

iii) In the particular case where the bond alone serves to hedge the interest rate risk, stock index demand still contains two components: a speculative component and an intertemporal hedging component relative to the stochastic market prices of risk. As a consequence, even in this particular case, the relation between the stock/bond mix and the parameter of risk aversion turns out to be arbitrary and not decreasing as shown in the literature.

In this paper, we solve for a CRRA investor who has to allocate her wealth between cash, a stock index and a discount bond. Both the interest rate risk and the market prices of risk are stochastic and imperfectly correlated. As a consequence, the discount bond dynamics are imperfectly correlated with the dynamics of the spot rate. Our solution to the optimal portfolio choice is explicit and does not rely on any approximation or on Monte Carlo simulations. Therefore, a methodological contribution of the paper is to extend studies that provided explicit solutions when i) only the market price of risk is stochastic (Kim and Omberg (1996) and Wachter (2002))[7] or ii) the only source of risk (state variable) in the economy is the interest rate risk that also affects the market price of risk (Lioui and Poncet (2001)). Our explicit solution allows us to study in depth the properties of the intertemporal hedging components in general and to contrast them with the hedging components encountered in the literature. We conduct extensive simulations of our results.

The remainder of the paper is planned as follows. In the next section we set out our framework, in the third section we solve for the dynamic portfolio choice and in the

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[7] In Kim and Omberg (1996), the power utility agent has utility only from terminal wealth. In Wachter (2002), the power utility agent has utility from intermediate consumption and terminal wealth.

fourth section we simulate our model for a particular set of parameters. The fifth section concludes and some mathematical derivations are gathered in the appendix.

2. Preliminaries

We consider a multi asset economy, with stochastic interest rates and a stochastic market price of risk. We restrict ourselves to an affine setting for the dynamics of the short rate. More precisely, we assume that the short term interest rate (hereafter the spot rate) follows an Ornstein – Uhlenbeck process à la Vasicek:

$$dr(t) = \theta_r(\mu_r - r(t))dt + \sigma_{r1}dZ_1(t) + \sigma_{r2}dZ_2(t) \quad (1)$$

where θ_r, μ_r, σ_{r1} and σ_{r2} are constants. Z_1 and Z_2 are two independent standard Brownian motions. The uncertainty is formalized by the complete filtered space (Ω, F, P), where Ω is the state space, F is the σ-algebra representing measurable events, P is the actual (historical) probability and the filtration is the augmented filtration generated by the Brownian motion assumed to satisfy the usual conditions[8].

Our modeling choice (1) deserves the following comment. Since the (non-stochastic) weighted sum of the two Brownian increments could be written as a single Brownian increment term, (1) could have been written with only one driving Brownian motion $Z_r(t)$ where the volatility of the short term interest rate is $\sigma_r \equiv \sqrt{\sigma_{r1}^2 + \sigma_{r2}^2}$. Similarly, the stock index dynamics would have included a source of risk $Z_S(t)$. For our purpose in this paper, the two Brownian motions $Z_r(t)$ and $Z_S(t)$ will have to be correlated. Our choice to model the dynamics of the interest rate according to (1) stems from the fact we

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[8] The σ-algebra contains the events whose probability with respect to P is null. See Karatzas ans Shreve (1991), p. 89.

are using the martingale approach both for bond pricing and portfolio choice in the sequel. Since most of the theory (change of measure, relation between traded assets volatility matrix and market completeness, transformation of a dynamic optimization problem into a static one,…) has been built for a standard *d*-dimensional Brownian motion where each element is independent from the other, we would not have been able to apply these results straightforwardly had we used $Z_r(t)$ and $Z_S(t)$ since they are correlated. Hence our choice to model r(t) according to (1). Of course, we will allow for a correlation between the stock index and the short term interest in our setting.

We assume that the market prices of risk associated with the two sources of risk driving (1) follow mean reverting processes such that:

$$d\kappa_i(t) = \theta_{\kappa_i}\left(\mu_{\kappa_i} - \kappa_i(t)\right)dt + \sigma_{\kappa_i} dZ_i(t) \qquad i = 1,2 \qquad (2)$$

where θ_{κ_i}, μ_{κ_i} and σ_{κ_i} are constants.

(1) has been extensively used in the literature on the term structure of interest rates, but also in the dynamic asset allocation literature[9]. A key difference between our setting and the existing ones is that both the interest rate and the related market prices of risk are assumed to be stochastic while only the interest rate was assumed to be stochastic in most previous works (see for example Bajeux – Besnainou et al. (2001, 2003) and Munk and Sorensen (2004))[10]. This modeling procedure is flexible enough to highlight cases previously dealt within the literature as particular cases of our results.

Agents trade continuously in a perfect and arbitrage free complete market. The pricing kernel in such an economy, denoted Λ, is known to follow the dynamics:

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[9] For a review of affine processes and their use in model building in Finance, see the recent paper by Duffie et al. (2003).

[10] For a critical review of the role of the market price of risk in models of the term structure of interest rates, see the recent survey by Dai and Singleton (2003).

$$\frac{d\Lambda(t)}{\Lambda(t)} = -r(t)dt - \kappa_1(t)dZ_1(t) - \kappa_2(t)dZ_2(t) \tag{3}$$

The martingale approach usually defines a martingale measure, Q, which is equivalent to P, the historical probability, such that discounted prices are martingales when the riskless asset is the numéraire. The no arbitrage and market completeness assumptions imply that Q exists and is unique. In addition, if a (locally) riskless asset is traded and the market prices of risk are such that their parameters satisfy the Novikov condition[11]:

$$E_0\left[\exp\left\{\int_0^t\left(\kappa_1^2(s)+\kappa_2^2(s)ds\right)\right\}\right] < \infty, \quad \text{for } 0 < t < \infty,$$

this martingale measure is such that:

$$\left.\frac{dQ}{dP}\right|_t = \eta(t) = \exp\left\{-\int_0^t \kappa_1(s)dZ_1(s) - \int_0^t \kappa_2(s)dZ_2(s) - \frac{1}{2}\int_0^t\left(\kappa_1(s)^2 + \kappa_2(s)^2\right)ds\right\} \tag{4}$$

Then it is well known that:

$$\Lambda(t) = B(t)^{-1}\eta(t) \equiv h(t)^{-1} \tag{5}$$

where:

$$\frac{dB(t)}{B(t)} = r(t)dt \tag{6}$$

and h(t) is the optimal portfolio of a logarithmic investor in the Economy.

Since we want our market to be complete, we assume that, in addition to the riskless asset (6), two risky assets are traded, a stock index the dynamics of which is:

$$\frac{dS(t)}{S(t)} = \mu_S(t)dt + \sigma_{S1}\, dZ_1(t) + \sigma_{S2}\, dZ_2(t) \tag{7}$$

[11] See Karatzas and Shreve (1991), Proposition 5.12, page 198.

where σ_{S1} and σ_{S2} are constant, and a discount bond maturing at time T solving:

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)dt + \sigma_{P1}(t,T)dZ_1(t) + \sigma_{P2}(t,T)dZ_2(t) \tag{8}$$

We do not assume any particular function for the drift of (7), but only that this stochastic differential equation has a unique solution[12]. The specification (7) implies that the stock index price process is only driven by the sources of risk that drive the discount bond/short rate. This is a simplifying assumption that we make since it seems reasonable to assume that some specific source of risk affects the stock index price dynamics. Nevertheless, a new source of risk can always be added in the previous specification (7) as long as an additional asset is introduced to allow for market completeness. Such an asset could be an option on the stock index or even an index futures contract. The latter will not be perfectly correlated with the stock index due to stochastic interest rates. Therefore, it will allow for market completeness. As long as markets are complete, derivatives prices can be calculated explicitly in our setting. While the presence of a third asset will certainly enrich the analysis, its absence does not affect the main message of the paper.

Given that (1) and that the market is complete and arbitrage free, the parameters of the dynamics (8) can be computed explicitly. The solution to (8) is given in the following

<u>Proposition 1:</u>

The price of a discount bond maturing at time T is:

$$P(t,T) = e^{A(t,T)+A_r(t,T)r(t)+A_{\kappa_1}(t,T)\kappa_1(t)+A_{\kappa_2}(t,T)\kappa_2(t)} \tag{9}$$

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[12] The main condition is that $\mu_S(t)$ is locally Lipschitz-continuous in the space variable. See Karatzas and Shreve (1991), Theorem 2.5, page 287. One can similarly put conditions on the parameters of the spot rate and the market prices of risk for this to hold since, when there is no arbitrage, we must have that: $\mu_S(t) = r(t) + \sigma_{S1}\kappa_1(t) + \sigma_{S2}\kappa_2(t)$.

where A(t,T) is explicitly given in the Appendix and

$$A_r(t,T) = -\frac{1}{\theta_r}\left(1 - e^{-\theta_r(T-t)}\right)$$

$$A_{\kappa_i}(t,T) = \frac{\sigma_{ri}}{\theta_r}\frac{1}{\theta_r - \theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\left(\frac{\theta_r}{\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\left(1 - e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-t)}\right) - \left(1 - e^{-\theta_r(T-t)}\right)\right)$$

The price (9) is of a standard form given our affine market: it is an exponential function of the spot rate and the market price of risk. Applying Ito's lemma to (9) yields the volatility parameters of the discount bond:

$$\begin{aligned} \sigma_{P1}(t,T) &= A_r(t,T)\sigma_{r1} + A_{\kappa_1}(t,T)\sigma_{\kappa_1} \\ \sigma_{P2}(t,T) &= A_r(t,T)\sigma_{r2} + A_{\kappa_2}(t,T)\sigma_{\kappa_2} \end{aligned} \qquad (10)$$

When the market price of risk is stochastic, it follows from (10) that the correlation between the bond and the spot rate is imperfect. An interesting particular case is when the spot rate is driven by a single source of risk (one factor term structure). Suppose for example that $\sigma_{r2} = 0$. It follows that the correlation between the spot rate and the discount bond is perfect and equal to:

$$\frac{A_r(t,T)\sigma_{r1} + A_{\kappa_1}(t,T)\sigma_{\kappa_1}}{\left|A_r(t,T)\sigma_{r1} + A_{\kappa_1}(t,T)\sigma_{\kappa_1}\right|} \qquad (11)$$

and therefore this correlation may be positive or negative depending on the parameters of the dynamics of the spot rate and the market price of risk, as well as on the bond maturity. These features have important consequences for dynamic hedging purposes.

When following a dynamic self-financing trading strategy that combines the available assets, an investor's wealth dynamics is:

$$\frac{dV(t)}{V(t)} = \alpha(t)\frac{dS(t)}{S(t)} + \beta(t)\frac{dP(t,T)}{P(t,T)} + \left(1 - \alpha(t) - \beta(t)\right)\frac{dB(t)}{B(t)} \qquad (12)$$

where $\alpha(t)$ and $\beta(t)$ are the proportions of the investor's wealth invested in the stock index and the discount bond, respectively. Using (7) and (8) yields:

$$\frac{dV(t)}{V(t)} = (.)dt + \begin{bmatrix} \sigma_{S1} & \sigma_{P1}(t,T) \\ \sigma_{S2} & \sigma_{P2}(t,T) \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} \begin{bmatrix} dZ_1(t) \\ dZ_2(t) \end{bmatrix} \tag{13}$$

Assuming that markets are complete is equivalent to assuming that the volatility matrix appearing in (13) is invertible. For further use, we will denote this volatility matrix as follows:

$$\Sigma(t,T) \equiv \begin{bmatrix} \sigma_{S1} & \sigma_{P1}(t,T) \\ \sigma_{S2} & \sigma_{P2}(t,T) \end{bmatrix}$$

We are now fully equipped to conduct our analysis that consists in finding the optimal dynamic asset allocation for a CRRA investor in the previous setting.

3. Dynamic Asset Allocation

In this section, we characterize an investor's optimal wealth and compute the optimal dynamic asset allocation. For ease of exposition, we use, in the standard framework of a complete and frictionless market, the martingale approach to portfolio choice developed by Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989). In this, we follow many recent contributions like Lioui and Poncet (2001), Wachter (2002), Detemple et al. (2003) and Cvitanic et al. (2003).

Following Cox and Huang (1989), the investor's problem is solved in two steps. First, the investor's optimal terminal wealth is determined, and then the dynamic strategy which replicates it is found. The investor's program is thus transformed into the following static problem:

$$\begin{cases} \max_{V(\tau)} E_0\left[\frac{V(\tau)^{1-\gamma}}{1-\gamma}\right] \\ \text{s.t.} \qquad E_0[\Lambda(\tau)V(\tau)] = V(0) \end{cases} \tag{14}$$

We limit our discussion to investors with $\gamma > 1$, i.e., those who are more risk averse than the logarithmic investor. This is the case investigated by all the papers cited above and that deal with asset allocation in an intertemporal setting. Cox and Huang (1989) showed that the solution to this program is unique and such that:

$$V(\tau) = \lambda^{-\frac{1}{\gamma}} \Lambda(\tau)^{-\frac{1}{\gamma}} \tag{15}$$

where the Lagrange multiplier λ relative to the budget constraint is characterized by:

$$\lambda^{-\frac{1}{\gamma}} E_0\left[\Lambda(\tau)^{1-\frac{1}{\gamma}}\right] = V(0) \tag{16}$$

The optimal wealth at each date t can be computed according to:

$$\Lambda(t)V(t) = E_t[\Lambda(\tau)V(\tau)] \tag{17}$$

where E_t is the expectation under P conditional on the information available at time t. This implies:

$$V(t) = \lambda^{-\frac{1}{\gamma}} \Lambda(t)^{-\frac{1}{\gamma}} E_t\left[\left(\frac{\Lambda(\tau)}{\Lambda(t)}\right)^{-\frac{1-\gamma}{\gamma}}\right] \tag{18}$$

Explicit computations yield:

Proposition 2:

The investor's optimal wealth is:

$$V(t) = \lambda^{-\frac{1}{\gamma}} \Lambda(t)^{-\frac{1}{\gamma}} \Pi(t, \tau, r(t), \kappa_1(t), \kappa_2(t); \gamma) \tag{19}$$

where:

$$\Pi\left(t,\tau,r(t),\kappa_1(t),\kappa_2(t)\right)=\exp\Big\{A_0(t,\tau)+A_1(t,\tau)r(t)$$
$$+\left(A_2(t,\tau)+A_4(t,\tau)\kappa_1(t)\right)\kappa_1(t)+\left(A_3(t,\tau)+A_5(t,\tau)\kappa_2(t)\right)\kappa_2(t)\Big\} \tag{20}$$

and

$$A_1(t,\tau)=\left(1-\frac{1}{\gamma}\right)\frac{1}{\theta_r}\left(e^{-\theta r(\tau-t)}-1\right)$$

$$A_2(t,\tau)=-\frac{4\theta_{\kappa_1}\mu_{\kappa_1}\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)}{\sqrt{\varphi_4}}A_4(t,\tau)+\frac{\sigma_{r1}}{\theta_r}\left(1-\frac{1}{\gamma}\right)^2\left(\frac{1}{\theta_r}\left(1-e^{\theta_r(\tau-t)}\right)+(\tau-t)\right)$$
$$-2\sigma_{r1}\sigma_{\kappa_1}A_1(t,\tau)A_4(t,\tau)$$

$$A_3(t,\tau)=-\frac{4\theta_{\kappa_2}\mu_{\kappa_2}\left(1-e^{-\sqrt{\varphi_5}(\tau-t)}\right)}{\sqrt{\varphi_5}}A_5(t,\tau)+\frac{\sigma_{r2}}{\theta_r}\left(1-\frac{1}{\gamma}\right)^2\left(\frac{1}{\theta_r}\left(1-e^{\theta_r(\tau-t)}\right)+(\tau-t)\right)$$
$$-2\sigma_{r2}\sigma_{\kappa_2}A_1(t,\tau)A_5(t,\tau)$$

$$A_4(t,\tau)=\frac{2a_4\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)}{2\sqrt{\varphi_4}-\left(b_4+\sqrt{\varphi_4}\right)\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)}$$

$$A_5(t,\tau)=\frac{2a_5\left(1-e^{-\sqrt{\varphi_5}(\tau-t)}\right)}{2\sqrt{\varphi_5}-\left(b_5+\sqrt{\varphi_5}\right)\left(1-e^{-\sqrt{\varphi_5}(\tau-t)}\right)}$$

$$a_4=\frac{1}{2}\frac{1}{\gamma}\frac{\gamma-1}{\gamma} \qquad b_4=2\left(\theta_{\kappa_1}+\frac{\gamma-1}{\gamma}\sigma_{\kappa_1}\right) \qquad c_4=-2\sigma_{\kappa_1}{}^2$$

$$\varphi_4=b_4{}^2-4a_4c_4=4\left(\theta_{\kappa_1}+\frac{\gamma-1}{\gamma}\sigma_{\kappa_1}\right)^2+4\frac{1}{\gamma}\frac{\gamma-1}{\gamma}\sigma_{\kappa_1}{}^2>0$$

$$a_5=\frac{1}{2}\frac{1}{\gamma}\frac{\gamma-1}{\gamma} \qquad b_5=2\left(\theta_{\kappa_2}+\frac{\gamma-1}{\gamma}\sigma_{\kappa_2}\right) \qquad c_5=-2\sigma_{\kappa_2}{}^2$$

$$\varphi_5 = b_5^{\;2} - 4a_5c_5 = 4\left(\theta_{\kappa_2} + \frac{\gamma-1}{\gamma}\sigma_{\kappa_2}\right)^2 + 4\frac{1}{\gamma}\frac{\gamma-1}{\gamma}\sigma_{\kappa_2}^{\;2} > 0$$

The investor's optimal wealth reflects the properties of our affine setting. It is an exponential function of the spot rate and the market prices of risk. Indeed, given (15), the investor's optimal wealth can be interpreted as a contingent claim on the pricing kernel. Therefore, its price at any time t is a function of the relevant sources of risk for this contingent claim. A crucial difference between result (19) and the standard features of affine models is that the term in the exponential is not a linear but a nonlinear function of the underlying state variables. This makes the derivation more complicated as shown in the Appendix.

The derivation of the optimal dynamic strategy follows from applying Ito's lemma to (19). As such, we first set the following:

Proposition 3:

When $\gamma > 1$, the functions appearing in (20) have the following signs:

$$\begin{aligned} A_1(t,\tau) &\leq 0 \\ A_4(t,\tau) &\geq 0 \\ A_5(t,\tau) &\geq 0 \end{aligned} \tag{21}$$

while the sign of $A_2(t,\tau)$ and $A_3(t,\tau)$ is indeterminate.

Let us now turn to the optimal strategy to be followed by the investor. Applying Ito's lemma to (19) yields:

$$\begin{aligned} \frac{dV(t)}{V(t)} = (.)dt \; &+ \left(\frac{1}{\gamma}\kappa_1(t) + A_1(t,\tau)\sigma_{r1} + \left(A_2(t,\tau) + 2A_4(t,\tau)\kappa_1(t)\right)\sigma_{\kappa_1}\right)dZ_1(t) \\ &+ \left(\frac{1}{\gamma}\kappa_2(t) + A_1(t,\tau)\sigma_{r2} + \left(A_3(t,\tau) + 2A_5(t,\tau)\kappa_2(t)\right)\sigma_{\kappa_2}\right)dZ_2(t) \end{aligned} \tag{22}$$

Identifying with (13) yields:

Proposition 4:

The optimal dynamic strategy is such that:

$$\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix}=\frac{1}{\gamma}\Sigma(t,T)^{-1}\begin{bmatrix}\kappa_1(t)\\ \kappa_2(t)\end{bmatrix}+A_1(t,\tau)\Sigma(t,T)^{-1}\begin{bmatrix}\sigma_{r1}\\ \sigma_{r2}\end{bmatrix}$$
$$+\Sigma(t,T)^{-1}\begin{bmatrix}(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t))\sigma_{\kappa_1}\\ (A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t))\sigma_{\kappa_2}\end{bmatrix} \tag{23}$$

or, equivalently,

$$\alpha(t)=\frac{1}{\gamma}\frac{\kappa_1(t)\sigma_{P2}(t,T)-\kappa_2(t)\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$
$$+A_1(t,\tau)\frac{\sigma_{r1}\sigma_{P2}(t,T)-\sigma_{r2}\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)} \tag{24}$$
$$+\frac{(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t))\sigma_{\kappa_1}\sigma_{P2}(t,T)-(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t))\sigma_{\kappa_2}\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$

$$\beta(t)=\frac{1}{\gamma}\frac{\sigma_{S1}\kappa_2(t)-\sigma_{S2}\kappa_1(t)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$
$$+A_1(t,\tau)\frac{\sigma_{S1}\sigma_{r2}-\sigma_{S2}\sigma_{r1}}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)} \tag{25}$$
$$+\frac{(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t))\sigma_{\kappa_2}\sigma_{S1}-(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t))\sigma_{\kappa_1}\sigma_{S2}}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$

As it appears in (23) or (24) and (25), the optimal strategy involves three components in the demand for each risky asset: a mean-variance component (the first component in the RHS of (23), (24) and (25)) and two intertemporal hedging components which are related to interest rate risk (the second component in the RHS of (23), (24) and (25)) and the risk stemming from the market prices of risk.

To disentangle the intuition behind (23), we first focus on a particular case in which the discount bond, the short rate and the corresponding market price of risk are all perfectly correlated. In addition, the stock index is not correlated at all with the spot rate and/or its corresponding market price of risk. This is to insure that the whole burden of interest rate risk hedging falls on the discount bond. It corresponds to the case $\sigma_{S1} = \sigma_{r2} = 0$ [13]. In such a case, (24) and (25) become:

$$\hat{\alpha}(t) = \frac{1}{\gamma}\frac{\kappa_2(t)}{\sigma_{S2}} + \left(A_3(t,\tau) + 2A_5(t,\tau)\kappa_2(t)\right)\frac{\sigma_{\kappa_2}}{\sigma_{S2}} \tag{26}$$

$$\hat{\beta}(t) = \frac{1}{\gamma}\frac{\kappa_1(t)}{\sigma_{P1}(t,T)} + A_1(t,\tau)\frac{\sigma_{r1}}{\sigma_{P1}(t,T)} + \left(A_2(t,\tau) + 2A_4(t,\tau)\kappa_1(t)\right)\frac{\sigma_{\kappa_1}}{\sigma_{P1}(t,T)} \tag{27}$$

The demand for the stock index has two components, a speculative component (the first component on the RHS of (26)) and a hedging component that is related to the market price of risk associated with the stock index risk. Naturally, this hedging component is also a perfect hedge component since the stock index is perfectly correlated with it. As to the bond demand, it has three components. Like in the stock index, there are one speculative component and two intertemporal hedging components. The first hedging component (the second term in the RHS of (27)) is a perfect hedge against the interest rate risk. The second intertemporal hedging component (the last component in the RHS of (27)) hedges against the risk stemming from the market price of risk of the interest rate risk. These hedging components have features very similar to those derived by Detemple et al. (2003) for example. The interest rate hedging component is very stable; it is actually deterministic in our setting. However, a notable difference with Detemple et al. (2003) is that its *sign* can be positive or negative and may change as time passes. The same holds for the market price of risk hedging component in (27). Since it depends

[13] It also correspond to the case $\sigma_{S2} = \sigma_{r1} = 0$.

on the market price of risk level, it is stochastic, although its sign is indeterminate and could change as time passes.

What does hedging activity imply for stock index and bond demand ? Actually, and perhaps fortunately, the results are not clear cut. As to the stock index, a negative market price of risk would always imply short selling the stock index, both for speculative and hedging components. However, if the market prices of risk were positive, this would imply buying stocks for speculation but does not necessarily imply buying stocks for hedging their corresponding market prices of risk. To see this, note that given (21), $A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t)$ may be positive or negative. As to the bond demand, hedging components may theoretically have any impact on it. It could even be reduced to the demand of a pure myopic investor. This uncertainty on the impact of the hedging components on the demand for the stock index and for the bond obviously translates into the stock/bond mix.

Let us now turn to the general case in (24) and (25). Its main feature is that the interest rate related hedging components are deterministic, both in the stock index demand and the bond demand. Both assets are used to hedge interest rate risk. Nevertheless the sign of these hedging components cannot be determined. Another interesting feature of (24) and (25) is that the market price of risk hedging component can also be decomposed into a deterministic component and a stochastic component. To see this, note that the third component on the RHS of (23) can be written as:

$$\Sigma(t,T)^{-1}\begin{bmatrix}(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t))\sigma_{\kappa_1}\\(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t))\sigma_{\kappa_2}\end{bmatrix}=\Sigma(t,T)^{-1}\begin{bmatrix}A_2(t,\tau)\sigma_{\kappa_1}\\A_3(t,\tau)\sigma_{\kappa_2}\end{bmatrix}+2\Sigma(t,T)^{-1}\begin{bmatrix}A_4(t,\tau)\kappa_1(t)\sigma_{\kappa_1}\\A_5(t,\tau)\kappa_2(t)\sigma_{\kappa_2}\end{bmatrix} \quad (28)$$

The last component on the RHS of (28) depends upon the level of the market price of risk and thus inherits their variability in our setting.

Finally, let us write (24) and (25) in the following way:

$$\alpha(t)=\hat{\alpha}(t)+\frac{1}{\gamma}\left(\frac{\sigma_{S2}\kappa_1(t)-\sigma_{S1}\kappa_2(t)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}\frac{\sigma_{P2}(t,T)}{\sigma_{S2}}\right)$$
$$+A_1(t,\tau)\frac{\sigma_{r1}\sigma_{P2}(t,T)-\sigma_{r2}\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)} \tag{29}$$
$$+\frac{(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t))\sigma_{S2}\sigma_{\kappa_1}-(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t))\sigma_{S1}\sigma_{\kappa_2}}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}\frac{\sigma_{P2}(t,T)}{\sigma_{S2}}$$

$$\beta(t)=\hat{\beta}(t)+\frac{1}{\gamma}\left(\frac{\sigma_{P1}(t,T)\kappa_2(t)-\sigma_{P2}(t,T)\kappa_1(t)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}\frac{\sigma_{S1}}{\sigma_{P1}(t,T)}\right)$$
$$-A_1(t,\tau)\frac{\sigma_{r1}\sigma_{P2}(t,T)-\sigma_{r2}\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}\frac{\sigma_{S1}}{\sigma_{P1}(t,T)} \tag{30}$$
$$+\frac{(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t))\sigma_{\kappa_2}\sigma_{P1}(t,T)-(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t))\sigma_{\kappa_1}\sigma_{P2}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}\frac{\sigma_{S1}}{\sigma_{P1}(t,T)}$$

Equations (29) and (30) clearly show the bias introduced by assuming perfect correlation between the spot rate and the bond. In general, any profile for the hedging components is possible. This is also true for the stock/bond mix.

In the general analysis above we have assumed that the discount bond is a long-lived asset and thus that its maturity is greater or equal to the investor's horizon. An interesting issue is to what extent our results would be affected had this maturity been equal to the investor's horizon. More generally, what is the impact of the investor's horizon on the structure of the optimal risky assets demands? It is now well documented that the optimal asset allocation of a rational investor will always include trading directly or synthetically in a discount bond maturing at the investor's horizon. To exhibit the importance of the investor's horizon, the alternative optimal portfolio strategy decomposition suggested by Lioui and Poncet (2001) or Detemple et al. (2003) can be used. Standard decomposition à la Merton decomposes the portfolio strategy into a speculative (mean – variance efficient) component and intertemporal hedging components related to the state variables. Alternatively, Lioui and Poncet (2001) or

Detemple et al. (2003) showed that the optimal strategy could be decomposed into a speculative (mean – variance efficient) component, an intertemporal hedging component related to the interest rate risk of a discount bond *maturing at the investor's horizon*[14], and another intertemporal hedging component related to the stochastic market prices of risk. In our specific setting, the optimal strategy in the following way can be written:

Proposition 5:

The optimal asset allocation is such that:

$$\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix}=\frac{1}{\gamma}\Sigma(t,T)^{-1}\begin{bmatrix}\kappa_1(t)\\ \kappa_2(t)\end{bmatrix}+\left(1-\frac{1}{\gamma}\right)\Sigma(t,T)^{-1}\begin{bmatrix}\sigma_{P1}(t,\tau)\\ \sigma_{P2}(t,\tau)\end{bmatrix}$$
$$+\Sigma(t,T)^{-1}\begin{bmatrix}\left(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_1}(t,\tau)\right)\sigma_{\kappa_1}\\ \left(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_2}(t,\tau)\right)\sigma_{\kappa_2}\end{bmatrix} \qquad (31)$$

The second component in (31) (the second term on the RHS) is now clearly related to the interest rate risk of a discount bond maturing at the investor's horizon. Moreover, both the stock and the bond are used to replicate this bond. (31) is usefdul because it shows the difference between a static and an intertermporal setting in explaining the Puzzle. In an intertermporal setting, what a model needs to explain the Puzzle is intertemporal hedging components. To see this, note that without state variables (i.e., when interest rate and market prices of risk are not stochastic) (31) becomes:

$$\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix}=\frac{1}{\gamma}\Sigma(t,T)^{-1}\begin{bmatrix}\kappa_1(t)\\ \kappa_2(t)\end{bmatrix} \quad \Leftrightarrow \quad \frac{\alpha(t)}{\beta(t)}=\frac{\kappa_1(t)\sigma_{P2}(t,T)-\kappa_2(t)\sigma_{P1}(t,T)}{\sigma_{S1}\kappa_2(t)-\sigma_{S2}\kappa_1(t)} \qquad (32)$$

[14] This is a generalization of Sorensen (1999) original result that was shown to hold only under a specific dynamics of the spot rate and constant market price(s) of risk.

Therefore, the stock/bond ratio is independent from the investor's parameter of risk aversion and therefore the intertemporal setting is unable to explain the Puzzle. It must be noted that whether the maturity of the traded bond T is greater or equal to the investor's horizon does not affect the conclusion. Now if we assume that only the spot rate is stochastic while the market prices of risk are deterministic, this introduces intertemporal hedging components so that (31) becomes:

$$\begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} = \frac{1}{\gamma}\Sigma(t,T)^{-1}\begin{bmatrix} \kappa_1(t) \\ \kappa_2(t) \end{bmatrix} + \left(1-\frac{1}{\gamma}\right)\Sigma(t,T)^{-1}\begin{bmatrix} \sigma_{P1}(t,\tau) \\ \sigma_{P2}(t,\tau) \end{bmatrix} \tag{33}$$

(33) shows that the relationship between the stock/bond ratio and the parameter of risk aversion is not clear. Therefore, (33) does not provide in general a solution to the Puzzle although this ratio depends *now* upon the parameter of risk aversion. Yet (33) allows us to understand why previous authors were able to explain the Puzzle in an intertemporal setting. Another advantage of (33) (and thus of our setting) is that it allows us to put the whole burden of intertemporal interest rate risk hedging on the bond without assuming that the bond and the stock index have correlation 0 or that the bond and the spot rate are perfectly correlated. Given (31), the same objective can be achieved (i.e., when the intertemporal interest rate risk can be hedged with the bond only) by assuming that the traded discount bond maturity is equal to the investor's horizon. In this case, the second term on the RHS of (33) becomes:

$$\left(1-\frac{1}{\gamma}\right)\Sigma(t,\tau)^{-1}\begin{bmatrix} \sigma_{P1}(t,\tau) \\ \sigma_{P2}(t,\tau) \end{bmatrix} = \left(1-\frac{1}{\gamma}\right)\begin{bmatrix} \sigma_{S1} & \sigma_{P1}(t,\tau) \\ \sigma_{S2} & \sigma_{P2}(t,\tau) \end{bmatrix}^{-1}\begin{bmatrix} \sigma_{P1}(t,\tau) \\ \sigma_{P2}(t,\tau) \end{bmatrix} = \left(1-\frac{1}{\gamma}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{34}$$

and therefore:

$$\begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} = \frac{1}{\gamma}\Sigma(t,\tau)^{-1}\begin{bmatrix} \kappa_1(t) \\ \kappa_2(t) \end{bmatrix} + \left(1-\frac{1}{\gamma}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{35}$$

The presence of a discount bond maturing at the investor's maturity thus put all the burden of *intertemporal interest rate risk hedging* on this bond. It must be noted that (25) and (26) above have been obtained under the assumption that the correlation between the stock index and the spot interest rate/traded discount bond is 0. Given (35), the stock/bond ratio is thus:

$$\frac{\alpha(t)}{\beta(t)}=\frac{\dfrac{\kappa_1(t)\sigma_{P2}(t,\tau)-\kappa_2(t)\sigma_{P1}(t,\tau)}{\sigma_{S1}\sigma_{P2}(t,\tau)-\sigma_{S2}\sigma_{P1}(t,\tau)}}{\dfrac{\sigma_{S1}\kappa_2(t)-\sigma_{S2}\kappa_1(t)}{\sigma_{S1}\sigma_{P2}(t,\tau)-\sigma_{S2}\sigma_{P1}(t,\tau)}+\gamma-1} \tag{36}$$

Therefore, the stock/bond ratio is a decreasing function of the investor's parameter of risk aversion and an elegant explanation for the Puzzle is obtained. This is exactly the way Bajeux – Besnainou et al. (2001) were able to explain the Puzzle. It must be reminded that in a static setting, as clearly explained by Bajeux – Besnainou et al. (2001), the Puzzle cannot be explained if a discount bond maturing at the investor's horizon is traded while in an intertemporal setting the Puzzle may still be explained. The reason is the presence of intertemporal hedging components in an intertemporal setting and their absence in a static setting.

Finally, when market prices of risk are stochastic, a simple decreasing relation between the stock/bond ratio and the parameter of risk aversion as the one in (36) fails to hold. Therefore, even with intertemporal hedging, no satisfactory explanation to the Puzzle can be provided when the stock market and the bond market are correlated.

We seek further insights from (24) and (25) through a simulation of our model.

4. Simulations

In this section, we report results obtained from simulations of the preceding findings. The basic set of parameters has been chosen as follows:

$$
\begin{array}{llll}
r(0)=0.04 & \kappa_1(0)=0.3 & \kappa_2(0)=0.3 & \sigma_{S1}=0.20 \\
\theta_r=0.08 & \theta_{\kappa 1}=0.09 & \theta_{\kappa 2}=0.07 & \sigma_{S2}=0.18 \\
\sigma_{r1}=0.04 & \mu_{\kappa 1}=0.06 & \mu_{\kappa 2}=0.065 & \\
\sigma_{r2}=0.035 & \sigma_{\kappa 1}=0.15 & \sigma_{\kappa 2}=0.16 &
\end{array}
\qquad (37)
$$

The choice of parameters is based on parameters used in previous studies like Detemple et al. (2003) and Cvitanic et al. (2003). We assumed that the traded bond has a maturity of 15 years and the results are qualitatively not affected by this choice. As to the investor's parameters, we use a horizon of 5 years and a coefficient of risk aversion of 5 when such parameters are kept constant for the analysis.

For comparison, we reported the components of the stock and bond demand, as well as the stock/bond mix in two cases. In the first case, the discount bond is the only instrument for interest rate risk hedging (the stock index and the bond return are not correlated); in the second case, the bond is not the only instrument for risk hedging and needs to be combined with the stock index. The former case has been considered in the literature when the discount bond is constrained to be negatively perfectly correlated with the spot rate. In our setting, the correlation is constrained to be perfect but its sign is endogenous and time dependent. MV, IR and MPR stand for the mean-variance, the interest rate hedging and the market prices of risk hedging components, respectively. In order to attenuate the impact of the particular choice of the parameters (37), we report the relative part of each component in the total demand for the risky asset. For example, the MV-S is the ratio of the mean variance component in the stock index demand over the total stock index demand. The stock/bond mix is simply the ratio of the total stock index demand to the total bond demand.

Our first results pertain to the behavior of the stock index demand and discount bond demand as functions of the investor's horizon.

Figure 1

In figure **1a** we report the risky assets demand components when the bond is the only instrument used to hedge the interest rate risk. We report the components of the demand for the stock index, the components of the demand for the discount bond and the stock demand to bond demand ratio. It must be noted that the dependence of the mean – variance component upon the horizon stems from the fact that we reported the ratio of this component to the total demand. Since the latter is horizon dependent, this ratio will also be horizon dependent. In absolute terms, the mean – variance component is horizon *independent*, both in the stock index and the bond demands.

The stock index demand has two components: a speculative component (MV-S) and a market price of risk hedging component (MPR-S). The interest rate risk hedging component (IR-S) is zero by construction in Figure 1a. The speculative component is a decreasing function of the horizon. Thus, the market price of risk hedging component is an increasing function of the investor's horizon. The latter component is predominant in the demand for the stock index after a horizon of around 3.5 years. The discount bond demand has three components: a speculative component and two intertemporal hedging components. A key feature of the latter hedging demands is that they are always of opposite sign. Interest rate risk hedging always commands buying bonds while the market price of risk hedging component always requires short selling of bonds. As a consequence, the total bond demand will be positive or negative depending on which component will be predominant. Since optimal asset allocation always requires buying the stock index, the sign of the stock/bond ratio depends upon the bond demand sign.

When the stock index is involved in interest rate hedging activity, the features of risky asset demands change considerably as shown in Figure **1b**. The additional interest rate risk hedging component in the stock index demand is always negative and thus requires shorting the stock index. However, since the speculative component and the market prices of risk related component are still positive in this case, the total stock index demand remains positive. In the bond demand, the more important change is that until an horizon of 9 years, all the bond demand components are positive and therefore total bond demand is positive. As a consequence, the stock/bond ratio is also always positive and increasing in the horizon. After a horizon of 9 years, the market prices of risk related components become negative and predominant in the bond demand. The bond demand thus becomes negative. As a consequence, the stock/bond ratio becomes negative and decreasing in absolute value.

While the optimal asset allocation always implies long positions in the stock index, the bond position depends on the correlation between the stock index and the bond. As a consequence, the behavior of the stock/bond mix will have an arbitrary behavior in general.

Figure 2

In Figure 2, we report the behavior of the components of the risky asset demand as functions of the investor's coefficient of risk aversion. In Figure 2a, the discount bond is the only instrument for interest rate hedging, while in Figure 2b, both the discount bond and the stock index are used for interest rate risk hedging.

As expected, the more risk averse the investor, the more predominant are the intertemporal hedging components. The stock/bond ratio turns out to be an inverse U-shaped function of the parameter of risk aversion in absolute terms when the bond and the stock index are not correlated (Figure **2a**). However, when the stock market and the bond market are correlated, the stock/bond ratio turns out to be a U-shaped function of the parameter of risk aversion in absolute terms (Figure **2b**). In the latter case and for low levels of risk aversion (up to 6), it is a decreasing function of the risk aversion; for levels greater than 6, it becomes an increasing function in absolute terms. It must be noted that in related literature, the ratio was found to be a monotonic decreasing function of the parameter of risk aversion.

5. Concluding Remarks

We have shown that assuming that interest rate risk can be perfectly hedged using a traded bond has important consequences as to the behavior of the intertemporal hedging components of a rational investor. In particular, any profile of the stock/bond mix could be reached and therefore intertemporal hedging activity seems unlikely to provide a convincing answer to the Asset Allocation Puzzle. Relative to a static setting, an intertemporal setting allows the stock/bond mix to be a function of investor's risk aversion parameter. Yet, only in particular cases will this relation be a decreasing relation that may rationalize empirical evidence.

The paper focuses on the "Asset Allocation Puzzle" and the notion that intertemporal hedging does not necessarily solve this puzzle. Several papers showed that hedging demands can have complex behavior relative to investment horizon and/or risk aversion when multiple assets and/or multiple state variables come into play[15]. The present paper enhances this conclusion by considering a setting with imperfect correlation between the interest rate and possible hedging instruments[16].

A possible direction for solving such a puzzle has been provided by Shalit and Yitzhaki (2003). They showed that the right way to look at the puzzle is from the point of view of the investment advisor and not of a particular investor. Therefore, what Shalit and Yitzhaki (2003) suggest is simply to check whether the suggested portfolio by the investment advisor is efficient at least for one rational risk averse investor. Their answer is positive when grounded on the stochastic dominance analysis. These findings are

[15] See Lynch and Tan (2005) and the References therein.

promising and pave the way for challenging questions for future research.

[16] I would like to thank a Referee for this suggested perspective on the paper contribution.

Appendix

Proof of Proposition 1:

The price at time t of a discount bond maturing at time T is:

$$\frac{P(t,T)}{B(t)} = E_t^Q\left[\frac{1}{B(T)}\right] \tag{A1}$$

where $E_t^Q[\ ]$ denotes expectation under the martingale measure Q and thus:

$$P(t,T) = E_t^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\right] \tag{A2}$$

Now under Q, the dynamics (1) become:

$$dr(t) = (\theta_r(\mu_r - r(t)) - \sigma_{r1}\kappa_1(t) - \sigma_{r2}\kappa_2(t))dt + \sigma_{r1}d\hat{Z}_1(t) + \sigma_{r2}d\hat{Z}_2(t) \tag{A3}$$

and

$$d\kappa_i(t) = \theta_{\kappa_i}(\mu_{\kappa_i} - (1+\sigma_{\kappa_i})\kappa_i(t))dt + \sigma_{\kappa_i}d\hat{Z}_i(t) \tag{A4}$$

where $\hat{Z}_i$ is a standard Brownian motion under Q. It follows that:

$$\begin{aligned}\int_t^T r(s)ds = \mu_r(T-t) - \frac{1}{\theta_r}(r(T)-r(t)) - \frac{\sigma_{r1}}{\theta_r}\int_t^T \kappa_1(s)ds - \frac{\sigma_{r2}}{\theta_r}\int_t^T \kappa_2(s)ds \\ + \frac{\sigma_{r1}}{\theta_r}\int_t^T d\hat{Z}_1(s) + \frac{\sigma_{r2}}{\theta_r}\int_t^T d\hat{Z}_2(s)\end{aligned} \tag{A5}$$

From (A3), it follows that:

$$\begin{aligned}r(T) = e^{-\theta_r(T-t)}r(t) + \theta_r\mu_r\int_t^T e^{-\theta_r(T-s)}ds - \sigma_{r1}\int_t^T e^{-\theta_r(T-s)}\kappa_1(s)ds - \sigma_{r2}\int_t^T e^{-\theta_r(T-s)}\kappa_2(s)ds \\ + \sigma_{r1}\int_t^T e^{-\theta_r(T-s)}d\hat{Z}_1(s) + \sigma_{r2}\int_t^T e^{-\theta_r(T-s)}d\hat{Z}_2(s)\end{aligned} \tag{A6}$$

Substituting into (A5) yields:

$$\begin{aligned}\int_t^T r(s)ds = \frac{1}{\theta_r}\left(1-e^{-\theta_r(T-t)}\right)r(t) + \mu_r\left((T-t) - \int_t^T e^{-\theta_r(T-s)}ds\right) \\ - \frac{\sigma_{r1}}{\theta_r}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)\kappa_1(s)ds - \frac{\sigma_{r2}}{\theta_r}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)\kappa_2(s)ds \\ + \frac{\sigma_{r1}}{\theta_r}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)d\hat{Z}_1(s) + \frac{\sigma_{r2}}{\theta_r}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)d\hat{Z}_2(s)\end{aligned} \tag{A7}$$

From (2), it follows that:

$$\int_t^T \left(1-e^{-\theta_r(T-s)}\right)\kappa_i(s)ds = -\left(\frac{1-e^{-\theta_r(T-t)}}{\theta_r-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\right)\kappa_i(t) - \frac{\theta_{\kappa_i}\mu_{\kappa_i}}{\theta_r-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\int_t^T \left(1-e^{-\theta_r(T-s)}\right)ds$$
$$+\frac{\theta_r}{\theta_r-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\int_t^T \kappa_i(s)ds - \frac{\sigma_{\kappa_i}}{\theta_r-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\int_t^T \left(1-e^{-\theta_r(T-s)}\right)d\hat{Z}_i(s) \quad \text{(A8)}$$

and from (2) it follows that:

$$\int_t^T \kappa_i(s)ds = \frac{\mu_{\kappa_i}}{\left(1+\sigma_{\kappa_i}\right)}(T-t) - \frac{1}{\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\left(\kappa_i(T)-\kappa_i(t)\right) + \frac{\sigma_{\kappa_i}}{\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\int_t^T d\hat{Z}_i(s) \quad \text{(A9)}$$

Since:

$$\kappa_i(T) = e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-t)}\kappa_i(t) + \theta_{\kappa_i}\mu_{\kappa_i}\int_t^T e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-s)}ds + \sigma_{\kappa_i}\int_t^T e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-s)}d\hat{Z}_i(s) \quad \text{(A10)}$$

Substituting into (A9) yields:

$$\int_t^T \kappa_i(s)ds = \frac{1-e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-t)}}{\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\kappa_i(t) + \frac{\mu_{\kappa_i}}{\left(1+\sigma_{\kappa_i}\right)}\left((T-t) - \int_t^T e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-s)}ds\right)$$
$$+\frac{\sigma_{\kappa_i}}{\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\int_t^T \left(1-e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-s)}\right)d\hat{Z}_i(s) \quad \text{(A11)}$$

and thus substituting into (A8) yields:

$$\int_t^T \left(1-e^{-\theta_r(T-s)}\right)\kappa_i(s)ds = \frac{1}{\theta_r-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\left(\frac{\theta_r}{\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\left(1-e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-t)}\right) - \left(1-e^{-\theta_r(T-t)}\right)\right)\kappa_i(t)$$
$$+\frac{\mu_{\kappa_i}}{\theta_r-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\left(\frac{\theta_r}{\left(1+\sigma_{\kappa_i}\right)}\left((T-t) - \int_t^T e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-s)}ds\right) - \theta_{\kappa_i}\int_t^T \left(1-e^{-\theta_r(T-s)}\right)ds\right) \quad \text{(A12)}$$
$$+\frac{\sigma_{\kappa_i}}{\theta_r-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\int_t^T \left(\frac{\theta_r}{\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)}\left(1-e^{-\theta_{\kappa_i}\left(1+\sigma_{\kappa_i}\right)(T-s)}\right) - \left(1-e^{-\theta_r(T-s)}\right)\right)d\hat{Z}_i(s)$$

Substituting into (A7) yields:

$$-\int_t^T r(s)ds = a_r(t,T) + A_r(t,T)r(t) + A_{r,\kappa_1}(t,T)\kappa_1(t) + A_{r,\kappa_2}(t,T)\kappa_2(t)$$
$$+\int_t^T \varepsilon_{r1}(t,T)d\hat{Z}_1(s) + \int_t^T \varepsilon_{r2}(t,T)d\hat{Z}_2(s) \quad \text{(A13)}$$

where:

$$A_r(t,T) = -\frac{1}{\theta_r}\left(1-e^{-\theta_r(T-t)}\right)$$

$$a_r(t,T) = -\mu_r\left((T-t) - \int_t^T e^{-\theta_r(T-s)}ds\right)$$

$$+\frac{\sigma_{r1}}{\theta_r}\frac{\mu_{\kappa_1}}{\theta_r - \theta_{\kappa_1}(1+\sigma_{\kappa_1})}\left(\frac{\theta_r}{(1+\sigma_{\kappa_1})}\left((T-t) - \int_t^T e^{-\theta_{\kappa_1}(1+\sigma_{\kappa_1})(T-s)}ds\right) - \theta_{\kappa_1}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)ds\right)$$

$$+\frac{\sigma_{r2}}{\theta_r}\frac{\mu_{\kappa_2}}{\theta_r - \theta_{\kappa_2}(1+\sigma_{\kappa_2})}\left(\frac{\theta_r}{(1+\sigma_{\kappa_2})}\left((T-t) - \int_t^T e^{-\theta_{\kappa_2}(1+\sigma_{\kappa_2})(T-s)}ds\right) - \theta_{\kappa_2}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)ds\right)$$

$$A_{\kappa_i}(t,T) = \frac{\sigma_{ri}}{\theta_r}\frac{1}{\theta_r - \theta_{\kappa_i}(1+\sigma_{\kappa_i})}\left(\frac{\theta_r}{\theta_{\kappa_i}(1+\sigma_{\kappa_i})}\left(1-e^{-\theta_{\kappa_i}(1+\sigma_{\kappa_i})(T-t)}\right) - \left(1-e^{-\theta_r(T-t)}\right)\right)$$

$$\varepsilon_{ri}(t,T) = \frac{\sigma_{ri}}{\theta_r}\left\{\frac{\sigma_{\kappa_i}}{\theta_r - \theta_{\kappa_i}(1+\sigma_{\kappa_i})}\left(\frac{\theta_r}{\theta_{\kappa_i}(1+\sigma_{\kappa_i})}\left(1-e^{-\theta_{\kappa_i}(1+\sigma_{\kappa_i})(T-s)}\right) - \left(1-e^{-\theta_r(T-s)}\right)\right) - \left(1-e^{-\theta_r(T-s)}\right)\right\}$$

Using (A13), it follows that:

$$P(t,T) = e^{a_r(t,T)+A_r(t,T)r(t)+A_{\kappa_1}(t,T)\kappa_1(t)+A_{\kappa_2}(t,T)\kappa_2(t)+\frac{1}{2}\int_t^T\left(\varepsilon_{r1}(s,T)^2+\varepsilon_{r2}(s,T)^2\right)ds} \tag{A14}$$

Defining

$$A(t,T) = a_r(t,T) + \frac{1}{2}\int_t^T\left(\varepsilon_{r1}(s,T)^2 + \varepsilon_{r2}(s,T)^2\right)ds \tag{A15}$$

and integrating yields

$$A(t,T) = -\mu_r\left((T-t) - \int_t^T e^{-\theta_r(T-s)}ds\right)$$

$$+\frac{\sigma_{r1}}{\theta_r}\frac{\mu_{\kappa_1}}{\theta_r - \theta_{\kappa_1}(1+\sigma_{\kappa_1})}\left(\frac{\theta_r}{(1+\sigma_{\kappa_1})}\left((T-t) - \int_t^T e^{-\theta_{\kappa_1}(1+\sigma_{\kappa_1})(T-s)}ds\right) - \theta_{\kappa_1}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)ds\right)$$

$$+\frac{\sigma_{r2}}{\theta_r}\frac{\mu_{\kappa_2}}{\theta_r - \theta_{\kappa_2}(1+\sigma_{\kappa_2})}\left(\frac{\theta_r}{(1+\sigma_{\kappa_2})}\left((T-t) - \int_t^T e^{-\theta_{\kappa_2}(1+\sigma_{\kappa_2})(T-s)}ds\right) - \theta_{\kappa_2}\int_t^T\left(1-e^{-\theta_r(T-s)}\right)ds\right)$$

$$+\frac{1}{2}\left(\frac{\sigma_{r1}}{\theta_r}\right)^2\left\{\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\frac{\theta_r}{\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\right)^2+\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}+1\right)^2\right.$$

$$\left.-2\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\frac{\theta_r}{\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}+1\right)\right\}(T-t)$$

$$+\frac{1}{2}\left(\frac{\sigma_{r1}}{\theta_r}\right)^2\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\frac{\theta_r}{\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\right)^2\int_t^T e^{-2\theta_{\kappa_1}(1+\sigma_{\kappa_1})(T-s)}ds$$

$$-\left(\frac{\sigma_{r1}}{\theta_r}\right)^2\left\{\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\frac{\theta_r}{\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\right)^2\right.$$

$$\left.-\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\frac{\theta_r}{\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}+1\right)\right\}\int_t^T e^{-\theta_{\kappa_1}(1+\sigma_{\kappa_1})(T-s)}ds$$

$$-\left(\frac{\sigma_{r1}}{\theta_r}\right)^2\left\{\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}+1\right)^2-\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\frac{\theta_r}{\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}+1\right)\right\}\int_t^T e^{-\theta_r(T-s)}ds$$

$$+\left(\frac{\sigma_{r1}}{\theta_r}\right)^2\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}+1\right)^2\int_t^T e^{-2\theta_r(T-s)}ds$$

$$-\left(\frac{\sigma_{r1}}{\theta_r}\right)^2\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\frac{\theta_r}{\theta_{\kappa_1}(1+\sigma_{\kappa_1})}\left(\frac{\sigma_{\kappa_1}}{\theta_r-\theta_{\kappa_1}(1+\sigma_{\kappa_1})}+1\right)\int_t^T e^{-(\theta_r+\theta_{\kappa_1}(1+\sigma_{\kappa_1}))(T-s)}ds$$

$$+\frac{1}{2}\left(\frac{\sigma_{r2}}{\theta_r}\right)^2\left\{\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\frac{\theta_r}{\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\right)^2+\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}+1\right)^2\right.$$

$$\left.-2\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\frac{\theta_r}{\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}+1\right)\right\}(T-t)$$

$$+\frac{1}{2}\left(\frac{\sigma_{r2}}{\theta_r}\right)^2\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\frac{\theta_r}{\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\right)^2\int_t^T e^{-2\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)(T-s)}ds$$

$$-\left(\frac{\sigma_{r2}}{\theta_r}\right)^2\left\{\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\frac{\theta_r}{\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\right)^2\right.$$

$$\left.-\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\frac{\theta_r}{\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}+1\right)\right\}\int_t^T e^{-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)(T-s)}ds$$

$$-\left(\frac{\sigma_{r2}}{\theta_r}\right)^2\left\{\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}+1\right)^2-\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\frac{\theta_r}{\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}+1\right)\right\}\int_t^T e^{-\theta_r(T-s)}ds$$

$$+\left(\frac{\sigma_{r2}}{\theta_r}\right)^2\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}+1\right)^2\int_t^T e^{-2\theta_r(T-s)}ds$$

$$-\left(\frac{\sigma_{r2}}{\theta_r}\right)^2\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\frac{\theta_r}{\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}\left(\frac{\sigma_{\kappa_2}}{\theta_r-\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)}+1\right)\int_t^T e^{-\left(\theta_r+\theta_{\kappa_2}\left(1+\sigma_{\kappa_2}\right)\right)(T-s)}ds$$

Substituting into (A14) yields the desired result.

Proof of Proposition 2:

We have:

$$E_t\left[\left(\frac{\Lambda(\tau)}{\Lambda(t)}\right)^{-\frac{1-\gamma}{\gamma}}\right]=E_t\left[\exp\left\{\frac{1-\gamma}{\gamma}\int_t^\tau r(s)ds+\frac{1}{2}\frac{1-\gamma}{\gamma}\int_t^\tau\left(\kappa_1(s)^2+\kappa_2(s)^2\right)ds\right.\right.$$
$$\left.\left.+\frac{1-\gamma}{\gamma}\int_t^\tau\kappa_1(s)dZ_1(s)+\frac{1-\gamma}{\gamma}\int_t^\tau\kappa_2(s)dZ_2(s)\right\}\right] \quad (A16)$$

Given the Markovian properety of our setting, it follows that:

$$E_t\left[\left(\frac{\Lambda(\tau)}{\Lambda(t)}\right)^{-\frac{1-\gamma}{\gamma}}\right]=\Pi\left(t,\tau,r(t),\kappa_1(t),\kappa_2(t),\kappa_1(t)^2,\kappa_2(t)^2\right) \quad (A17)$$

and we guess a solution like:

$$\Pi(t,\tau,r(t),\kappa_1(t),\kappa_2(t))=$$
$$\exp\left\{A_0(t,\tau)+A_1(t,\tau)r(t)+A_2(t,\tau)\kappa_1(t)+A_3(t,\tau)\kappa_2(t)+A_4(t,\tau)\kappa_1(t)^2+A_5(t,\tau)\kappa_2(t)^2\right\} \quad (A18)$$

Substituting into (18) yields:

$$V(t) = \lambda^{-\frac{1}{\gamma}} \Lambda(t)^{-\frac{1}{\gamma}} \exp\left\{ A_0(t,\tau) + A_1(t,\tau) r(t) + A_2(t,\tau)\kappa_1(t) + A_3(t,\tau)\kappa_2(t) + A_4(t,\tau)\kappa_1(t)^2 + A_5(t,\tau)\kappa_2(t)^2 \right\} \tag{A19}$$

and then:

$$\Lambda(t)V(t) = \lambda^{-\frac{1}{\gamma}} \Lambda(t)^{1-\frac{1}{\gamma}} \exp\left\{ A_0(t,\tau) + A_1(t,\tau) r(t) + A_2(t,\tau)\kappa_1(t) + A_3(t,\tau)\kappa_2(t) + A_4(t,\tau)\kappa_1(t)^2 + A_5(t,\tau)\kappa_2(t)^2 \right\} \tag{A20}$$

Applying Ito's lemma to (A20) and using (1), (2), and (3) yields:

$$\begin{aligned}
\frac{d\Lambda(t)V(t)}{\Lambda(t)V(t)} = &\left\{ \frac{\partial A_0(t,\tau)}{\partial t} + \frac{\partial A_1(t,\tau)}{\partial t} r(t) + \frac{\partial A_2(t,\tau)}{\partial t}\kappa_1(t) + \frac{\partial A_3(t,\tau)}{\partial t}\kappa_2(t) + \frac{\partial A_4(t,\tau)}{\partial t}\kappa_1(t)^2 \right. \\
&+ \frac{\partial A_5(t,\tau)}{\partial t}\kappa_2(t)^2 + A_1(t,\tau)\theta_r(\mu_r - r(t)) + (A_2(t,\tau) + 2\kappa_1(t)A_4(t,\tau))\theta_{\kappa_1}(\mu_{\kappa_1} - \kappa_1(t)) \\
&- \left(1-\frac{1}{\gamma}\right) r(t) + (A_3(t,\tau) + 2\kappa_2(t)A_5(t,\tau))\theta_{\kappa_2}(\mu_{\kappa_2} - \kappa_2(t)) + \frac{1}{2}A_1(t,\tau)^2\left(\sigma_{r1}{}^2 + \sigma_{r2}{}^2\right) \\
&+ \frac{1}{2}\left(-\frac{1}{\gamma}\right)\left(1-\frac{1}{\gamma}\right)\left(\kappa_1(t)^2 + \kappa_2(t)^2\right) + \frac{1}{2}\left(2A_4(t,\tau) + (A_2(t,\tau) + 2\kappa_1(t)A_4(t,\tau))^2\right)\sigma_{\kappa_1}{}^2 \\
&+ \frac{1}{2}\left(2A_5(t,\tau) + (A_3(t,\tau) + 2\kappa_2(t)A_5(t,\tau))^2\right)\sigma_{\kappa_2}{}^2 - \left(1-\frac{1}{\gamma}\right)A_1(t,\tau)(+\sigma_{r1}\kappa_1(t) + \sigma_{r2}\kappa_2(t)) \\
&- \left(1-\frac{1}{\gamma}\right)(A_2(t,\tau) + 2\kappa_1(t)A_4(t,\tau))\sigma_{\kappa_1}\kappa_1(t) - \left(1-\frac{1}{\gamma}\right)(A_3(t,\tau) + 2\kappa_2(t)A_5(t,\tau))\sigma_{\kappa_2}\kappa_2(t) \\
&\left. + A_1(t,\tau)(A_2(t,\tau) + 2\kappa_1(t)A_4(t,\tau))\sigma_{r1}\sigma_{\kappa_1} + A_1(t,\tau)(A_3(t,\tau) + 2\kappa_2(t)A_5(t,\tau))\sigma_{r2}\sigma_{\kappa_2} \right\} dt \\
&+ \left\{ -\left(1-\frac{1}{\gamma}\right)\kappa_1(t) + A_1(t,\tau)\sigma_{r1} + (A_2(t,\tau) + 2\kappa_1(t)A_4(t,\tau))\sigma_{\kappa_1} \right\} dZ_1(t) \\
&+ \left\{ -\left(1-\frac{1}{\gamma}\right)\kappa_2(t) + A_1(t,\tau)\sigma_{r2} + (A_3(t,\tau) + 2\kappa_2(t)A_5(t,\tau))\sigma_{\kappa_2} \right\} dZ_2(t)
\end{aligned} \tag{A21}$$

Since (A21) is a martingale, its drift should be zero. Therefore:

$$\frac{\partial A_1(t,\tau)}{\partial t} - \left(1-\frac{1}{\gamma}\right) - A_1(t,\tau)\theta_r = 0 \tag{A22}$$

$$\begin{aligned}
&\frac{\partial A_2(t,\tau)}{\partial t} + 2\sigma_{\kappa_1}{}^2 A_2(t,\tau)A_4(t,\tau) + 2\theta_{\kappa_1}\mu_{\kappa_1}A_4(t,\tau) - \theta_{\kappa_1}A_2(t,\tau) \\
&- \left(1-\frac{1}{\gamma}\right)A_1(t,\tau)\sigma_{r1} - \left(1-\frac{1}{\gamma}\right)\sigma_{\kappa_1}A_2(t,\tau) + 2\sigma_{r1}\sigma_{\kappa_1}A_1(t,\tau)A_4(t,\tau) = 0
\end{aligned} \tag{A23}$$

$$\frac{\partial A_3(t,\tau)}{\partial t}+2\sigma_{\kappa_2}{}^2A_3(t,\tau)A_5(t,\tau)+2\theta_{\kappa_2}\mu_{\kappa_2}A_5(t,\tau)-\theta_{\kappa_2}A_3(t,\tau)$$
$$-\left(1-\frac{1}{\gamma}\right)A_1(t,\tau)\sigma_{r2}-\left(1-\frac{1}{\gamma}\right)\sigma_{\kappa_2}A_3(t,\tau)+2\sigma_{r2}\sigma_{\kappa_2}A_1(t,\tau)A_5(t,\tau)=0 \quad \text{(A24)}$$

$$\frac{\partial A_4(t,\tau)}{\partial t}+\frac{1}{2}\left(-\frac{1}{\gamma}\right)\left(1-\frac{1}{\gamma}\right)+2\sigma_{\kappa_1}{}^2A_4(t,\tau)^2-2\theta_{\kappa_1}A_4(t,\tau)-\left(1-\frac{1}{\gamma}\right)2\sigma_{\kappa_1}A_4(t,\tau)=0 \quad \text{(A25)}$$

$$\frac{\partial A_5(t,\tau)}{\partial t}+\frac{1}{2}\left(-\frac{1}{\gamma}\right)\left(1-\frac{1}{\gamma}\right)+2\sigma_{\kappa_2}{}^2A_5(t,\tau)^2-2\theta_{\kappa_2}A_5(t,\tau)-\left(1-\frac{1}{\gamma}\right)2\sigma_{\kappa_2}A_5(t,\tau)=0 \quad \text{(A26)}$$

$$\frac{\partial A_0(t,\tau)}{\partial t}+\frac{1}{2}A_1(t,\tau)^2\left(\sigma_{r1}{}^2+\sigma_{r2}{}^2\right)+\sigma_{\kappa_1}{}^2A_4(t,\tau)+\frac{1}{2}\sigma_{\kappa_1}{}^2A_2(t,\tau)^2+\sigma_{\kappa_2}{}^2A_5(t,\tau)$$
$$+\frac{1}{2}\sigma_{\kappa_2}{}^2A_3(t,\tau)^2+A_1(t,\tau)\theta_r\mu_r+\theta_{\kappa_1}\mu_{\kappa_1}A_2(t,\tau)+\theta_{\kappa_2}\mu_{\kappa_2}A_3(t,\tau)+\sigma_{r1}\sigma_{\kappa_1}A_1(t,\tau)A_2(t,\tau) \quad \text{(A27)}$$
$$+\sigma_{r2}\sigma_{\kappa_2}A_1(t,\tau)A_3(t,\tau)=0$$

We have:

$$A_1(t,\tau)=\left(1-\frac{1}{\gamma}\right)\frac{1}{\theta_r}\left(e^{-\theta_r(\tau-t)}-1\right) \quad \text{(A28)}$$

Let's focus now on (A25). It could be written as:

$$\frac{\partial A_4(t,\tau)}{\partial t}=c_4A_4(t,\tau)^2+b_4A_4(t,\tau)+a_4 \quad \text{(A29)}$$

where

$$c_4=-2\sigma_{\kappa_1}{}^2$$
$$b_4=2\left(\theta_{\kappa_1}+\frac{\gamma-1}{\gamma}\sigma_{\kappa_1}\right) \quad \text{(A30)}$$
$$a_4=\frac{1}{2}\frac{1}{\gamma}\frac{\gamma-1}{\gamma}$$

Since:

$$\varphi_4=b_4{}^2-4a_4c_4=4\left(\theta_{\kappa_1}+\frac{\gamma-1}{\gamma}\sigma_{\kappa_1}\right)^2+4\frac{1}{\gamma}\frac{\gamma-1}{\gamma}\sigma_{\kappa_1}{}^2>0 \quad \text{(A31)}$$

when $\gamma>1$, it follows that:

$$A_4(t,\tau)=\frac{2a_4\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)}{2\sqrt{\varphi_4}-\left(b_4+\sqrt{\varphi_4}\right)\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)} \quad \text{(A32)}$$

and similarly

$$A_5(t,\tau)=\frac{2a_5\left(1-e^{-\sqrt{\varphi_5}(\tau-t)}\right)}{2\sqrt{\varphi_5}-\left(b_5+\sqrt{\varphi_5}\right)\left(1-e^{-\sqrt{\varphi_5}(\tau-t)}\right)} \tag{A33}$$

where

$$\begin{aligned} c_5 &= -2\sigma_{\kappa_2}{}^2 \\ b_5 &= 2\left(\theta_{\kappa_2}+\frac{\gamma-1}{\gamma}\sigma_{\kappa_2}\right) \\ a_5 &= \frac{1}{2}\frac{1}{\gamma}\frac{\gamma-1}{\gamma} \end{aligned} \tag{A34}$$

$$\varphi_5 = b_5{}^2-4a_5c_5 = 4\left(\theta_{\kappa_2}+\frac{\gamma-1}{\gamma}\sigma_{\kappa_2}\right)^2+4\frac{1}{\gamma}\frac{\gamma-1}{\gamma}\sigma_{\kappa_2}{}^2>0 \tag{A35}$$

On the other hand, we have:

$$\begin{aligned} \frac{\partial A_2(t,\tau)}{\partial t} &= c_4A_2(t,\tau)A_4(t,\tau)+\frac{b_4}{2}A_2(t,\tau)-2\theta_{\kappa_1}\mu_{\kappa_1}A_4(t,\tau) \\ &+\left(1-\frac{1}{\gamma}\right)A_1(t,\tau)\sigma_{r1}-2\sigma_{r1}\sigma_{\kappa_1}A_1(t,\tau)A_4(t,\tau) \end{aligned} \tag{A36}$$

and

$$\begin{aligned} \frac{\partial A_3(t,\tau)}{\partial t} &= c_5A_3(t,\tau)A_5(t,\tau)+\frac{b_5}{2}A_3(t,\tau)-2\theta_{\kappa_2}\mu_{\kappa_2}A_5(t,\tau) \\ &+\left(1-\frac{1}{\gamma}\right)A_1(t,\tau)\sigma_{r2}-2\sigma_{r2}\sigma_{\kappa_2}A_1(t,\tau)A_5(t,\tau) \end{aligned} \tag{A37}$$

Using standard integral tables yields:

$$\begin{aligned} A_2(t,\tau) &= -\frac{4\theta_{\kappa_1}\mu_{\kappa_1}\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)}{\sqrt{\varphi_4}}A_4(t,\tau)+\left(1-\frac{1}{\gamma}\right)^2\left[\frac{\sigma_{r1}}{\theta_r{}^2}\left(1-e^{\theta_r(\tau-t)}\right)+\frac{\sigma_{r1}}{\theta_r}(\tau-t)\right] \\ &-2\sigma_{r1}\sigma_{\kappa_1}A_1(t,\tau)A_4(t,\tau) \end{aligned} \tag{A38}$$

<u>Proof of Proposition 3:</u>

For $A_1(t,\tau)$, the result follows immediately from (A28). As to $A_4(t,\tau)$, first note that since $a_4>0$ when $\gamma > 1$, then the numerator is always positive. As to the denominator, it could be rewritten as follows:

$$2\sqrt{\varphi_4}-\left(b_4+\sqrt{\varphi_4}\right)\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)=\sqrt{\varphi_4}-b_4\left(1-e^{-\sqrt{\varphi_4}(\tau-t)}\right)+\sqrt{\varphi_4}e^{-\sqrt{\varphi_4}(\tau-t)} \tag{A39}$$

From (A31), it follows that $\sqrt{\varphi_4}>b_4$ and therefore (A39) is always positive. The same hold for $A_5(t,\tau)$. Finally, concerning $A_2(t,\tau)$, note that the second term is negative. To see this, note that this holds when:

$$\frac{1}{\theta_r}\left(1-e^{\theta_r(\tau-t)}\right)+(\tau-t)<0 \Leftrightarrow 1+\theta_r(\tau-t)<e^{\theta_r(\tau-t)} \tag{A40}$$

which is always satisfied. The last term being positive, the sign of $A_2(t,\tau)$ is indetermined.

<u>Proof of Proposition 5:</u>

Using (A28), one can write (23) in the following way:

$$\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix}=\frac{1}{\gamma}\Sigma(t,T)^{-1}\begin{bmatrix}\kappa_1(t)\\ \kappa_2(t)\end{bmatrix}+\left(1-\frac{1}{\gamma}\right)\Sigma(t,T)^{-1}\begin{bmatrix}\frac{1}{\theta_r}\left(e^{-\theta_r(\tau-t)}-1\right)\sigma_{r1}\\ \frac{1}{\theta_r}\left(e^{-\theta_r(\tau-t)}-1\right)\sigma_{r2}\end{bmatrix}$$
$$+\Sigma(t,T)^{-1}\begin{bmatrix}\left(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t)\right)\sigma_{\kappa_1}\\ \left(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t)\right)\sigma_{\kappa_2}\end{bmatrix} \tag{A41}$$

Given (10), the volatility of a discount bond maturing at the investor's horizon writes:

$$\begin{aligned}\sigma_{P1}(t,\tau)&=A_r(t,\tau)\sigma_{r1}+A_{\kappa_1}(t,\tau)\sigma_{\kappa_1}\\ \sigma_{P2}(t,\tau)&=A_r(t,\tau)\sigma_{r2}+A_{\kappa_2}(t,\tau)\sigma_{\kappa_2}\end{aligned} \tag{A42}$$

Using the fact that:

$$A_r(t,\tau)=-\frac{1}{\theta_r}\left(1-e^{-\theta_r(\tau-t)}\right)$$

we have:

$$\begin{aligned}\sigma_{P1}(t,\tau)&=-\frac{1}{\theta_r}\left(1-e^{-\theta_r(\tau-t)}\right)\sigma_{r1}+A_{\kappa_1}(t,\tau)\sigma_{\kappa_1}\\ \sigma_{P2}(t,\tau)&=-\frac{1}{\theta_r}\left(1-e^{-\theta_r(\tau-t)}\right)\sigma_{r2}+A_{\kappa_2}(t,\tau)\sigma_{\kappa_2}\end{aligned} \tag{A43}$$

Substituting for (A43) into (A41) yields:

$$\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix}=\frac{1}{\gamma}\Sigma(t,T)^{-1}\begin{bmatrix}\kappa_1(t)\\ \kappa_2(t)\end{bmatrix}+\left(1-\frac{1}{\gamma}\right)\Sigma(t,T)^{-1}\begin{bmatrix}\sigma_{P1}(t,\tau)-A_{\kappa_1}(t,\tau)\sigma_{\kappa_1}\\ \sigma_{P2}(t,\tau)-A_{\kappa_2}(t,\tau)\sigma_{\kappa_2}\end{bmatrix}$$
$$+\Sigma(t,T)^{-1}\begin{bmatrix}\left(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t)\right)\sigma_{\kappa_1}\\ \left(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t)\right)\sigma_{\kappa_2}\end{bmatrix} \tag{A44}$$

or else:

$$\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix}=\frac{1}{\gamma}\Sigma(t,T)^{-1}\begin{bmatrix}\kappa_1(t)\\ \kappa_2(t)\end{bmatrix}+\left(1-\frac{1}{\gamma}\right)\Sigma(t,T)^{-1}\begin{bmatrix}\sigma_{P1}(t,\tau)\\ \sigma_{P2}(t,\tau)\end{bmatrix}$$
$$+\Sigma(t,T)^{-1}\begin{bmatrix}\left(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_1}(t,\tau)\right)\sigma_{\kappa_1}\\ \left(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_2}(t,\tau)\right)\sigma_{\kappa_2}\end{bmatrix} \tag{A45}$$

This is (31). Explicit calculations yield:

$$\alpha(t)=\frac{1}{\gamma}\frac{\kappa_1(t)\sigma_{P2}(t,T)-\kappa_2(t)\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}+\left(1-\frac{1}{\gamma}\right)\frac{\sigma_{P1}(t,\tau)\sigma_{P2}(t,T)-\sigma_{P2}(t,\tau)\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$
$$+\frac{\left(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_1}(t,\tau)\right)\sigma_{\kappa_1}\sigma_{P2}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$
$$-\frac{\left(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_2}(t,\tau)\right)\sigma_{\kappa_2}\sigma_{P1}(t,T)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)} \tag{A46}$$

$$\beta(t)=\frac{1}{\gamma}\frac{\sigma_{S1}\kappa_2(t)-\sigma_{S2}\kappa_1(t)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}+\left(1-\frac{1}{\gamma}\right)\frac{\sigma_{S1}\sigma_{P2}(t,\tau)-\sigma_{S2}\sigma_{P1}(t,\tau)}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$
$$+\frac{\left(A_3(t,\tau)+2A_5(t,\tau)\kappa_2(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_2}(t,\tau)\right)\sigma_{\kappa_2}\sigma_{S1}}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)}$$
$$-\frac{\left(A_2(t,\tau)+2A_4(t,\tau)\kappa_1(t)-\left(1-\frac{1}{\gamma}\right)A_{\kappa_1}(t,\tau)\right)\sigma_{\kappa_1}\sigma_{S2}}{\sigma_{S1}\sigma_{P2}(t,T)-\sigma_{S2}\sigma_{P1}(t,T)} \tag{A47}$$

When the traded discount bond has a maturity equal to the investor's horizon, then:

$$\frac{\sigma_{P1}(t,\tau)\sigma_{P2}(t,\tau)-\sigma_{P2}(t,\tau)\sigma_{P1}(t,\tau)}{\sigma_{S1}\sigma_{P2}(t,\tau)-\sigma_{S2}\sigma_{P1}(t,\tau)}=0$$

$$\frac{\sigma_{S1}\sigma_{P2}(t,\tau)-\sigma_{S2}\sigma_{P1}(t,\tau)}{\sigma_{S1}\sigma_{P2}(t,\tau)-\sigma_{S2}\sigma_{P1}(t,\tau)}=1$$

and therefore the second component in (A46) cancels out while the second component in (A47) is equal to $\left(1-\frac{1}{\gamma}\right)$.

References

Bajeux-Besnainou, I., J. Jordan and R. Portait, 2001, An Asset Allocation Puzzle: A Comment, *American Economic Review* 91(4), 1170-1180.

Bajeux-Besnainou, I., J. Jordan and R. Portait, 2003, Dynamic Asset Allocation for Stocks, Bonds and Cash, *Journal of Business* 76 (2), 263 – 287.

Brennan, M. and J. Xia, 2000, Stochastic Interest Rates and the Bond-Stock Mix, *European Finance Review* 4, 197-210.

Brennan, M. and J. Xia, 2002, Dynamic Asset Allocation Under Inflation, *Journal of Finance* 57(3), 1201-1238.

Campbell, J. and L. Viceira, 2001, Who should buy long term bonds ? *American Economic Review* 91, 99 – 127.

Canner, N., G. Mankiw and D.Weil, 1997, An Asset Allocation Puzzle, *American Economic Review* 87 (1), 181-91.

Collin-Dufresne, P. and R. Goldstein, 2002, Do bonds span the fixed income market? Theory and evidence for unspanned stochastic volatility, *Journal of Finance* 57(4), 1685-1730.

Cox, J., and C.F. Huang, 1989, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33-83.

Cvitanic, J., L. Goukasian and F. Zapatero, 2003, Monte Carlo computation of optimal portfolios in complete markets, *Journal of Economic Dynamics and Control* 27(6), 971-986.

D'Adona, S. and A. Kind, 2005, International Stock – Bond correlations in a simple affine asset pricing model, Working Paper.

Dai, Q. and K. Singleton, 2003, Term structure dynamics in Theory and Practice, *Review of Financial Studies* 16(3), 631-678.

Detemple, J., R. Garcia and M. Rindisbacher, 2003, A Monte Carlo Method for Optimal Portfolios, *Journal of Finance* 58(1), 401-446.

Duffie, D., D. Filipovic and W. Schachermayer, 2003, Afine Processes and Applications in Finance, *Annals of Applied Probability* 13, 984-1053.

Fan, R., A. Gupta and P. Ritchken, 2003, Hedging in the possible presence of unspanned stochastic volatility: Evidence from Swaption Markets, *Journal of Finance* 58(5), 2219 - 2248.

Karatzas, I., J. Lehoczky and S. Shreve, 1987, Optimal portfolio and consumption decisions for a small investor on a finite horizon*, SIAM Journal of Control and Optimization* 25, 1557-1586.

Kim, T. and E. Omberg, 1996, Dynamic Nonmyopic Portfolio Behavior, *Review of Financial Studies* 9 (1), 141-61.

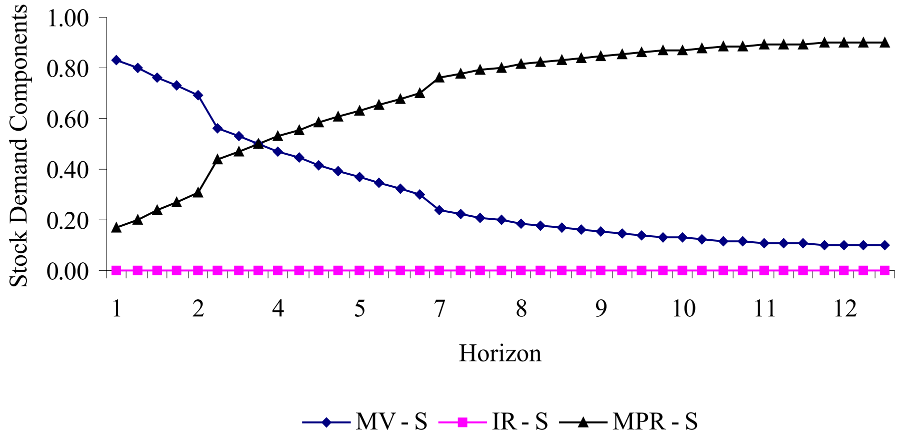
Li, L., 2002, Macroeconomic Factors and the correlation of stock and bond returns, Working Paper, Yale International Center for Finance.

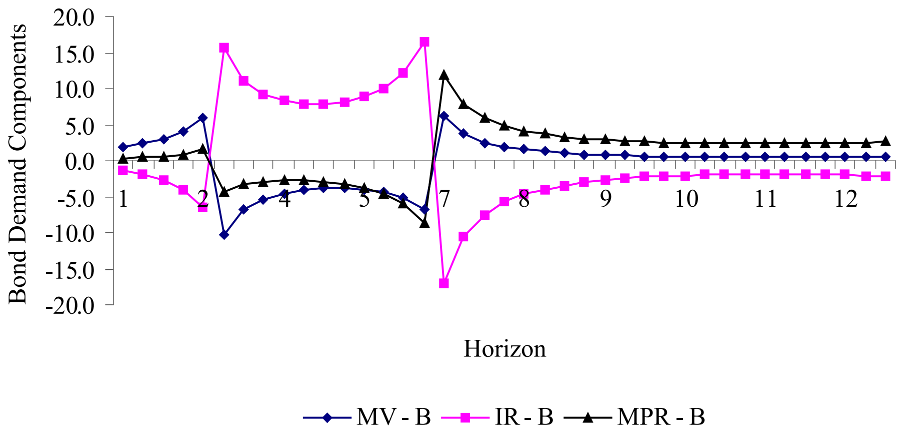
Lioui, A., and P. Poncet, 2001, On Optimal Portfolio Choice Under Stochastic Interest Rates, *Journal of Economic Dynamics and Control* 25, 1141-1865.

Liu, J., 2001, Portfolio Selection In Stochastic Environments, Working Paper.

Lynch, A. and S. Tan, 2005, Multiple Risky Assets, Transaction Costs, and Return Predictability: Implications for Portfolio Choice, Working Paper.
Mele, A., 2003, Fundamental Properties of Bond Prices in Models of the Short-Term Rate, *Review of Financial Studies* 16, 679-716.
Merton, R., 1973, An Intertemporal Capital Asset Pricing Model, *Econometrica* 41 (5), 867-87.
Mougeot, N., 2003, Interest Rate Risk, Learning and the Asset Allocation Puzzle, Working Paper.
Munk, C. and C. Sorensen, 2004, Optimal Consumption and Investment Strategies with Stochastic Interest Rates, *Journal of Banking and Finance* 28(8), 1987 – 2013.
Pliska, S., 1986, A Stochastic Calculus Model of Continuous Trading : Optimal Portfolios, *Mathematics of Operations Research* 11, 371-382.
Sangvinatsos, A. and J. Wachter, 2005, Does the failure of the Expectations Hypothesis matter for long term investors?, *Journal of Finance* LX(1), 179 – 230.
Shalit, H. and S. Yitzhaki, 2003, An Asset Allocation Puzzle: Comment, *American Economic Review* 93(3), 1002-1008.
Sorensen, C., 1999, Dynamic asset allocation and fixed income management, *Journal of Financial and Quantitative Analysis* 34, 513–532.
Wachter, J., 2002, Portfolio and Consumption Decisions under Mean Reverting Returns: An Exact Solution for Complete Markets, *Journal of Financial and Quantitative Analysis* 37, 63 – 91.

Figure 1a: Effects of investment horizon on Intertemporal Hedging components and the stock and bond mix when the discount bond allows a perfect hedge of interest rate risk.





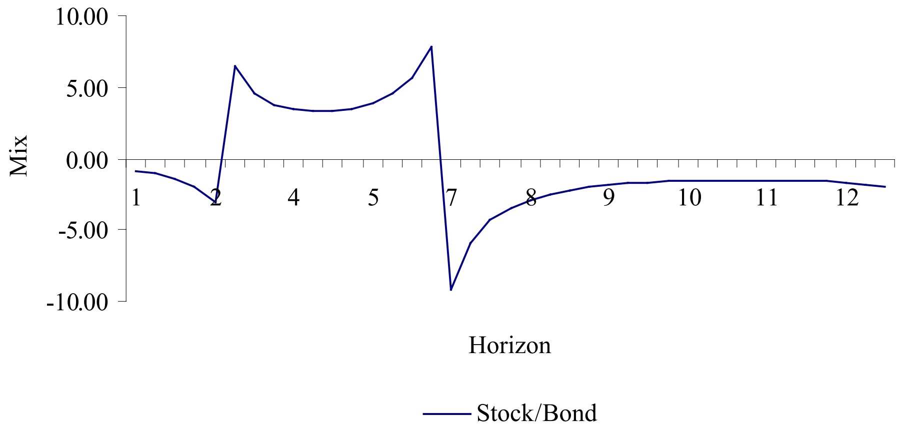


Figure 1b: Effects of investment horizon on Intertemporal Hedging components and the stock and bond mix when the discount bond does *not* allow a perfect hedge of interest rate risk.

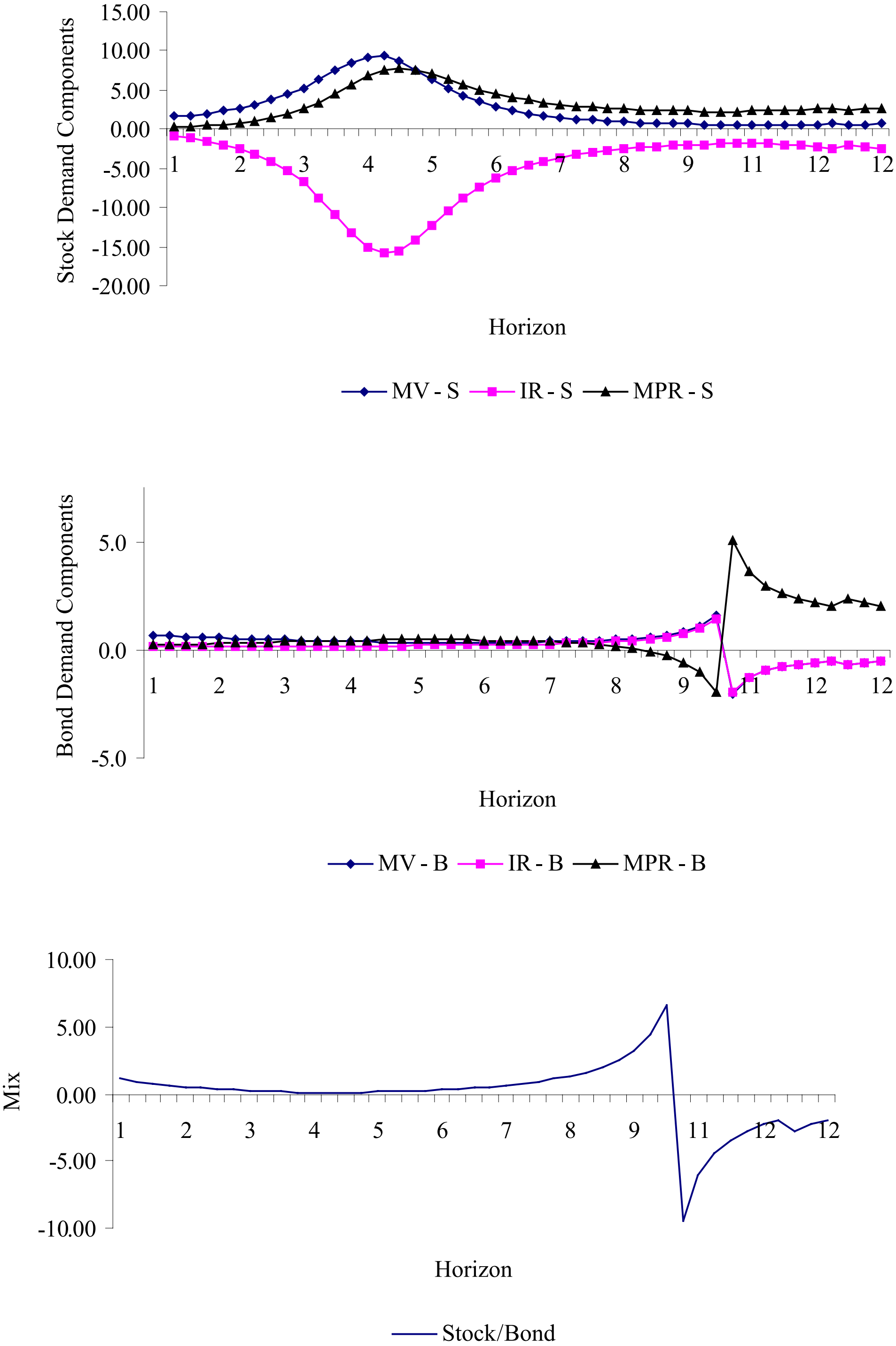


Figure 2a: Effects of the risk aversion parameter on Intertemporal Hedging components and the stock and bond mix when the discount bond allows a perfect hedge of interest rate risk.

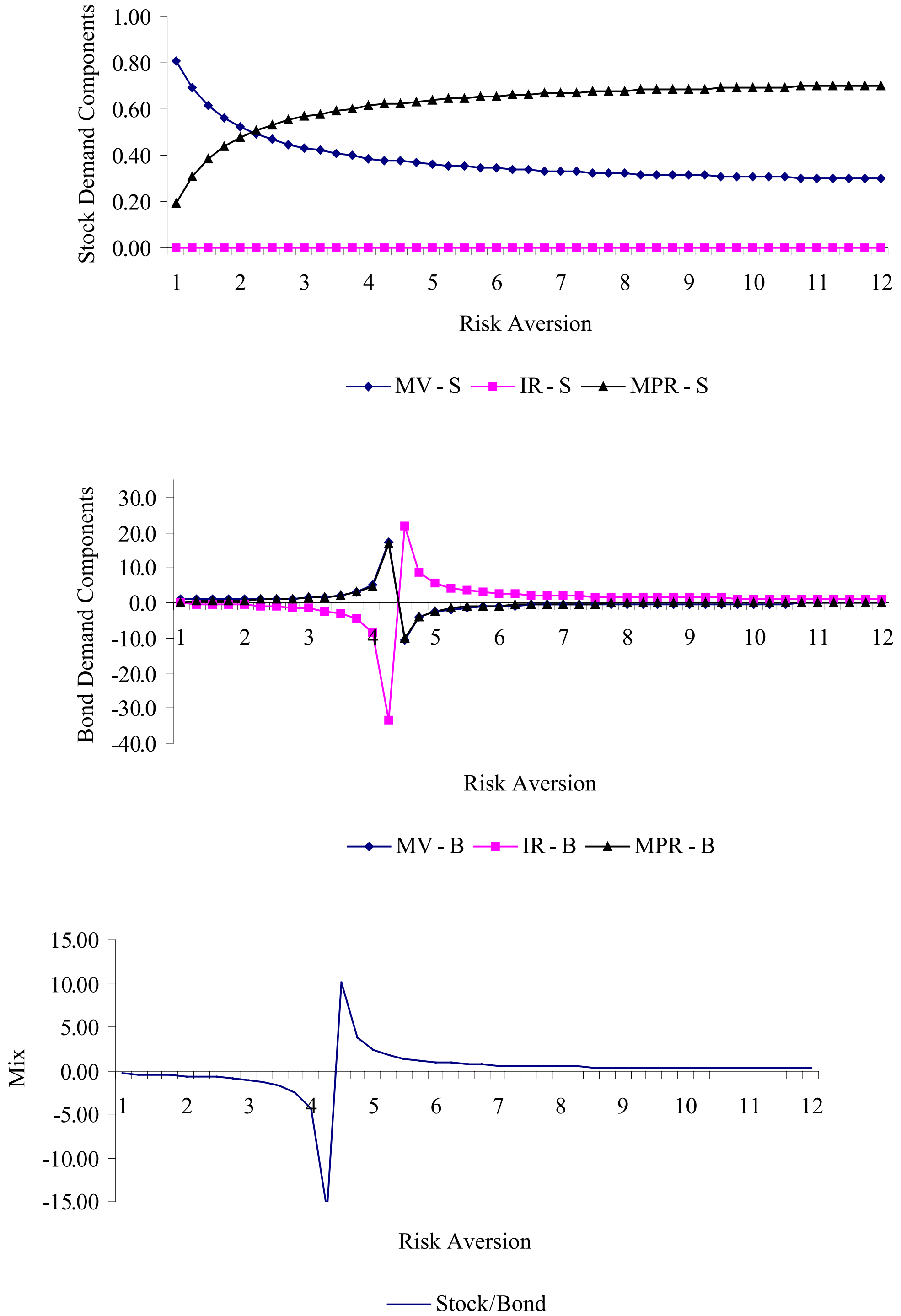


Figure 2b: Effects of the risk aversion parameter on Intertemporal Hedging components and the stock and bond mix when the discount bond does *not* allow a perfect hedge of interest rate risk.